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Sequential-◊-Henstock Integrals for Locally Convex Space-valued Functions on Time Scale

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Abstract. Let X be a Hausdorff locally convex topological vector space with its topology Ω and Topological dual X^* Suppose $f : [0, 1] \to X$ be a function defined on X. Let $\rho(X)$ be a family of ρ-continuous seminorms on X so that the topology is generated by $\rho(X)$. Is f Sequential Mcshane(SMcS) and Sequentia-◊- Henstock(◊ SH) integrable with respect to the semi-norm on time scale? Do these integrals coincide and relate to other integrals such as Pettis and Bochner for which the Sequential Henstock lemma holds for the characterization of locally Convex space on time scale? It is the purpose of this paper to give affirmative answer to these questions.

Keywords: Hausdorff Topological vector Space, Guages, Topological dual, Semi-norms, Time scale.

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1. Introduction

The concept of time scales calculus was introduced by Stefan Hilger [8] in his Ph.D. thesis in 1988. Thompson [21] introduced time scales theory to the study of Henstock-Kurzweil integral. Some of the basic properties of Henstock delta integral on time scales were introduced by Peterson and Thompson [15]. Recently, Afariogun et al. [2], studied some properties of l_p-valued functions for Henstock-Kurzweil-Stieltjes-$\diamond$-double integrals on time scales. Henstock-Kurzweil integral has been studied for different space-valued functions on time scales. For some of the articles on time scales calculus, we refer the readers to see ([1], [2], [3], [5], [21]). The sequential approach to Henstock integral was introduced in the late twenty-first century by Paxton [14] in 2016 as a more generalized method of integrating different class of functions which was initially evaluated by Henstock integral studied by Ralph Henstock and Jaroslav Kursweil in 1955 and 1957 respectively (See [1], [2], [9], [10], [6],[7], [14] and [21]). They introduced and applied a lemma which holds for Real space-valued functions but failed for the Banach space function setting. However, Skvortsov and Solodov in [20], and Di Piazza and Musial in [16] developed a characteristic Banach space-valued function for which the lemma holds. Sakurada and Nakanishi [18] established the McShane and Henstock integrals taking values in vector spaces called (UCs-N) spaces. The Banach space, the Frechet spaces and the strict inductive limits of the Frechet spaces can be defined as complete (UCs-N) spaces. Sergio [19] also studied Henstock ϕ integrals to funtions taking values from Topological vector spaces.(TVS) thereby proving that the Henstock lemma holds if a function $f;[a,b] \to X$ is a McShane or Henstock integrable function and X is a Hibertian (UCs-N) space endowed with nuclearity. So which locally convex spaces is this lemma true? For some of the integrals in locally convex space, see ([4], [11], [12], [13] and [17]).
In this paper, we establish the Sequential McShane and Sequential $\Diamond$ Henstock integrals for functions taking values in a locally convex space, discuss their properties, prove that the Sequential Mcshane integral lies between the Bochner and the Pettis integral on time scale and show in our main results that in a Frechet space, Henstock Lemma holds true if and only if the space is endowed with nuclearity.

2. Definition and Notation

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ equipped with the topology inherited from the standard topology on $\mathbb{R}$. We use $[a,b]_T$ to denote a time scale interval where $[a,b]_T = \mathbb{T} \bigcap [a,b]$, and $a, b \in \mathbb{T}$. Suppose $t \in T$ such that $a \leq t \leq b.$, then A time scale $\mathbb{T}$ is said to be isolated if t is right-scattered and left-scattered for all $t \in \mathbb{T}$ where $t \neq \sup\{\mathbb{T}\}$ and $t \neq \inf\{\mathbb{T}\}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-$continuous provided it is continuous at all right-dense $t \in \mathbb{T}$ and its' left sided limit exists (finite) at left dense points in $\mathbb{T}$.

