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On the determination of asymptotic formula of the nodal points for the Sturm-Liouville equation with one turning point

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Abstract. In this paper, the asymptotic representation of the corresponding eigenfunctions of the eigenvalues has been investigated. Furthermore, we obtain the zeros of eigenfunctions.

Keywords: Turning point; Inverse nodal problem; Nodal Points; Eigenvalues; Eigenfunctions.

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1. **Introduction**

In the literature review of mathematics, a large number of research studies has been devoted entirely or partially to the study of the Sturm-Liouville i.e.,

$$y''(x) + (\lambda\phi^2(x) - q(x))y = 0, \tag{1.1}$$

where $\lambda = \rho^2$ and the real valued functions ϕ^2 and q are said to be the coefficients of the problem, ϕ^2 is the weight and q is the potential function. The zeros of ϕ^2 are called turning points of (1). Differential equations with turning points play an important role in various areas of mathematics and other branches of natural sciences. For example in

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elasticity, optics, geophysics(see [6,8,11] and the references therein).
Inverse spectral problems consist in recovering operators from their spectral characteristics. The first spectral problem was given by Ambarzumyan [3]. Since 1946, various forms of the inverse problem have been considered by several authors [4,10]. In later years, these problems was studied for Sturm-Liouville operators with turning points (see [7,13]).
Recently, some researchers have paid attention to a new class of inverse problems. This is the so-called inverse nodal problem. Inverse nodal problems consist in recovering operators from given nodes (zeros) of their eigenfunctions.
In 1988, it seems that J.R. Mclaughlin [12] to be the first to consider this sort of inverse problem. She showed that the nodal set of the Dirichlet problem alone can determine the potential function of the Sturm-Liouville problem up to a constant. Yang [15] showed that this uniqueness result is valid for any q.
In resent years, some interesting results of inverse nodal problems of the Sturm-Liouville operators were obtained(for example, refer to[5,9,14]) .In this work, we consider the following Sturm-Liouville equation

$$y''(x) + (\lambda x - q(x))y = 0, \qquad -1 \leq x \leq 1, \tag{1.2}$$

where $q \in L[-1,1]$ and λ is a real parameter.
In this paper, we obtain the eigenvalues and eigenfunctions corresponding to large modulus eigenvalues and we calculate an asymptotic of the nodal points.

2. Main result

Let $C(x,\lambda)$ is a solution for Eq.(2) with the initial conditions $C(-1,\lambda) = 0$, $C'(-1,\lambda) = 1$.
In [2], it was shown that

$$C(x,\lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}} \sinh(p(x)\sqrt{\lambda})(1 + O(\frac{1}{\sqrt{\lambda}})), & -1 \leq x < 0, \\ \frac{x^{-\frac{1}{4}}}{\sqrt{\lambda}}\{e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4}) + e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{2}\sin(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4})\}(1 + O(\frac{1}{\sqrt{\lambda}})), & x > 0, \end{cases}$$

where $p(x) = \int_{-1}^{x} \sqrt{-\nu}\, d\nu$.
We have the integral equations

$$C(x,\lambda) = \begin{cases} \frac{1}{\sqrt{\lambda}}(-x)^{-\frac{1}{4}} \sinh p(x)\sqrt{\lambda} \\ + \frac{1}{\sqrt{\lambda}} \int_{-1}^{x} (xt)^{-\frac{1}{4}} \sinh(p(x) - p(t))\sqrt{\lambda} q(t) C(t,\sqrt{\lambda}) dt, & -1 \leq x < 0, \\ \frac{x^{-\frac{1}{4}}}{\sqrt{\lambda}}\{e^{\frac{2}{3}\sqrt{\lambda}} \cos(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4}) + e^{-\frac{2}{3}\sqrt{\lambda}}\frac{1}{2}\sin(\frac{2}{3}x^{\frac{3}{2}}\sqrt{\lambda} - \frac{\pi}{4})\} \\ + \frac{1}{\sqrt{\lambda}} \int_{0}^{x} (xt)^{-\frac{1}{4}} \sin(f(x) - f(t))\sqrt{\lambda} q(t) C(t,\sqrt{\lambda}) dt, & x > 0, \end{cases} \tag{2.1}$$

where $f(x) = \int_0^x \sqrt{\nu} d\nu$.
We consider Eq.(2) with boundary conditions

$$y(-1,\lambda) = 0, \qquad y'(-1,\lambda) = 1, \qquad y(b,\lambda) = 0.$$

The problem corresponding to Eq.(2) on $[-1, b]$ where $b < 0$ is fixed, has an infinite number of negative eigenvalues $\{\lambda_n^{(1)}(b)\}$. The asymptotic distribution of each function $\lambda_n^{(1)}(b)$ is of the form

$$\sqrt{-\lambda_n^{(1)}(b)} = \frac{n\pi}{\int_{-1}^{b} \sqrt{-t} dt} + O(\frac{1}{n}), \qquad b < 0. \tag{2.2}$$

