On the determination of asymptotic formula of the nodal points for the Sturm-Liouville equation with one turning point

A. Dabbaghian\(^\text{1}\) and A. Neamaty \(^\text{2}\)
\(^{1}\) Islamic Azad University, Neka Branch, Neka, Iran
\(^{2}\) Department of Mathematics, University of Mazandaran, Babolsar, Iran

Abstract. In this paper, the asymptotic representation of the corresponding eigenfunctions of the eigenvalues has been investigated. Furthermore, we obtain the zeros of eigenfunctions.

Keywords: Turning point; Inverse nodal problem; Nodal Points; Eigenvalues; Eigenfunctions.

2000 Mathematics subject classification: 34A55, 34B24, 47E05, 34E20

1. Introduction

In the literature review of mathematics, a large number of research studies has been devoted entirely or partially to the study of the Sturm-Liouville i.e.,

\[ y''(x) + (\lambda \phi^2(x) - q(x))y = 0, \tag{1.1} \]

where \( \lambda = \rho^2 \) and the real valued functions \( \phi^2 \) and \( q \) are said to be the coefficients of the problem, \( \phi^2 \) is the weight and \( q \) is the potential function. The zeros of \( \phi^2 \) are called turning points of (1). Differential equations with turning points play an important role in various areas of mathematics and other branches of natural sciences. For example in

\(^{1}\) Corresponding author: a.dabbaghian@iauneka.ac.ir
Received: 19 November 2013
Revised: 21 January 2014
Accepted: 25 January 2014
elasticity, optics, geophysics (see [6,8,11] and the references therein).

Inverse spectral problems consist in recovering operators from their spectral characteristics. The first spectral problem was given by Ambarzumyan [3]. Since 1946, various forms of the inverse problem have been considered by several authors [4,10]. In later years, these problems were studied for Sturm-Liouville operators with turning points (see [7,13]).

Recently, some researchers have paid attention to a new class of inverse problems. This is the so-called inverse nodal problem. Inverse nodal problems consist in recovering operators from given nodes (zeros) of their eigenfunctions.

In 1988, it seems that J.R. McLaughlin [12] to be the first to consider this sort of inverse problem. She showed that the nodal set of the Dirichlet problem alone can determine the potential function of the Sturm-Liouville problem up to a constant. Yang [15] showed that this uniqueness result is valid for any $q$.

In resent years, some interesting results of inverse nodal problems of the Sturm-Liouville operators were obtained (for example, refer to [5,9,14]). In this work, we consider the following Sturm-Liouville equation

$$y''(x) + (\lambda x - q(x))y = 0, \quad -1 \leq x \leq 1,$$

where $q \in L[-1,1]$ and $\lambda$ is a real parameter.

In this paper, we obtain the eigenvalues and eigenfunctions corresponding to large modulus eigenvalues and we calculate an asymptotic of the nodal points.

2. Main result

Let $C(x, \lambda)$ is a solution for Eq. (2) with the initial conditions $C(-1, \lambda) = 0$, $C'(-1, \lambda) = 1$.

In [2], it was shown that

$$C(x, \lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}} \sinh(p(x)\sqrt{\lambda})(1 + O(\frac{1}{\sqrt{\lambda}})), & -1 \leq x < 0, \\ \frac{1}{\sqrt{\lambda}} \int_{-1}^{x}(xt)^{-\frac{1}{4}} \sinh((\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4})) \frac{1}{\sqrt{\lambda} q(t)}C(t, \sqrt{\lambda})dt, & x > 0, \end{cases}$$

where $p(x) = \int_{-1}^{x} \sqrt{-\nu} d\nu$.

We have the integral equations

$$C(x, \lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}} \sinh(p(x)\sqrt{\lambda}) \\ + \frac{1}{\sqrt{\lambda}} \int_{-1}^{x}(xt)^{-\frac{1}{4}} \sinh(p(x) - p(t))\sqrt{\lambda} q(t)C(t, \sqrt{\lambda})dt, & -1 \leq x < 0, \\ \frac{1}{\sqrt{\lambda}} \int_{0}^{x}(xt)^{-\frac{1}{4}} \sin(f(x) - f(t))\sqrt{\lambda} q(t)C(t, \sqrt{\lambda})dt, & x > 0, \end{cases}$$

(2.1)
where \( f(x) = \int_0^x \sqrt{t} \, dt \).