Let X be a Hausdorff locally convex topological vector space(briefly a locally convex space) with its topology τ and topological dual X^*. $\rho(X)$ denotes a family of τ-continuous seminorms on X so that the topology is generated by $\rho(X)$. For $p \in \rho(X)$. Let $V_p = x \in X : \rho(X) \leq 1$, so that V_p^0, the polar of $V_p \in X^*$ is a *weak**-closed, absolutely convex equicontinuous set on X^*
For a set E of the real numbers $|E|, \lambda E$ and $\delta(E)$ denote respectively the Lebesgue outer measure, the characteristic function and the boundary of E. A set $E \subset R$ is called negligible If $|E| = 0$. ϱ denotes the family of all Lebesgues measurable subsets of $[0,1]_{\mathbb{T}}$. An interval is a compact subinterval of R. A collection of intervals is called nonoverlapping if their interiors are disjoint. The symbol $\mathit{\Phi}$ denotes the family of all subintervals of $[0,1]_{\mathbb{T}}$. A sequence of partitions P_n in $[0,1]$ is a collection $\{(t_{(i-1)_n}), t_{i_n}), \zeta_{i_n}\}$ and $t_{(i-1)_n} \leq \zeta_{i_n} \leq t_{i_n}$ where $[t_{(i-1)_n}, t_{i_n}]$ are nonoverlapping subintervals of $[0,1]_{\mathbb{T}}$ and $t_{1_n}, ..., t_{i_n} \in [0,1]_{\mathbb{T}}$. Given a set $E \subset R$, we say that P_n is
(i) a sequence of partition in E if $\bigcup_{i=1}^{n}(t_{i_n} - t_{(i-1)_n}) \subset E$;
(ii) a sequence of partition in E if $\bigcup_{i=1}^{n}(t_{i_n} - t_{(i-1)_n}) = E$;
(iii) a sequence of Perron partition if $t_{(i-1)_n} \leq \zeta_{i_n} \leq t_{i_n}$.
Given $f : [0,1]_{\mathbb{T}} \to X$ and a sequence of partition $P_n = \{(t_{(i-1)_n}), t_{i_n}), \zeta_{i_n}\}$ in $[0,1]$, we set

$$S(f, P_n) = \sum_{i=1}^{m_n \in N} f(\zeta_{i_n})(t_{i_n} - t_{(i-1)_n}).$$

A sequence of gauge $\{\delta_n(x)\}_{n-1}^{\infty}$ on $E \subset [0,1]_{\mathbb{T}}$ are positive functions on E. For a given sequence of gauges $\{\delta_n(x)\}_{n=1}^{\infty}$ on E a sequence of partition $P_n = \{(t_{(i-1)_n}), t_{i_n}), \zeta_{i_n}\}$ in $[0,1]_{\mathbb{T}}$ is $\delta_n(x)-fine$ if $(\zeta_{i_n} - \zeta_{(i-1)_n}) \subset (\zeta_{i_n} - \delta(\zeta_{i_n}), \zeta_{i_n} + \delta(\zeta_{i_n}))$.
Let $\Diamond I = (t_{i_n} - t_{(i-1)_n})$ for $i = 1, 2, ..., m$ where I is an identity function in $\mathbb{R}$, then the Sequential-$\Diamond$-Henstock sum of f with respect to functions I is denoted by $S(f, P_n)$ is written

$$S(f, P_n) = \sum_{i=1}^{n \in N} f(\zeta_{i_n}).\Diamond I.$$

A function $f : [0,1]_{\mathbb{T}} \to X$ is called simple if it is a finite sum, where the functions are characteristic functions on a set.
A function $f : [0,1]_{\mathbb{T}} \to X$ is called weakly-measurable if the function $x^* f$ is measurable for every $x^* \in X^*$
We recall the following definitions (see [19]):

Definition 2.1. . A function $f : [0,1]_{\mathbb{T}} \to X$ is said to be strongly (or Bochner) integrable if there exists a sequence$(f_n)_n$ of simple function such that

(i) $f_n(t) - f(t)$ a.e.,i.e. f is strongly measurable.
(ii) $p(f(t) - f_n(t)) \in L^1([0,1])$ for each $n \in N$ and $p \in \rho(X)$ and for all $p \in \rho(X)$

$$\lim_{n\to\infty} \int_0^1 p(f(t) - f_n(t))dt = 0;$$

(iii) $\int_A f_n$ converges to X for each measurable subset A of $[0,1]_{\mathbb{T}}$.
In this case, we put $(B)\int_A f = lim_{n\to\infty} \int_A f_n$.

