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# Study of geodesic equations of Randers metrics on para-hypercomplex Lie groups

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Abstract. In this paper, we study the 4-dimensional para-hypercomplex Lie groups equipped with a left invariant Randers metric. We first get all the equations related to geodesic vectors on four dimensional para-hypercomplex Lie groups. Finally, we show the condition of the equivalence of geodesic vectors in the Riemannian and Finslerian spaces.

Keywords: Geodesic equation, Geodesic vector, Lie groups, Para-hypercomplex structure, Randers metric.

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## 1. Introduction

A geodesic is the shortest curve among all piece-wise differentiable curves on any surface connecting two points. For the first time in 1697, Bernoulli study about concept of geodesics. Let $S$ be a surface on $\mathbb{R}^3$ parameterized by a regular patch $X$. Then the geodesics on $S$ are determined by the system of

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two second order differential equations

$$\begin{cases} u'' + \Gamma^1_{11}u'^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}v'^2 = 0, \\ v'' + \Gamma^2_{11}u'^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}v'^2 = 0, \end{cases}$$

where the $\Gamma^i_{jk}$ are the Christoffel symbols of $X$ and $u, v : (a, b) \to \mathbb{R}$ are differentiable functions. For example the three straight lines $v \mapsto (0, v, 0)$, $v \mapsto (v\sqrt{3}, v, 0)$ and $v \mapsto (-v\sqrt{3}, v, 0)$ are geodesics on monkey saddle $z = x^3 - 3xy^2$ that satisfies the differential equation

$$uu'^2 - 2vu'v' - uv'^2 = 0.$$

Now assume that $M$ be a smooth manifold and let $\nabla$ be a connection in $TM$. Let $\sigma : I \to M$ be a smooth curve. In the term of smooth coordinates $(x^i)$ we can write $\sigma(t) = (x^1(t), \ldots, x^n(t))$. Then $\sigma$ is a geodesic if and only if its component functions satisfy the following geodesic equation

$$\frac{d^2\sigma^k}{dt^2} + \Gamma^k_{ij}\frac{d\sigma^i}{dt}\frac{d\sigma^j}{dt} = 0.$$

In 2003, the authors in [2] study the para-hypercomlex structure and they have classified four dimensional real Lie algebras which admit a para-hypercomlex structure. Later in [6], the author study properties of left invariant Riemannian metrics on para-hypercomlex four dimensional Lie groups. Also in [6], the explicit formulas for computing flag curvature have been obtained for left invariant Randers metrics of Berwald type. In this paper, we consider the left invariant Randers metrics and we give all geodesics equations on the para-hypercomplex 4-dimensional Lie groups. We note that in [7], we describe all geodesic vectors of the invariant infinite series metric on the left invariant hypercomplex four dimensional simply connected Lie groups and in [5], we study geodesic vectors of the left invariant $(\alpha, \beta)$-metrics on nilpotent Lie groups of five dimensional.

## 2. Preliminaries

Assume that $M$ be a smooth $n$-dimensional $C^\infty$ manifold and $TM$ be its tangent bundle. A Finsler metric on $M$ is a non-negative function $F : TM \to \mathbb{R}$ with the following properties [1]:

(1) $F$ is smooth on the slit tangent bundle $TM^0 := TM\backslash\{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The following bilinear symmetric form $g_y : T_xM \times T_xM \to \mathbb{R}$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}. \tag{2.1}$$

Let $\alpha = \sqrt{\widetilde{a}_{ij}(x)\, y^i y^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta(x,y) = b_i(x)y^i$ be a 1-form on an $n$-dimensional manifold $M$. Let

$$b := \|\beta(x)\|_\alpha := \sqrt{\widetilde{a}^{ij}(x) b_i(x) b_j(x)}.$$

Now, assume that the function $F$ is defined as follows:

$$F := \alpha\varphi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\varphi = \varphi(s)$ is a positive $C^\infty$ function on $(-b_0, b_0)$ satisfying:

$$\varphi(s) - s\varphi'(s) + \left(b^2 - s^2\right)\varphi''(s) > 0, \qquad |s| \leq \ b < b_0.$$