We consider Eq. (2) with boundary conditions
\[
y(-1, \lambda) = 0, \quad y'(-1, \lambda) = 1, \quad y(b, \lambda) = 0.
\]

The problem corresponding to Eq. (2) on \([-1, b] \) where \( b < 0 \) is fixed, has an infinite number of negative eigenvalues \( \{ \lambda_n^{(1)}(b) \} \). The asymptotic distribution of each function \( \lambda_n^{(1)}(b) \) is of the form
\[
\sqrt{-\lambda_n^{(1)}(b)} = \frac{n\pi}{\int_{-1}^b \sqrt{-t} \, dt} + O\left(\frac{1}{n}\right), \quad b < 0.
\]
(2.2)

For more details see [1].

For \( b \in (0, 1] \), fixed, the problem for (2) on \([-1, b] \) has an infinite number of positive and negative eigenvalues which we denote by \( \{ \lambda_n^{(2)}(b) \} \), \( \{ \lambda_n^{(3)}(b) \} \), respectively.

The positive eigenvalues \( \lambda_n^{(2)}(b) \) admit the asymptotic representation
\[
\sqrt{\lambda_n^{(2)}(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_0^b \sqrt{t} \, dt} + \frac{1}{2n\pi} T_1 + O\left(\frac{1}{n^2}\right),
\]
(2.3)

where
\[
T_1 = \frac{5}{72 \int_0^b \sqrt{t} \, dt} + \frac{1}{2} \int_0^b \frac{q(t)}{\sqrt{t}} \, dt.
\]

Similarly, the negative eigenvalues, \( \lambda_n^{(3)}(b) \), admit the asymptotic representation of the form
\[
\sqrt{-\lambda_n^{(3)}(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_{-1}^0 \sqrt{-t} \, dt} + \frac{1}{2n\pi} T_2 + O\left(\frac{1}{n^2}\right),
\]
(2.4)

where
\[
T_2 = \frac{5}{72 \int_{-1}^0 \sqrt{-t} \, dt} + \frac{1}{2} \int_{-1}^0 \frac{q(t)}{\sqrt{-t}} \, dt.
\]

We now state a theorem which gives asymptotic approximation for the eigenfunctions of the Sturm-Liouville equation in one turning point case.

Let \( C(x, \lambda_n^{(i)}) \) be the eigenfunction corresponding to the eigenvalue \( \lambda_n^{(i)} \) where \( i \in \{1, 2, 3\} \).

**Theorem 2.1.** a) For \( b \in [-1, 0) \) fixed, the corresponding eigenfunctions of the negative eigenvalues \( \lambda_n^{(1)}(b) \), has asymptotic representation,
\[
C(x, \lambda_n^{(1)}(b)) = \frac{p(b)(-x)^{-\frac{1}{4}}}{n\pi} \sin \left( \frac{n\pi p(x)}{p(b)} \right)
\]
b) For $b \in (0,1]$ fixed, the corresponding eigenfunctions of the positive eigenvalues, $\lambda_n^{(2)}(b)$, admit asymptotic representation,

$$C(x, \lambda_n^{(2)}(b)) = \frac{x^{-\frac{1}{4}} e^{\frac{2}{3}(\frac{n\pi x}{f(b)})}}{3(n\pi - \frac{\pi}{4} i)} \sin\left( x^{\frac{3}{4}} (n\pi - \frac{\pi}{4} i) \right)$$

$$- \frac{x^{-\frac{1}{4}} e^{(\frac{n\pi x}{f(b)})}}{3(n\pi - \frac{\pi}{4} i)^2} \sin\left( x^{\frac{3}{4}} (n\pi - \frac{\pi}{4} i) \right) - \frac{\pi}{4} \int_0^x t^{-\frac{1}{3}} q(t) \cos\left( t^{\frac{3}{4}} (n\pi - \frac{\pi}{4} i) \right) dt + o\left( \frac{1}{n^2} \right)$$

\[ \text{Proof.} \]

a) In this case the eigenvalues are negative. Substituting the asymptotic form (4) in (3) and noting that $\sqrt{\lambda} = i \sqrt{-\lambda_n^{(1)}(b)}$ we can get

$$C(x, \lambda_n^{(1)}(b)) = \frac{1}{i \sqrt{-\lambda_n^{(1)}(b)}} (-x)^{-\frac{1}{4}} \sinh(ip(x)\sqrt{-\lambda_n^{(1)}(b)})$$

$$+ \frac{1}{i \sqrt{-\lambda_n^{(1)}(b)}} \int_{-1}^x (xt)^{-\frac{1}{4}} \sinh(ip(x) - p(t)) \sqrt{-\lambda_n^{(1)}(b)} q(t) C(t, \lambda_n^{(1)}(b)) dt$$