Definition 2.2. A function $f : [0,1]_{\mathbb{T}} \to X$ is said to be integrable by semi-norm if for any $p \in \rho(X)$ there exists a sequence $(f_n^p)_n$ of simple functions and a subset $X_0^p \subset [0,1]$ with $X_0^p = 0$ such that
(i) $\lim_{n\to\infty} p(f_n^p(t) - f(t))dx = 0$ for all $t \in [0,1]$ X_0^p, i.e. f is measurable by semi-norms.

(ii) $p(f(t) - f_n^p(t)) \in L^1([0,1]_{\mathbb{T}})$ for each $n \in N$ and $p \in \rho(X)$ and for all $p \in \rho(X)$

$$\lim_{n\to\infty} \int_0^1 p(f(t) - f_n^p(t))dt = 0$$

(iii) For each measurable subset $A \in [0,1]_{\mathbb{T}}$there exists an element $\kappa_A \in X$ such that

$$\lim_{n\to\infty} p(\int_A f_n^p(t)) - \kappa_A) = 0.$$

for every $p \in \rho(X)$. We then put $\int_A f = \kappa_A$.

Definition 2.3. A function $f : [0,1]_{\mathbb{T}} \to X$ is said to be Pettis integrable X^*f is Lebesgue integrable on $[0,1]_{\mathbb{T}}$ for each $x \in X^*f$ and if for every measurable set $E \subset [0,1]_{\mathbb{T}}$ there is a vector $\nu(E) \in X$ such that $x^*(\nu(E)) = \int_E x^*f(t)dt$ for $x^* \in X$.
The set function $f : \rho \to X$ is called indefinite Pettis integral of f. It is known that ν is a countably additive vector measure, continuous with respect to the Lebesgues measure (in the sense that $|E| = 0$, then $\nu(E) = 0$) (see [11]).

Definition 2.4. Let X be a locally convex space. Y is a Banach space. A linear operator $T : X \to Y$ is called nuclear operator if

$$T(X) = \sum_{n=1}^{\infty} c_n f_n(x) y_n$$

where (f_n) is an equi-continuous linear functionals on X. (y_n) is a bounded sequence of elements in Y and (c_n) is a sequence of non-negative numbers with $\sum_{n=1}^{\infty} c_n < \infty$.

Definition 2.5. . A locally convex space X is called nuclear space if for any convex balanced neighbourhood V of φ, there exists another convex balanced neighbourhood $U \subseteq V$ of φ such that the canonical mapping

$$T : X_U \to \hat{X}_V$$

where $\hat{X}_V$ is the completion of X_V, is nuclear.
We recall that a locally convex space X is a Frechet space or simply an F-space if it is a complete space in which the topology is induced by a sequence of pseudo-norms. A series $\sum_i x_i$ in X unconditionally convergent if for each permutation $n(i)$ of positive integers the series $\sum_i x_{n(i)}$ converges. (see [12])

Definition 2.6. A function $f : [0,1]_{\mathbb{T}} \to X$ is called simple if there exists $x_1, x_2, ..., x_n \in X$ and $E_1, E_2, ..., E_n \in \rho$ such that $\sum_{i=1}^{n} x_i X_{E_i}$.

Definition 2.7. The function $f : [0,1]_{\mathbb{T}} \to X$ is said to be McShane integrable(respectively Henstock integrable) on $[0,1]$ if there exists a vector $\omega \in X$ such that for every $\varepsilon > 0$ and $p \in \rho(X)$ there exists a gauge $\delta_p > 0$ on $[0,1]$ such that for each $\delta_p - fine$ partition $P = \{(t_{i-1}, t_i), \zeta_i\}$ where $[t_{i-1}, t_i] \in [0,1]_{\mathbb{T}}$ we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \omega| < \varepsilon.$$

Definition 2.8. The function $f : [0,1]_{\mathbb{T}} \to X$ is said to be Sequential McShane integrable(respectively Sequential Henstock integrable) on $[0,1]$ if there exists a vector $\omega \in X$ such that for every $\varepsilon > 0$ and $p \in \rho(X)$ there exists a sequence of gauges $\delta_{\mu p}(x) \in \{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]$ for $n > \mu \in N$ on [0,1] such that for all $\delta_{np}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \omega| < \varepsilon.$$

We denote by $SMcS([0,1]_{\mathbb{T}}, X)$ respectively, $SH([0,1]_{\mathbb{T}}, X)$ the family of all Sequential McShane and Sequential Henstock integrable functions on $[0, 1_{\mathbb{T}}]$ and we set $\omega = (SMcS) \int_{[0,1]_{\mathbb{T}}} f$ $(\omega = (SH) \int_{[0,1]_{\mathbb{T}}} f$.