For more details see [1].
For $b \in (0, 1]$, fixed, the problem for (2) on $[-1, b]$ has an infinite number of positive and negative eigenvalues which we denote by $\{\lambda_n^{(2)}(b)\}$, $\{\lambda_n^{(3)}(b)\}$, respectively.
The positive eigenvalues $\lambda_n^{(2)}(b)$ admit the asymptotic representation

$$\sqrt{\lambda_n^{(2)}(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_0^b \sqrt{t} dt} + \frac{1}{2n\pi} T_1 + O(\frac{1}{n^2}), \tag{2.3}$$

where

$$T_1 = \frac{5}{72 \int_0^b \sqrt{t} dt} + \frac{1}{2} \int_0^b \frac{q(t)}{\sqrt{t}} dt.$$

Similarly, the negative eigenvalues, $\lambda_n^{(3)}(b)$, admit the asymptotic representation of the form

$$\sqrt{-\lambda_n^{(3)}(b)} = \frac{n\pi - \frac{\pi}{4}}{\int_{-1}^{0} \sqrt{-t} dt} + \frac{1}{2n\pi} T_2 + O(\frac{1}{n^2}), \qquad . \tag{2.4}$$

where

$$T_2 = \frac{5}{72 \int_{-1}^0 \sqrt{-t} dt} + \frac{1}{2} \int_{-1}^0 \frac{q(t)}{\sqrt{-t}} dt.$$

We now state a theorem which gives asymptotic approximation for the eigenfunctions of the Sturm-Liouville equation in one turning point case. Let $C(x, \lambda_n^{(i)})$ be the eigenfunction corresponding to the eigenvalue $\lambda_n^{(i)}$ where $i \in \{1, 2, 3\}$.

Theorem 2.1. *a) For $b \in [-1, 0)$ fixed, the corresponding eigenfunctions of the negative eigenvalues $\lambda_n^{(1)}(b)$, has asymptotic representation,*

$$C(x, \lambda_n^{(1)}(b)) = \frac{p(b)(-x)^{-\frac{1}{4}}}{n\pi} \sin \frac{n\pi p(x)}{p(b)}$$

$$-\frac{p^2(b)(-x)^{-\frac{1}{4}}}{n^2\pi^2}\cos\frac{n\pi p(x)}{p(b)}\int_{-1}^{x}(-t)^{-\frac{1}{2}}q(t)\sin^2\frac{n\pi p(t)}{p(b)}dt+o(\frac{1}{n^2}).\ (2.5)$$

b) For $b \in (0,1]$ fixed, the corresponding eigenfunctions of the positive eigenvalues, $\lambda_n^{(2)}(b)$, admit asymptotic representation,

$$C(x,\lambda_n^{(2)}(b))=\frac{x^{-\frac{1}{4}}e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(b)})}\cos[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(b)})-\frac{\pi}{4}]}{(\frac{n\pi-\frac{\pi}{4}}{f(b)})}$$

$$-\frac{x^{-\frac{1}{4}}e^{\frac{2}{3}(\frac{n\pi-\frac{\pi}{4}}{f(b)})}}{(\frac{n\pi-\frac{\pi}{4}}{f(b)})^2}\sin[f(x)(\frac{n\pi-\frac{\pi}{4}}{f(b)})-\frac{\pi}{4}]\int_0^x t^{-\frac{1}{2}}q(t)\cos^2[f(t)(\frac{n\pi-\frac{\pi}{4}}{f(b)})-\frac{\pi}{4}]dt+o(\frac{1}{n^2}).(2.6)$$

c) For $b \in (0,1]$ fixed, the corresponding eigenfunctions of the negative eigenvalues, $\lambda_n^{(3)}(b)$, admit asymptotic representation,

