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Fredholm multiplication operators between Hölder spaces

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Abstract. Let U be an open bounded subset of $\mathbb{R}^d$. We aim to investigate the multiplication operator on Hölder space $C^{1,1}(\overline{U})$, specifically focusing on Fredholm operators. Our goal is to provide an interpretation of these operators in these spaces and to discuss and prove their properties. In the case where U is a subset of $\mathbb{R}$, we present and prove results regarding these operators. Furthermore, we establish a necessary condition for the Fredholm property of these operators.

Keywords: Banach space, Hölder space, Differentiable functions, multiplication operator, Fredholm operator.

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1. Introduction

In functional analysis, composition operators serve as essential tools for examining the behavior of functions within Banach spaces. These operators emerge when a mapping is applied to functions, providing valuable insights in areas such as operator theory and complex analysis. To enhance the analysis of these operators, the concept of weighting enables us to modify the impact

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of the composed functions. This leads to the introduction of weighted composition operators, which integrate the process of composition with a weighting function, thus allowing for a more generalized approach to manipulating functions in Banach spaces.

Let A and B be two Banach spaces of complex-valued functions on Hausdorff topological spaces X and Y, respectively. For a complex-valued function u on Y and a map $\varphi : Y \to X$, if $u(f \circ \varphi) \in B$ whenever $f \in A$, then the operator $uC_\varphi : A \to B$ defined by $uC_\varphi(f) = u(f \circ \varphi)$ is called a weighted composition operator from A into B. In the special case where $u = 1$ or $X = Y$ and $\varphi(x) = x$, the weighted composition operator uC_φ reduces to the composition operator C_φ or the multiplication operator M_u, respectively. If the pointwise convergence topology on A and B is weaker than their norm topology, then using the closed graph theorem, every weighted composition operator $uC_\varphi : A \to B$ is automatically continuous, equivalently, it is a bounded linear operator.

The general form of numerous operators between various spaces of functions is in the form of weighted composition operators. Recently, the boundedness, compactness, Fredholmness, and essential norm of such operators have been studied intensively. For example, in [6] a complete description of weighted composition operators on Lipschitz algebras has been obtained, and then necessary and sufficient conditions for the injectivity, the surjectivity, and the compactness of these operators on Lipschitz algebras have been established. In [1], the study focuses on weighted composition operators defined on the space of differentiable functions $D^n(X)$, where X is a subset of the complex numbers. In that paper, a necessary and sufficient condition for these operators to be compact is provided and power compact composition operators on these algebras are characterized. Fredholm composition operators on $L^2(\mu)$ [10, 11], and on a variety of spaces of analytic functions [4, 5, 8, 12] have been characterized. Fredholm weighted composition operators on Diriclet spaces [16], on Hardy spaces [17] and on weighted Hardy spaces [18] have been studied. For disjointness preserving Fredholm operators between some spaces of continuous or Lipschitz functions see [2, 9, 13]. A complete description of Fredholm weighted composition operators between Lipschitz algebras has been provided in [14]. Also, that paper characterized Fredholm multiplication operators on little Lipschitz algebras and showed that the Fredholm index of such operators is zero. In this paper, we begin by introducing the concept of Hölder spaces, providing a detailed explanation of their properties and significance. We then present several preliminary theorems related to these spaces, which have already been established in the literature. These theorems serve as the foundational building blocks for our work in this paper. We will then introduce the Fredholm multiplication operator in this space and discuss the properties of these operators and the zero set of u and show that the Fredholm index of such operators is equal to the negative of its corank. We then focus on these types of operators in a specific case where the function space is defined on a subset of $\mathbb{R}$. In this context, we demonstrate that the zero set of u is empty under certain conditions. Then as the main theorem (Theorem 5.1), we give a description of Fredholm multiplication operators on the Hölder spaces.

2. **Fredholm Operators**

In functional analysis, Fredholm operators play a significant role in the study of linear operators on Banach and Hilbert spaces. A bounded linear operator $T : X \longrightarrow Y$ between Banach spaces X and Y is called a Fredholm operator if it satisfies the following conditions:

1) The kernel (null space) of T, denoted by $\ker(T)$, is finite-dimensional.
2) The quotient space $Y/R(T)$, where $R(T) = T(X)$, is finite-dimensional.

We define the nullity of T to be $\text{null}(T) = \dim(\ker(T))$. The corank of T is the codimension of $R(T)$ in Y, that is, $\text{corank}(T) = \text{codim}(R(T)) = \dim(Y/R(T))$. These properties allow us to analyze the operator in a more structured way, particularly concerning the solvability of equations and the dimensionality of various related spaces.