Clearly, a SMcS integrable function is also SH integrable. The vector ω in Definition 2.8 is uniquely determined by $f \in SMcS([0,1]_{\mathbb{T}}, X)$

$(SH([0,1]_{\mathbb{T}}, X))$. Indeed, let $f \in SMcS([0,1]_{\mathbb{T}}, X)$ (respectively, $SH([0,1]_{\mathbb{T}}, X))$ and let ω_1 and ω_2 be two vectors satisfying definition 5. Choose $\varepsilon > 0$, $p \in \rho(X)$ and find a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ on $[0,1]_{\mathbb{T}}$, i = 1,2 such that for all $\delta^i_{np}(x)-fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \omega_i| < \frac{\varepsilon}{2}.$$

If $\delta_{np} = min(\delta^1_{np}, \delta^2_{np})$ for all $\delta^i_{np}(x) - fine$ tagged partitions on $[0,1]_{\mathbb{T}}, (i = 1,2)$, we have

$$\begin{aligned} |(\omega_1 - \omega_2)| &= |\sum_{i=1}^{m_n \in N} f(\zeta_{i_n}).\Diamond I - \omega_1| + |\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \omega_2| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

From the arbitrariness of ε, we obtain that $p(\omega_1 + \omega_2) = 0$ for all $p \in \rho(X)$. Since the space X is Hausdorff, it is separated. Hence $\omega_1 = \omega_2$.

Now, we state and prove the following lemmas which are useful in the proofs of the theorems in our main results.

Lemma 2.9. *(Saks-Sequential $\Diamond$ Henstock) Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be such that the integral $\int_{[a,b]_{\mathbb{T}}} f$ exists, then for any $\varepsilon > 0$, there exists a sequence of gauges $\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a,b]$ for on $[a,b]_{\mathbb{T}}$ such that for all $\delta_n(x)-fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [a,b]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have*

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| < \varepsilon.$$

If $\{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\} : i = 1,2,...,n$ is an arbitrary system satisfying

$$a \leq t_0 \leq \zeta_{i_n} \leq t_{1_n} \leq t_{2_n} \leq ... \leq t_{(i-1)_n} \leq \zeta_{i_n} \leq t_{i_n} \leq b. \tag{2.1}$$

and

$$[t_{(i-1)_n}, t_{i_n}] \subset (\zeta_{i_n} - \delta_n(\zeta_{i_n}), \zeta_{i_n} + \delta_n(\zeta_{i_n})),$$

then

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| < \varepsilon. \tag{2.2}$$

Proof. Assume the system $\{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n} : i \in 1,2,...,n\}$ satisfies (1) . We set $t_{(i-1)_n} = a$ and $t_{i_n} = b$. Now, let $\beta_n = 0$ and $i \in 0,1,...,n$ be given. Assume that $t_{i_n} < t_{(i+1)_n}$, then if the sequence of gauges δ_n, δ_0 are such that

$\delta_0 < \delta_n$ on $[a,b]$, then every $\delta_0 - fine$ partition of $[a,b]$ is also $\delta_n - fine$, so there are sequence of gauges $\{\delta_{i_n}(x)\}_{n-1}^{\infty}$. So for every $\delta_{i_n} - fine$ partitions on $[t_{i_n} < t_{(i+1)_n}]$ and $\delta_{i_n} - fine$ partitions $P_{i_n} = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $t_{(i-1)_n} \leq \zeta_{i_n} \leq t_{i_n}$ of $[t_{i_n}, t_{(i+1)_n}]$ such that $\delta_{i_n}(x) \leq \delta_n(x)$ for $x \in [t_{(i-1)_n}, t_{i_n}]$ and