Then $F$ is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.2) is called an $(\alpha,\beta)$-metric [7]. A Finsler space having the Finsler function $F(x,y) = \alpha(x,y) + \beta(x,y)$, is called a Randers space. We note that the Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space $T_x^*M$ such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on $T_x^*M$ induces a linear isomorphism between $T_x^*M$ and $T_xM$. Then the 1-form $\beta$ corresponds to a vector field $\tilde{X}$ on $M$ such that $\tilde{a}(y, \tilde{X}(x)) = \beta(x,y)$. Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Thus we can write Randers metric as $F(x,y) = \sqrt{\tilde{a}(y,y)} + \tilde{a}(X,y)$. Consider the Chern connection on $\pi^*TM$ whose coefficients are denoted by $\Gamma^i_{jk}$. Let $\gamma(t)$ be a smooth regular curve in $M$ with velocity field $V$. Suppose $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along $\gamma$. Then the covariant derivative $D_V W$ with reference vector $V$ have the form

$$\Big[\frac{dW^i}{dt} + W^j V^k (\Gamma^i_{jk})_{(\gamma,V)}\Big]\frac{\partial}{\partial x^i}|_{\gamma(t)}.$$

A curve $\gamma(t)$ with the velocity $V$, is a Finslerian geodesic if

$$D_V\Big[\frac{V}{F(V)}\Big] = 0, \quad \text{with reference vector } \ V.$$

We recall that, in Riemannian setting the authors in [3], proved that a $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$\langle [X,Y]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = 0, \quad \forall Y \in \mathfrak{m}. \tag{2.3}$$

After this, the second author in Finsler setting shown that:

**Lemma 2.1.** [4] *Suppose $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

*Therefore, $Y \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y,Z]_{\mathfrak{m}}) = 0, \quad \forall Z \in \mathfrak{m}, \tag{2.4}$$

*where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.*

## 3. Geodesic vectors of Randers metric on para-hypercomplex Lie groups

Assume that $M$ be a smooth manifold and $\{J_i\}_{i=1,2,3}$ be a family of fiberwise endomorphism of $TM$ such that

$$J_1^2 = -Id_{TM}, \quad J_2^2 = Id_{TM},$$
$$J_2 \neq \pm Id_{TM}, \quad J_1J_2 = -J_2J_1 = J_3,$$

and $N_i = 0, i = 1, 2, 3$ where $N_i$ is the Nijenhuis tensor corresponding to $J_i$ defined as follows:

$$N_1(X,Y) = [J_1X, J_1Y] - J_1([X, J_1Y] + [J_1X, Y]) - [X, Y],$$
$$N_i(X,Y) = [J_iX, J_iY] - J_i([X, J_iY] + [J_iX, Y]) + [X, Y], i = 2, 3,$$

for all vector fields $X, Y$ on $M$. Then the family $\{J_i\}_{i=1,2,3}$ is called a para-hypercomplex structure on $M$. Recall that, indeed a para-hypercomplex structure on a smooth manifold $M$ is a triple $\{J_i\}_{i=1,2,3}$ such that $J_1$ is a complex structure and $J_2, J_3$ are two non-trivial integrable product structures on $M$ satisfying $J_1J_2 = -J_2J_1 = J_3$.

A para-hypercomplex structure $\{J_i\}_{i=1,2,3}$ on a Lie group $G$ is said to be left invariant if for any $t \in G$, $J_i = Tl_t \circ J_i \circ Tl_{t^{-1}}, i = 1, 2, 3,$ where $Tl_t$ is the differential function of the left translation $l_t$. A Riemannian metric $g$ on a Lie group $G$ is called left invariant if

$$g(t)(X,Y) = g(e)(T_tl_{t^{-1}}X, T_tl_{t^{-1}}Y), \quad \forall t \in G, X, Y \in T_tG,$$

where $e$ is the unit element of $G$. Like as above a Finsler metric $F$ on a Lie group $G$ is called left invariant if $F(t, X) = F(e, T_tl_{t^{-1}}X), \quad \forall t \in G, X \in T_tG.$

In [2], the authors classified four dimensional Lie algebras admitting left invariant para-hypercomplex structures. In each case, let $G_i$ be the connected four dimensional Lie group corresponding to the considered Lie algebra $\mathfrak{g}_i$ and $\langle , \rangle$ is an inner product on $\mathfrak{g}_i$ such that $\{E_1, E_2, E_3, E_4\}$ is an orthonormal basis for $\mathfrak{g}_i$. In the following we have these cases [2]:

(1)
$$[E_1, E_2] = E_2, \quad [E_1, E_4] = E_4. \tag{3.1}$$

(2)
$$[E_1, E_2] = E_3. \tag{3.2}$$

(3)
$$[E_1, E_2] = E_1. \tag{3.3}$$

(4)
$$[E_1, E_3] = E_1, \;\; [E_1, E_4] = E_2, \;\; [E_2, E_3] = E_2, \;\; [E_2, E_4] = \lambda E_1 + \gamma E_2, \;\; \lambda, \gamma \in \mathbb{R}. \tag{3.4}$$