\[ = \left( \frac{n\pi}{p(b)} + O\left( \frac{1}{n} \right) \right) \left[ \sin \frac{p(x)n\pi}{p(b)} \cos O\left( \frac{1}{n} \right) + \cos \frac{p(x)n\pi}{p(b)} \sin O\left( \frac{1}{n} \right) \right]$$

$$- \left( \frac{n\pi}{p(b)} + O\left( \frac{1}{n} \right) \right) \int_{-1}^x (-t)^{-\frac{1}{4}} q(t) \sin \frac{p(t)n\pi}{p(b)} \cos O\left( \frac{1}{n} \right) + \cos \frac{p(t)n\pi}{p(b)} \sin O\left( \frac{1}{n} \right) dt$$

\[ \left[ \cos \frac{p(x)n\pi}{p(b)} \cos O\left( \frac{1}{n} \right) - \sin \frac{p(x)n\pi}{p(b)} \sin O\left( \frac{1}{n} \right) \right] + o\left( \frac{1}{n^2} \right) \]
Using the following facts for large $n$:
\[
\cos O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n^2}\right), \quad \sin O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right)
\]
we get the result.

By inserting the asymptotic formulae (5) and (6) into (3) we get the results (b) and (c).

Suppose $\{x_j^{(i)n}\}$ is the $j$th nodal point of the eigenfunction $C(x, \lambda_j^{(i)})$ in $(-1, 1)$. In other words, $C(x_j^{(i)n}, \lambda_j^{(i)}) = 0$. Denote $X^{(i)} = \{x_j^{(i)n}\}_{n \geq 1, j = \overline{1,n}}$. $X^{(i)}$ is called the set of nodal points.

**Theorem 2.2.** We take $x_j^{(i)n}, n \geq 1, j = \overline{1,n}$ as the nodal points of problem. Then

\[
\int_{-1}^{x_j^{(1)n}} \sqrt{-v} dv = \frac{jp(b)}{n} + \frac{p^2(b)}{n^2\pi^2} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{1/2}} \sin^2\left(\frac{p(t)n\pi}{p(b)}\right) dt + o\left(\frac{1}{n^2}\right), \quad x < 0 (2.8)
\]

\[
\int_{0}^{x_j^{(2)n}} \sqrt{v} dv = \frac{(j - \frac{1}{4})f(b)}{n - \frac{1}{4}}
\]

\[+ \frac{f^2(b)}{(n\pi - \frac{n\pi}{4})^2} \int_{0}^{x_j^{(2)n}} \frac{q(t)}{t^{1/2}} \cos^2\left(f(t)\left(\frac{n\pi - \frac{n\pi}{4}}{f(b)} - \frac{\pi}{4}\right) \right) dt + o\left(\frac{1}{n^2}\right), (2.9) > 0,
\]
as $n \to \infty$ uniformly in $j$.

**Proof.** Since $\{x_j^{(i)n}\}$ are zeros of eigenfunctions, in the case $x < 0$,

\[
C(x_j^{(1)n}, \lambda_j^{(1)}(b)) = \frac{p(b)(-x_j^{(1)n})^{-1/4}}{n\pi} \sin \left(\frac{n\pi p(x_j^{(1)n})}{p(b)}\right)
\]

\[-\frac{p^2(b)(-x_j^{(1)n})^{-1/4}}{n^2\pi^2} \cos \left(\frac{n\pi p(x_j^{(1)n})}{p(b)}\right) \int_{-1}^{x_j^{(1)n}} (-t)^{-1/2} q(t) \sin^2\left(\frac{n\pi p(t)}{p(b)}\right) dt + o\left(\frac{1}{n^2}\right) = 0.
\]

Thus,

\[
\tan \left(\frac{n\pi p(x_j^{(1)n})}{p(b)}\right) = \frac{p(b)}{n\pi} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{1/2}} \sin^2\left(\frac{n\pi p(t)}{p(b)}\right) dt + o\left(\frac{1}{n}\right).
\]

Using Taylor’s expansions for the arctangent, we obtain that

\[
p(x_j^{(1)n}) = \frac{jp(b)}{n} + \frac{p^2(b)}{n^2\pi^2} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{1/2}} \sin^2\left(\frac{n\pi p(t)}{p(b)}\right) dt + o\left(\frac{1}{n^2}\right),
\]
and hence formula (10) holds.

Using (8), by the same arguments as above, one can show that (11) holds.