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| < \frac{\beta_n}{n+1}, \tag{2.3}$$

now, we form $\delta_n(x) - fine$ tagged partitions $Q_{i_n} = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ of the interval $[a,b]_{\mathbb{T}}$, such that

$$S(Q, P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} Q(\zeta_{i_n}).\Diamond I - + \sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I.$$

If $([t_{i_n} < t_{(i+1)_n}])$ and we set $\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond = 0$, then

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I + \sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f + \sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f|$$

$$= |\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| < \varepsilon.$$

This together with (2.3) yields

$$\begin{aligned} |\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| &\leq |\sum_{i=1}^{m_n \in \mathbb{N}} Q(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| \\ &\quad +|\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[a,b]_{\mathbb{T}}} f| \\ &< \varepsilon + \beta_n. \end{aligned}$$

Since $\beta_n > 0$ is arbitrary, (2.2) follows for each $\delta_n(x) - fine$ partition P_n of $[a,b]$ and each $n \in \mathbb{N}, f_n$ is uniformly Sequential-◊-Henstock integrable with respect to I. □

Lemma 2.10. *A function $f : [0,1]_{\mathbb{T}} \to X$ is a simple function then $f \in SMcS([0,1]_{\mathbb{T}}, X)$.*

Proof. Since SMcS-integral is linear, It is sufficient to study the case $f \in X_{E_i}(x).\omega$ where E is a set in $[0,1]_{\mathbb{T}}$ and ω is a non null vector in X. For each $[t_{(i-1)_n}, t_{i_n}] \in P_n$, put $f([t_{(i-1)_n}, t_{i_n}]) = |E \cap ([t_{(i-1)_n}, t_{i_n})].\omega$. Choose an open set G and a closed set F such that $F \subset E \subset G$. Define a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ for on $[0,1]_{\mathbb{T}}$,

$$\delta_{np}(x) = \begin{cases} dist(x,G), & \text{if } x \in F \\ inf\{dist(x,\delta(G)); dist(x,F)\}, & \text{if } x \in G \setminus F \\ dist(x,F), & \text{if } x \in [0,1]_{\mathbb{T}} \setminus G \end{cases}$$

such that for all $\delta_{np}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}]$ on $[0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$\begin{aligned} |\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - E\omega| &= |\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - E(t_{i_n} - t_{(i-1)_n})| \\ &\leq |\sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I - E(t_{i_n} - t_{(i-1)_n})| \\ &\quad +|\sum_{t_{i_n} \notin E} E(t_{i_n} - t_{(i-1)_n})| \\ &= |\sum_{t_{i_n} \in E} (t_{i_n} - t_{(i-1)_n}).\omega - E \cap (t_{i_n} - t_{(i-1)_n}).\omega| \\ &\quad +|\sum_{t_{i_n} \notin E} E \cap (t_{i_n} - t_{(i-1)_n}).\omega| \\ &\leq (\omega) \sum_{t_{i_n} \in E} |(t_{i_n} - t_{(i-1)_n}) - E \cap (t_{i_n} - t_{(i-1)_n})| \\ &\quad +(\omega) \sum_{t_{i_n} \notin E} |(E \cap (t_{i_n} - t_{(i-1)_n}))|. \\ &\leq 2|(\omega)|.|G \setminus F|. \end{aligned}$$

If $p(\omega) = 0$, the assertion follows trivially. Otherwise we choose F and G such that $|G \setminus F| < \dfrac{\varepsilon}{2|(\omega)|}$. Therefore $f \in SMcS([0,1]_{\mathbb{T}}, X)$ and
$E(t_{i_n} - t_{(i-1)_n}) = |E \cap (t_{i_n} - t_{(i-1)_n}).\omega|$. □

Lemma 2.11. *Let $f : [0,1]_{\mathbb{T}} \to X$ be a function. Given $\varepsilon > 0$ and $p \in \rho(X)$, there is a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ on $[0,1]$ such that for all $\delta_{np}^{i}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have*

$$\sum_{i=1}^{m_n \in \mathbb{N}} pf(\zeta_{i_n}).\Diamond I \leq \overline{\int}_{[0,1]_{\mathbb{T}}} p(f(t)dt).$$

where the integral in the last inequality is the upper Lebesgues integral.