A crucial aspect of Fredholm operators is their index, defined as:

$$\text{ind}(T) = \text{null}(T) - \text{corank}(T).$$

The index provides valuable information about the operator's behavior, particularly in relation to perturbations; small perturbations of Fredholm operators remain Fredholm, and their index remains invariant.

Fredholm operators arise naturally in various contexts, including the study of integral equations, differential equations, and the analysis of linear systems. They are particularly important in the formulation of the Fredholm alternative, which provides conditions under which a linear equation has solutions and relates to the spectral theory of operators.

Overall, Fredholm operators serve as a bridge between functional analysis and the solutions of linear problems, enabling a deeper understanding of the structure and properties of linear operators in infinite-dimensional spaces.

For Banach spaces X and Y, we denote by $\mathcal{B}(X,Y)$ and $\mathcal{K}(X,Y)$ the Banach space of all bounded linear operators and compact linear operators from X into Y, respectively. In addition, the space of all finite rank operators from X into Y will be denoted by $\mathcal{F}(X,Y)$. If $X = Y$, then we write $\mathcal{B}(X) = \mathcal{B}(X,Y)$, $\mathcal{K}(X) = \mathcal{K}(X,Y)$ and $\mathcal{F}(X) = \mathcal{F}(X,Y)$ for short. It is known that $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$, and $\mathcal{F}(X,Y)$ is a subspace of $\mathcal{B}(X,Y)$, moreover, $\overline{\mathcal{F}(X,Y)} \subseteq \mathcal{K}(X,Y)$. The topological dual space of X is the Banach space X^* whose elements are the bounded linear functionals on X. The topological adjoint of $T \in \mathcal{B}(X,Y)$ is the uniquely determined operator $T^* \in \mathcal{B}(Y^*, X^*)$ satisfying $T^*(\Lambda) = \Lambda \circ T$ for every $\Lambda \in Y^*$.

In connection with a bounded Fredholm operator $T \in \mathcal{B}(X\ ,\ Y)$ the condition $\operatorname{codim}(R(T)) < \infty$ implies that the range of T is closed in Y. It is known that $T \in \mathcal{B}(X, Y)$ is Fredholm if and only if $T^* \in \mathcal{B}(Y^*, X^*)$ is Fredholm. Also if $T \in \mathcal{B}(Y, X)$ is Fredholm then for each $K \in \mathcal{K}(Y, X)$ the operator $T + K$ is Fredholm. Moreover, $T \in \mathcal{B}(X, Y)$ is Fredholm if and only if there exist $S \in \mathcal{B}(Y, X)$, $F_1 \in \mathcal{F}(X)$ and $F_2 \in \mathcal{F}(Y)$ such that $ST = I_V + F_1$ and $TS = I_Y + F_2$; and if and only if there exist $S \in \mathcal{B}(Y, X)$, $K_1 \in \mathcal{K}(X)$ and $K_2 \in \mathcal{K}(Y)$ such that $ST = I_X + K_1$ and $TS = I_Y + K_2$ where I_X is the identity operator on X. Note that compactness of an operator between infinite-dimensional Banach spaces does not imply the Fredholm property. The book [15, Chapter III] is a good reference for the proofs of these results.

3. **Hölder space**

Assume $U \subseteq \mathbb{R}^d$ is open and $0 < \alpha \leq 1$. We consider the space of all Lipschitz functions $f : U \to \mathbb{R}$, that is, functions satisfying the estimate

$$|f(x) - f(y)| \leq C\|x - y\| \quad (x, y \in U), \tag{3.1}$$

for some constant C. The estimate of course implies that f is continuous and more importantly provides a uniform modulus of continuity. It turns out to be useful to consider also functions f satisfying a variant of (3.1), namely

$$|f(x) - f(y)| \leq C\|x - y\|^\alpha \quad (x, y \in U), \tag{3.2}$$

for some $0 < \alpha \leq 1$ and a constant C. Such function is said to be Hölder continuous with exponent α.

Definition 1. *Let U be an open subset of $\mathbb{R}^d$. In this case:*

(*i*) *If $f : U \to \mathbb{R}$ is bounded and continuous, we write*

$$\|f\|_U = \sup_{x \in U} |f(x)|.$$

(*ii*) *For the $f : U \to \mathbb{R}$ defined as*

$$[f]_{0,\alpha} := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

and

$$\|f\|_{0,\alpha} := \|f\|_U + [f]_{0,\alpha}.$$

Definition 2. *Let U be an open bounded subset of $\mathbb{R}^d$. The Hölder space $C^{k,\alpha}(\overline{U})$ consists of all function $f \in C^k(\overline{U})$ for which the norm*

$$\|f\|_{k,\alpha} := \sum_{|\gamma| \leq k} \|D^\gamma f\|_U + \sum_{|\gamma| = k} [D^\gamma f]_{0,\alpha} \tag{3.3}$$

is finite.