$$C(x,\lambda_n^{(3)}(b))=\frac{2x^{-\frac{1}{4}}e^{(n\pi-\frac{\pi}{4})i}\cos[x^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i-\frac{\pi}{4}]}{3(n\pi-\frac{\pi}{4})i}$$

$$-\frac{x^{-\frac{1}{4}}e^{(n\pi-\frac{\pi}{4})i}}{(\frac{n\pi-\frac{\pi}{4}}{\frac{2}{3}})^2}\sin[x^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i-\frac{\pi}{4}]\int_0^x t^{-\frac{1}{2}}q(t)\cos^2[t^{\frac{3}{2}}(n\pi-\frac{\pi}{4})i-\frac{\pi}{4}]dt+o(\frac{1}{n^2}).(2.7)$$

Proof. a) In this case the eigenvalues are negative. Substituting the asymptotic form (4) in (3) and noting that $\sqrt{\lambda}=i\sqrt{-\lambda_n^{(1)}(b)}$ we can get

$$C(x,\lambda_n^{(1)}(b))=\frac{1}{i\sqrt{-\lambda_n^{(1)}(b)}}(-x)^{-\frac{1}{4}}\sinh(ip(x)\sqrt{-\lambda_n^{(1)}(b)})$$

$$+\frac{1}{i\sqrt{-\lambda_n^{(1)}(b)}}\int_{-1}^{x}(xt)^{-\frac{1}{4}}\sinh(i(p(x)-p(t))\sqrt{-\lambda_n^{(1)}(b)})q(t)C(t,\lambda_n^{(1)}(b))dt$$

$$=\frac{(-x)^{-\frac{1}{4}}}{(\frac{n\pi}{p(b)}+O(\frac{1}{n}))}[\sin\frac{p(x)n\pi}{p(b)}\cos O(\frac{1}{n})+\cos\frac{p(x)n\pi}{p(b)}\sin O(\frac{1}{n})]$$

$$-\frac{(-x)^{-\frac{1}{4}}}{(\frac{n\pi}{p(b)}+O(\frac{1}{n}))^2}(\int_{-1}^{x}(-t)^{-\frac{1}{2}}q(t)[\sin\frac{p(t)n\pi}{p(b)}\cos O(\frac{1}{n})+\cos\frac{p(t)n\pi}{p(b)}\sin O(\frac{1}{n})]^2dt)$$

$$[\cos\frac{p(x)n\pi}{p(b)}\cos O(\frac{1}{n})-\sin\frac{p(x)n\pi}{p(b)}\sin O(\frac{1}{n})]+o(\frac{1}{n^2})$$

Using the following facts for large n:

$$\cos O(\frac{1}{n}) = 1 + O(\frac{1}{n^2}), \quad \sin O(\frac{1}{n}) = O(\frac{1}{n})$$

we get the result.
By inserting the asymptotic formulae (5) and (6) into (3) we get the results (b) and (c).$\diamond$
Suppose $\{x_j^{(i)n}\}$ is the jth nodal point of the eigenfunction $C(x, \lambda_n^{(i)})$ in (-1,1). In other words, $C(x_j^{(i)n}, \lambda_n^{(i)}) = 0$. Denote $X^{(i)} = \{x_j^{(i)n}\}_{n\geq 1, j=\overline{1,n}}$. $X^{(i)}$ is called the set of nodal points.

Theorem 2.2. *We take $x_j^{(i)n}, n \geq 1, j = \overline{1,n}$ as the nodal points of problem. Then*

$$\int_{-1}^{x_j^{(1)n}} \sqrt{-\nu} d\nu = \frac{jp(b)}{n} + \frac{p^2(b)}{n^2\pi^2} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \sin^2(\frac{p(t)n\pi}{p(b)}) dt + o(\frac{1}{n^2}), \quad x < 0, \tag{2.8}$$

$$\int_{0}^{x_j^{(2)n}} \sqrt{\nu} d\nu = \frac{(j-\frac{1}{4})f(b)}{n-\frac{1}{4}}$$

$$+ \frac{f^2(b)}{(n\pi - \frac{\pi}{4})^2} \int_0^{x_j^{(2)n}} \frac{q(t)}{t^{\frac{1}{2}}} \cos^2(f(t)(\frac{n\pi - \frac{\pi}{4}}{f(b)}) - \frac{\pi}{4}) dt + o(\frac{1}{n^2}), \quad x > 0, \tag{2.9}$$

as $n \to \infty$ uniformly in j.