Proof. Let $p \in \rho(X)$. We consider the case $\overline{\int}_{[0,1]_{\mathbb{T}}} p(f(t)dt) < \infty$. Otherwise the inequality is obvious. Choose a real-valued function h on $[0,1]_{\mathbb{T}}$ such that $h(t) \geq p(f(t))$ for all t and $\int_{[0,1]_{\mathbb{T}}} h(t)dt = \overline{\int}_{[0,1]_{\mathbb{T}}} p(f(t))dt$. Given $\varepsilon > 0$, there is a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ for on $[0,1]_{\mathbb{T}}$ such that for all $\delta_{np}^{i}(x) - fine$ tagged partitions
$P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}, (i = 1, 2)$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} h(\zeta_{i_n}).\Diamond I| \leq \overline{\int}_{[0,1]_{\mathbb{T}}} |(f(t)dt)| < \varepsilon. \tag{2.4}$$

Hence from (2.2), we have

$$\sum_{i=1}^{m_n \in \mathbb{N}} h(\zeta_{i_n}).\Diamond I| \leq \sum_{i=1}^{m_n \in \mathbb{N}} h(\zeta_{i_n}).\Diamond I| \leq \overline{\int}_{[0,1]_{\mathbb{T}}} |(f(t)dt)| + \varepsilon.$$

□

Lemma 2.12. *Let $f : [0,1]_{\mathbb{T}} \to X$ be a function which is integrable by semi-norm. Then it is $\Diamond SMcS$-integrable (and also $\Diamond SH$-integrable)and the two integrals coincide.*

Proof. Choose $p \in \rho(X)$ and fix $\varepsilon > 0$. Let $\varphi_p : [0,1]_{\mathbb{T}} \to X$ be a simple function such that

$$\int_{[0,1]_{\mathbb{T}}} |(f(t) - \varphi_p(t))dt| < \frac{\varepsilon}{4}. \tag{2.5}$$

The function φ_p is $\Diamond SMcS$-integrable as already proved. Thus there is a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ for on $[0,1]$ such that for all $\delta_{np}^{1}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|\sum_{i=1}^{m_n \in N} (\varphi_p(\zeta_{i_n})\Diamond I - \int_{[0,1]_{\mathbb{T}}} \varphi_p| < \frac{\varepsilon}{4}. \tag{2.6}$$

By Lemma 2.11, there is a sequence of gauges $\{\delta_{np}^{i}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ for $i = 1, 2$ such that for all $\delta_{np}^{i}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|\sum_{i=1}^{m_n \in N} (\varphi_p(t_{i_n}) - \varphi_p(t_{i_n})\Diamond I| < \int_{[0,1]} |(f(t) - \varphi_p(t))dt| + \frac{\varepsilon}{4}. \tag{2.7}$$

If $\delta_{np} = min(\delta^i_{np})$ for $i = 1, 2$ all $\delta^i_{np}(x) - fine$ tagged partitions on $[0,1]_{\mathbb{T}}$, then by (2.7),(2.6) and (2.5), we have

$$\begin{aligned}
|\sum_{i=1}^{m_n\in\mathbb{N}} f(\zeta_{i_n}).\Diamond I - \int_{[0,1]_{\mathbb{T}}} f| &\le \sum_{i=1}^{m_n\in N} |f(t_{i_n})\Diamond I - \varphi_p(t_{i_n})\Diamond I| \\
&\quad + \sum_{i=1}^{n} |\varphi_p(t_{i_n})\Diamond I| - \int_{[0,1]_{\mathbb{T}}} |\varphi_p| \\
&\quad + p\int_{[0,1]_{\mathbb{T}}} |\varphi_p| - \int_{[0,1]_{\mathbb{T}}} |f| \\
&\le \int_{[0,1]_{\mathbb{T}}} |(f(t) - \varphi_p(t))|dt + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&\quad + \int_{[0,1]_{\mathbb{T}}} |(f(t) - \varphi_p(t))|dt \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Which implies the $\Diamond SMcS$-integrability of f. □

Since Bochner integrable function is integrable by semi-norm (see [4] and [17]), we get

Corollary 2.13. *If $f : [0,1]_{\mathbb{T}} \to X$ is a $\Diamond$-Bochner-integrable function, then it is $\Diamond SMcS$-integrable (and also $\Diamond SH$-integrable)and the two integrals coincide.*