(5)

$$[E_1, E_3] = E_1, \quad [E_2, E_4] = E_2. \tag{3.5}$$

(6)

$$[E_1, E_2] = E_4, \quad [E_1, E_4] = -E_2, \quad [E_2, E_4] = -E_1. \tag{3.6}$$

where $\{E_1, E_2, E_3, E_4\}$ is an orthonormal basis. Now we describe all geodesic vectors of left invariant Randers metrics $F$ defined by $F(x, y) = \sqrt{\tilde{a}(y, y)} + \tilde{a}(X, y)$. By using (2.1) and some computations for the Randers metric $F$ we have:

$$\begin{aligned} g_Y(U, V) =& \tilde{a}(U, V) + \tilde{a}(X, U)\tilde{a}(X, V) - \frac{\tilde{a}(X, Y)\tilde{a}(Y, U)\tilde{a}(Y, V)}{\tilde{a}(Y, Y)^{3/2}} \\ &+ \frac{1}{\sqrt{\tilde{a}(Y, Y)}}\Big\{\tilde{a}(X, U)\tilde{a}(Y, V) + \tilde{a}(X, Y)\tilde{a}(U, V) + \tilde{a}(X, V)\tilde{a}(Y, U)\Big\}. \end{aligned}$$

So for all $Z \in \mathfrak{g}$ we have:

$$g_Y(Y, [Y, Z]) = \tilde{a}\Big(X + \frac{Y}{\sqrt{\tilde{a}(Y, Y)}}, [Y, Z]\Big)F(Y). \tag{3.7}$$

By using Lemma 2.1 and equation (3.7), a vector $Y = a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 \in \mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}\left(\sum_{i=1}^{4} b_iE_i + \frac{\sum_{i=1}^{4} a_iE_i}{\sqrt{\sum_{i=1}^{4} a_i^2}}, \Big[\sum_{i=1}^{4} a_iE_i, E_j\Big]\right) = 0, \quad j = 1, 2, 3, 4. \tag{3.8}$$

3.1. **Case (1).** In this case by using equation (3.1) we have:

$$\begin{cases} a_2b_2 + a_4b_4 + \frac{a_2^2+a_4^2}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0, \\ a_1b_2 + \frac{a_1a_2}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0, \\ a_1b_4 + \frac{a_1a_4}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0. \end{cases}$$

**Corollary 3.1.** *Assume that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_1$. Then geodesic vectors depending only on $\tilde{a}(X, E_2)$ and $\tilde{a}(X, E_4)$.*

**Theorem 3.2.** *Suppose that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = b_1E_1 + b_3E_3$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_1$. Then $Y \in \mathfrak{g}_1$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* Let $Y \in \sum_{i=1}^{4} a_iE_i \in \mathfrak{g}_1$. Suppose that $Y$ is a geodesic vector of $(M, \tilde{a})$. By using equation (2.3) we have $\tilde{a}(Y, [Y, E_i]) = 0$ for each $i = 1, 2, 3, 4$. Therefore by using (3.8), $Y$ is a geodesic of $(M, F)$.

Conversely, let $Y = \sum_{i=1}^{4} a_i E_i \in \mathfrak{g}_1$ is a geodesic vector of $(M, F)$, because $\tilde{a}(X, [Y, E_i]) = 0$ for each $i = 1, 2, 3, 4$, by using (3.8) we have $\tilde{a}(Y, [Y, E_i]) = 0$. This proof the assertion. □

3.2. **Case (2).** In this case by using equation (3.2) we have:

$$\begin{cases} a_2 b_3 + \frac{a_2 a_3}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0, \\ a_1 b_3 + \frac{a_1 a_3}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0. \end{cases}$$

**Corollary 3.3.** *Assume that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_2$. Then geodesic vectors depending only on $\tilde{a}(X, E_3)$.*

**Theorem 3.4.** *Suppose that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = b_1 E_1 + b_2 E_2 + b_4 E_4$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_2$. Then $Y \in \mathfrak{g}_2$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

3.3. **Case (3).** In this case by using equation (3.3) we have:

$$\begin{cases} a_2 b_1 + \frac{a_1 a_2}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0, \\ a_1 b_1 + \frac{a_1^2}{\sqrt{\sum_{i=1}^{4} a_i^2}} = 0. \end{cases}$$