So the space $C^{k,\alpha}(\overline{U})$ consists of those functions f that are k-times continuously differentiable and whose k-partial derivatives are bounded and Hölder continuous with exponent α. Such functions are well-behaved, and furthermore the space $C^{k,\alpha}(\overline{U})$ itself possesses a good mathematical structure.

Theorem 3.1. [3, Theorem 5.1.1]*[Hölder space as function spaces] The space of functions $C^{k,\alpha}(\overline{U})$ is a Banach space.*

4. Fredholm multiplication operator

Let U be an open bounded subset of $\mathbb{R}^d$ and let u be a real-valued function on $\overline{V}$. By the definition of the norm on $C^{1,1}(\overline{U})$, the pointwise convergence topology is weaker than the norm topology, hence any multiplication operator $T = M_u$ between Hölder spaces is a bounded linear operator. Moreover, if M_u is a multiplication operator on $C^{1,1}(\overline{U})$, since $C^{1,1}(\overline{U})$ contains the constant function 1 we have $u \in C^{1,1}(\overline{U})$.

For a surjective multiplication operator $M_u : C^{1,1}(\overline{U}) \longrightarrow C^{1,1}(\overline{U})$, the zero set of u is empty. We want to show in general, that if the map M_u has a finite corank, then the zero set of u is finite. As a result of [15, Lemma 16.2], we know that if a bounded operator has finite corank, then its range is closed. For now on, we suppose that U is an open bounded subset of $\mathbb{R}^d$, unless otherwise stated.

Theorem 4.1. *Let $T = M_u : C^{1,1}(\overline{U}) \longrightarrow C^{1,1}(\overline{U})$ be a multiplication operator with a finite corank m. Then the zero set of u,*

$$z(u) = \{y \in U : \quad u(y) = 0\},$$

is a finite subset of V with cardinality less than or equal to m.

Proof. The multiplication operator $T = M_u : C^{1,1}(\overline{U}) \longrightarrow C^{1,1}(\overline{U})$ is bounded with finite corank, so $R(T)$ is closed and $(C^{1,1}(\overline{U})/R(T))^* = (R(T))^{\perp}$. Thus

$$\begin{aligned}\dim(R(T))^{\perp} &= \dim((C^{1,1}(\overline{U})/R(T))^*) \\ &= \dim(C^{1,1}(\overline{U})/R(T)) = \operatorname{corank}(R(T)) = m.\end{aligned}$$

Now, suppose that $x_0 \in z(u)$, then $Tf(x_0) = u(x_0)f(x_0) = 0$, for all $f \in C^{1,1}(\overline{U})$. This means that $\delta_{x_0}(Tf) = 0$, then $\delta_{x_0} \in R(T)^{\perp}$. If $x_1, x_2, ..., x_k \in z(u)$ are distinct points, then the evaluation functional $\delta_{x_1}, \delta_{x_2}, ..., \delta_{x_k}$ are linearly independent in $R(T)^{\perp}$, so $k \leq m$ and thus $\operatorname{card}(z(u)) \leq m = \operatorname{corank}(R(T))$. □

Theorem 4.2. *Let M_u be a Fredholm multiplication operator on $C^{1,1}(\overline{U})$. Then the nullity of M_u is zero.*

Proof. According to theorem 4.1 the zero set $z(u)$ is finite, and if $M_u f = 0$ then $uf = 0$, and hence for all $x \in U \setminus z(u)$, $f(x) = 0$. Since every point in U is a limit point of U and $f \in C^{1,1}(\overline{U})$, in particular f is continuous on U, it follows that $f = 0$ on U. Hence, the operator M_u is injective, and thus its nullity is zero. □

For each $y \in \overline{U}$, let δ_y be the evaluation functional on $C^{1,1}(\overline{U})$ define by $\delta_y(f) = f(y)$. Clearly that δ_y is an element of the dual space $C^{1,1}(\overline{U})^*$ and $\|\delta_y\| = 1$. Moreover, by the definition of the adjoint of $T = M_u$ we have

$$T^*(\delta_y) = \delta_y \circ T = \delta_y \circ M_u = u(y)\delta_y. \tag{4.1}$$

In the final part of this article, our aim is to present results for the case where Hölder functions are defined on a subset of the real numbers. In fact, we consider the behavior of multiplication operators between two Hölder spaces, where the domain of the Hölder functions is a bounded interval of real numbers.