Proof. Since $\{x_j^{(i)n}\}$ are zeros of eigenfunctions, in the case $x < 0$,

$$C(x_j^{(1)n}, \lambda_n^{(1)}(b)) = \frac{p(b)(-x_j^{(1)n})^{-\frac{1}{4}}}{n\pi} \sin \frac{n\pi p(x_j^{(1)n})}{p(b)}$$

$$-\frac{p^2(b)(-x_j^{(1)n})^{-\frac{1}{4}}}{n^2\pi^2} \cos \frac{n\pi p(x_j^{(1)n})}{p(b)} \int_{-1}^{x_j^{(1)n}} (-t)^{-\frac{1}{2}} q(t) \sin^2 \frac{n\pi p(t)}{p(b)} dt + o(\frac{1}{n^2}) = 0.$$

Thus,

$$\tan \frac{n\pi p(x_j^{(1)n})}{p(b)} = \frac{p(b)}{n\pi} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \sin^2 \frac{n\pi p(t)}{p(b)} dt + o(\frac{1}{n}).$$

Using Taylor's expansions for the arctangent, we obtain that

$$p(x_j^{(1)n}) = \frac{jp(b)}{n} + \frac{p^2(b)}{n^2\pi^2} \int_{-1}^{x_j^{(1)n}} \frac{q(t)}{(-t)^{\frac{1}{2}}} \sin^2 \frac{n\pi p(t)}{p(b)} dt + o(\frac{1}{n^2}),$$

and hence formula (10) holds.
Using (8), by the same arguments as above, one can show that (11) holds.◇

References. [1] A.J. Akbarfam, *A Higher-order asymptotic approximations to the eigenvalues of the Sturm-Liouville problem in one turning point case* , AJSE, **24:**1A,January, (1999).

[2] A.J. Akbarfam, E. Pourreza, *A Research note on the second order differential equation* , J. Sci. I. R. Iran , **10**(3)(1999) 185-192.

[3] V.A. Ambarzumyan, *Über eine frage der eigenwerttheorie*, Z. Phys.**53** (1929) 690-695.

[4] G. Borg, *Eine umkehrung der Sturm-Liouvillesehen eigenwertaufgabe* , Acta Math., **78** (1945) 1-96.

[5] S.A. Buterin, C.T. Shieh , *Incomplete inverse spectral and nodal problems for differential pencils* , Results. Math. , **62**(2012) 167-179.

[6] W. Eberhard, G. Freiling, A. Schneider, *Connection formulae for second-order differential equations with acomplex parameter and having an arbitrary number of turning points* , Geophys. J. R. Astr. Soc. **50** (1997) 303-309.

[7] G. Freiling, V.A. Yurko, *Inverse problems for differential equations with turning points* , Inverse Problems., **13** (1997) 1247-1263.

[8] A.L. Goldenveizer, V.B. Lidsky, and P.E. Tovstik, *Free Vibration of Thin Elastic Shells* , Nauka, Moscow, 1979.

[9] H. Koyunbakan, E.S. Panakhov, *Inverse problem for a singular differential operator* , Mathematical and Computer Modeling., **47** (2008) 178-185.

[10] B.M. Levitan, *On the determination of the Sturm-Liouville operator from one and two spectra* , Mathematics of the USSR Izvestija., **12** (1978) 179-193.

[11] J. Mc Hugh, *An historical survey of ordinary linear differential equations with a large parameter and turning points* , Arch. Hist. Exact Sci, **7** (1970) 277-324.

[12] J.R. McLaughlin, *Inverse spectral theory using nodal points as a data-a uniqueness result* , J. differential Equations , **73**(1988) 354-362.

[13] A. Neamaty and S. Mosazadeh, *On the canonical solution of Sturm-Liouville with singularity and turning point of even order* , Canad. Math. Bull. , **54**(3) (2011) 506-518.

[14] C.T. Shieh, V. Yurko, *Inverse nodal and inverse spectral problems for discontinuous boundary value problems* , J. Mmath. Anal. Appl. , **374**(2008) 266-272.

[15] X.F. Yang, *A solution of the inverse nodal problem* , Inverse Problems , **13**(1997) 203-213.