**Corollary 3.5.** *Assume that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_3$. Then geodesic vectors depending only on $\tilde{a}(X, E_1)$.*

**Theorem 3.6.** *Suppose that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = b_2 E_2 + b_3 E_3 + b_4 E_4$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_3$. Then $Y \in \mathfrak{g}_3$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

3.4. **Case (4).** In this case by using equation (3.4) we have:

$$\begin{cases} a_3b_1 + a_4b_2 + \frac{a_1a_3+a_2a_4}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_4b_1\lambda + b_2(a_3 + a_4\gamma) + \frac{a_1a_4\lambda+a_2(a_3+a_4\gamma)}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_1b_1 + a_2b_2 + \frac{a_1^2+a_2^2}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_2b_1\lambda + b_2(a_1 + a_2\gamma) + \frac{a_1a_2\lambda+a_2(a_1+a_2\gamma)}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0. \end{cases}$$

**Corollary 3.7.** *Assume that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_4$. Then geodesic vectors depending only on $\tilde{a}(X, E_1)$, $\tilde{a}(X, E_2)$, $\lambda$ and $\gamma$.*

**Theorem 3.8.** *Suppose that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = b_3E_3 + b_4E_4$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_4$. Then $Y \in \mathfrak{g}_4$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

3.5. **Case (5).** In this case by using equation (3.5) we have:

$$\begin{cases} a_3b_1 + \frac{a_1a_3}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_4b_2 + \frac{a_2a_4}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_1b_1 + \frac{a_1^2}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_2b_2 + \frac{a_2^2}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0. \end{cases}$$

**Corollary 3.9.** *Assume that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_5$. Then geodesic vectors depending only on $\tilde{a}(X, E_1)$ and $\tilde{a}(X, E_2)$.*

**Theorem 3.10.** *Suppose that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = b_3E_3 + b_4E_4$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_5$. Then $Y \in \mathfrak{g}_5$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

3.6. **Case (6).** In this case by using equation (3.6) we have:

$$\begin{cases} a_4b_2 - a_2b_4 = 0, \\ a_4b_1 + a_1b_4\frac{2a_1a_4}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0, \\ a_2b_1 + a_1b_2 + \frac{2a_1a_2}{\sqrt{\sum_{i=1}^4 a_i^2}} = 0. \end{cases}$$

**Corollary 3.11.** *Assume that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_6$. Then geodesic vectors depending only on $\tilde{a}(X, E_1)$, $\tilde{a}(X, E_2)$ and $\tilde{a}(X, E_4)$.*

**Theorem 3.12.** *Suppose that $(M, F)$ be a Finsler space with Randers metric defined by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = b_3E_3$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with Lie algebra $\mathfrak{g}_6$. Then $Y \in \mathfrak{g}_6$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

## 4. Conclusion

In this paper, we show that for a Finsler space $(M, F)$ with Randers metric defined by the Riemannian metric $\tilde{a}$ and the special left invariant vector field $X$ on the left invariant connected 4-dimensional para-hypercomplex Lie group with algebra $\mathfrak{g}_i, i = 1, 2, 3, 4, 5, 6$, $Y \in \mathfrak{g}_i$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.

## References

[1] D. Bao, C.C. Chern and Z. Shen, An introduction to Riemann-Finsler geometry, Springer-Verlage, 2000.

[2] N. Blazic and S. Vukmirovic, Four-dimensional Lie algebras with a para-hypercomplex structure. preprint, *arxiv:math/0310180v1 [math.DG], (2003).*

[3] O. Kowalski and J. Szenthe, On the Existence of Homogeneous Geodesics in Homogeneous Riemannian manifolds, *Geom. Dedicata, **81**(2000), 209-214.*

[4] D. Latifi, Homogeneous geodesics in homogeneous Finsler spaces, *J. Geom. Phys., **57**(2007), 1421-1433.*

[5] D. Latifi and M. Zeinali Laki, Geodesic vectors of invariant $(\alpha, \beta)$-metrics on nilpotent Lie groups of five dimensional, *Caspian Journal of Mathematical Sciences, **12**(2)(2023), 211-223.*

[6] H. R. Salimi Moghadam, On The geometry of some para-hypercomplex Lie groups. *ARCHIVUM MATHEMATICUM (BRNO), Tomus, **45**(2009),159-170.*

[7] M. Zeinali Laki, Geodesic vectors of infinite series $(\alpha, \beta)$-metric on hypercomplex four dimensional Lie groups, *Journal of Finsler Geometry and its Applications, **4**(2)(2023), 103-112.*