Let U be an open bounded interval in $\mathbb{R}$. Then it follows from the mean value theorem,

$$\begin{aligned}\|\delta_x - \delta_y\| &= \sup_{\substack{f \in C^{1,1}(\overline{U}) \\ \|f\|_{1,1} \le 1}} |f(x) - f(y)| \\ &\le \sup_{\substack{f \in C^{1,1}(\overline{U}) \\ \|f\|_{1,1} \le 1}} |f'(z)||x-y| \le \|f\|_{1,1}|x-y| \le |x-y|,\end{aligned}$$

For each $x, y \in U$, and for some point z between x and y. The following lemma can be proved using the same argument as in [7].

Lemma 4.3. *Let U be an open bounded interval in $\mathbb{R}$. Then*

$$\|\delta_x - \delta_y\| \ge \frac{2|x-y|}{(2+2M) + |x-y|}, \tag{4.2}$$

for each distinct $x, y \in U$, where δ_x and δ_y are regarded as elements of the dual space $C^{1,1}(\overline{U})^*$ and $M = \operatorname{diam} U$.

Proof. We show that for each $x, y \in U$, there exist $f \in C^{1,1}(\overline{U})$ such that $\|f\| \le 1$ and $|(\delta_x - \delta_y)(f)| \ge \frac{2|x-y|}{(2+2M)+|x-y|}$. To this end, define $f_{x,y} : U \to \mathbb{R}$ as follows,

$$f_{x,y}(t) = \frac{2t - x - y}{(2+2M) + |x-y|},$$

for each $x, y \in U$ with $x \neq y$. Thus for every $t \in U$ we have,

$$|f_{x,y}(t)| = \frac{|2t - x - y|}{(2+2M) + |x-y|} \le \frac{2M + |x-y|}{2 + 2M + |x-y|}.$$

Also $f'_{x,y}(t) = \frac{2}{(2+2M)+|x-y|}$ hence $\|f'_{x,y}\|_U = \frac{2}{(2+2M)+|x-y|}$ and $[f'_{x,y}]_{0,1} = 0$, therefore

$$\begin{aligned}\|f_{x,y}\|_{1,1} &= \|f_{x,y}\|_U + \|f'_{x,y}\|_U + [f'_{x,y}]_{0,1} \\ &\le \frac{2M + |x-y|}{(2+2M) + |x-y|} + \frac{2}{(2+2M) + |x-y|} = 1.\end{aligned}$$

On the other hand,

$$|(\delta_x - \delta_y)(f_{x,y})| = \frac{2|x-y|}{(2+2M)+|x-y|},$$

hence, $\|\delta_x - \delta_y\| \geq \frac{2|x-y|}{(2+2M)+|x-y|}$. □

We will now show that under certain conditions, the set $z(u)$ is empty. To prove this, we need the following proposition from functional analysis, which is established in [14].

Proposition 4.4. [14, proposition 3.2] *Let B be a Banach space and let $\{\lambda_n\}$ be a sequence in B^* with $\lambda_n x \to 0$ for each $x \in B$. Then $\|F^*(\lambda_n)\| \to 0$ for each $F \in \mathcal{F}(B)$.*

Theorem 4.5. *Let U be an open interval in $\mathbb{R}$. Let $T = M_u : C^{1,1}(\overline{U}) \to C^{1,1}(\overline{U})$ be a multiplication operator. Suppose that the identity function lies in the range of T. If $z(u) = z(u')$, then either $z(u)$ is empty or T is not Fredholm.*

Proof. We suppose that T is Fredholm, hence according to theorem 4.1, $z(u)$ is finite. We want to show that $z(u)$ is empty.

For each $y, z \in U$, with $y \neq z$ consider the bounded linear functional

$$\tilde{\delta}_{y,z} = \frac{\delta_y - \delta_z}{|y-z|} : C^{1,1}(\overline{U}) \to \mathbb{R}.$$

According to lemma 4.3, when $M = \operatorname{diam} U$ we have,

$$\|\tilde{\delta}_{y,z}\| \geq \frac{\|\delta_y - \delta_z\|}{|y-z|} \geq \frac{2}{(2+2M)+|y-z|},$$

The adjoint operator $T^* = (M_u)^* : C^{1,1}(\overline{U})^* \longrightarrow C^{1,1}(\overline{U})^*$ is well defined, hence according to equation (4.1), $T^*(\delta_y) = \delta_y \circ T = u(y)\delta_y$ for each $y \in U$. Therefore

$$\begin{aligned} T^*(\tilde{\delta}_{y,z}) &= T^*(\frac{\delta_y - \delta_z}{|y-z|}) \\ &= \frac{u(y)\delta_y - u(z)\delta_z}{|y-z|} \\ &= \frac{u(y)-u(z)}{|y-z|}\delta_y + u(z)\frac{\delta_y - \delta_z}{|y-z|}, \end{aligned}$$

for $y, z \in U$ with $y \neq z$. Thus for each $y \in U$ and $y_0 \in z(u)$ with $y \neq y_0$, we have

$$T^*(\tilde{\delta}_{y,y_0}) = \frac{u(y)-u(y_0)}{|y-y_0|}\delta_y.$$

By the hypotheses, $T = M_u : C^{1,1}(\overline{U}) \longrightarrow C^{1,1}(\overline{U})$ is a bounded Fredholm operator. Therefore there exist a nonzero operator $S \in \mathcal{B}(C^{1,1}(\overline{U}))$ and $F \in$

$\mathcal{F}(C^{1,1}(\overline{U}))$ such that $TS = I + F$, where I is the identity map on $C^{1,1}(\overline{U})$. The letter I is used for the identity operator on both $C^{1,1}(\overline{U})$ and $C^{1,1}(\overline{U})^*$. Thus

$$\begin{aligned}\tilde{\delta}_{y,y_0} + F^*(\tilde{\delta}_{y,y_0}) &= (I+F)^*(\tilde{\delta}_{y,y_0}) = S^*T^*(\tilde{\delta}_{y,y_0}) \\ &= \frac{u(y)-u(y_0)}{|y-y_0|} S^*(\delta_y),\end{aligned}$$

for $y \in U$ and $y_0 \in z(u)$ with $y \neq y_0$.

Now, suppose that $z(u) \neq \emptyset$, and define

$$B = R(T) \oplus \operatorname{span}\{1\}.$$

In this case, B is a Banach subspace of $C^{1,1}(\overline{U})$ containing the function $f_{x,y}$ introduced in the proof of Lemma 4.3. Therefore,

$$\begin{aligned}\frac{2}{(2+2M)+|y-y_0|} - \|(F^*(\tilde{\delta}_{y,y_0}))|_B\| &\leq \|(\tilde{\delta}_{y,y_0})|_B\| - \|(F^*(\tilde{\delta}_{y,y_0}))|_B\| \\ &\leq \|(\tilde{\delta}_{y,y_0} + F^*(\tilde{\delta}_{y,y_0}))|_B\| \\ &\leq \frac{|u(y)-u(y_0)|}{|y-y_0|} \|S^*(\tilde{\delta}_{y,y_0})\| \\ &\leq \frac{|u(y)-u(y_0)|}{|y-y_0|} \|S^*\|,\end{aligned} \tag{4.3}$$

for each $y \in U$ and $y_0 \in z(u)$ with $y \neq y_0$.

we can choose an element y_0 of $z(u)$. According to theorem 4.1 $z(u)$ is finite and since U is an open interval, so there exists a sequence $\{y_n\} \subseteq \operatorname{coz}(u)$ converges to y_0. By using the equation (4.3),

$$\frac{2}{(2+2M)+|y_n-y_0|} - \|(F^*(\tilde{\delta}_{y_n,y_0}))|_B\| \leq \frac{|u(y_n)-u(y_0)|}{|y_n-y_0|} \|S^*\|.$$

Therefore if n tends to infinity, by proposition 4.4, the left side of the above equation tends to be $\frac{1}{1+M}$, in the event that the right side tends to zero, and this is a contradiction. Thus $z(u)$ is an empty set. □

5. conclusion

Theorem 5.1. *Let U be an open bounded subset of $\mathbb{R}^d$. Suppose u is a real-valued function on U such that $T = M_u$ is a Fredholm multiplication operator on $C^{1,1}(\overline{U})$, with finite nullity N and finite corank m. Then,*

1. *$z(u)$ is a finite subset of U such that the cardinality of $z()$ is less than or equal to m.*
2. *Nullity of M_u is zero, in fact M_u is injective.*

(3) *In a special case where U be an open intervals in $\mathbb{R}$ and the identity function lies in the range of T, if $z(u) = z(u')$ then $z(u)$ is empty.*

Under these assumptions, the operator satisfies a necessary condition for being Fredholm.

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