Remark 2.14. . If $f : [0,1]_{\mathbb{T}} \to X$ and $h : [0,1]_{\mathbb{T}} \to X$ are two functions such that $f = h$ a.e, then $f \in \Diamond SMcS([0,1]_{\mathbb{T}}, X)$ (respectively $\Diamond SH([0,1]_{\mathbb{T}}, X)$) if and only if $h \in \Diamond SMcS([0,1]_{\mathbb{T}}, X)(SH([0,1]_{\mathbb{T}}, X))$. In this case, we have $\int_{[0,1]_{\mathbb{T}}} f = \int_{[0,1]_{\mathbb{T}}} h$. which is integrable by semi-norm. Then it is $\Diamond SMcS$-integrable (and also $\Diamond SH$-integrable)and the two integrals coincide.

.

3. Main result

Theorem 3.1. . *Let $f : [0,1]_{\mathbb{T}} \to X$ be a measurable by semi-norm function. Then f is integrable by seminorm if and only if f is SH-integrable and for each $p \in \rho(X)$, the real valued function $p(f(x))$ is SH-integrable.*

Proof. The necessary condition has been proved in lemma 4. We now prove the converse implication. Observe that if $f : [0,1]_{\mathbb{T}} \to X$ is a measurable by semi-norm function such that for each $p \in \rho(X)$, the real valued function $p(f(x))$ is SH-integrable, then $p(f(x))$ is integrable. So it follows from the assertion in ([4], Theorem 2.10). This completes the proof. □

Now, we discuss the connection between Pettis and Sequential McShane integral. For each $p \in \rho(X), p^{-1}(0)$ is a vector subspace and p defines the norm on $X/p^{-1}(0)$. Let X_p be the associated Banach space, namely the completion of the normed linear space $X/p^{-1}(0)$ and π_p the canonical mapping of X into X_p(see [19]). Given a function $f : [0,1]_{\mathbb{T}} \to X$ and a seminorm $p \in \rho(X)$, define a function $f_p : [0,1]_{\mathbb{T}} \to X_p$ by

$$f_p(\xi_{i_n}) = (\pi_p \circ f(\xi_{i_n})) = \pi_p(f(\xi_{i_n}))$$

for $t_{i_n} \in [0,1]$

Remark 3.2. . If $f : [0,1]_{\mathbb{T}} \to X$ is SMcS-integrable(respectively SH-integrable), then $f_p : [0,1]_{\mathbb{T}} \to X_p$ is SMcS-integrable (respectively SH-integrable). Indeed let ω denote the SMcS-integral (SH-integral) of f. Choose $p \in \rho(X)$ and fix $\varepsilon > 0$ and find a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ such that for all $\delta^1_{np}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$| \sum_{i=1}^{m_n \in \mathbb{N}} pf(\zeta_{i_n}).\Diamond I - \omega| < \varepsilon \tag{3.1}$$

Since

$$|\pi_p \circ \sum_{i=1}^{m_n \in \mathbb{N}} pf(\zeta_{i_n}).\Diamond I - \pi_p(\omega)| = | \sum_{i=1}^{m_n \in \mathbb{N}} pf(\zeta_{i_n}).\Diamond I - \omega|$$

From (3.1), we obtain

$$|\pi_p \circ \sum_{i=1}^{m_n \in \mathbb{N}} pf(\zeta_{i_n}).\Diamond I - \pi_p(\omega)| < \varepsilon$$

This completes the proof.

Theorem 3.3. *Let $f : [0,1]_{\mathbb{T}} \to X$ be a $\Diamond SMcS$-integrable function. Then the function f is $\Diamond$Pettis integrable.*

Proof. From Remark 3.2, we obtain that for each $p \in \rho(X)$, the function $f_p : [0,1]_{\mathbb{T}} \to X$ is SMcS-integrable. Then f_p is Pettis integrable. Since X is complete, it follows from ([11], Lemma 2.9) that the function f is Pettis integrable. □

Theorem 3.4. *Let $f : [0,1]_{\mathbb{T}} \to X$ be a function which is measurable by semi-norm. Then f is integrable by semi-norm if and only if f is SMcS-integrable and for each if for each $p \in \rho(X)$, the real valued function $p(f(X))$ is SMcS-integrable.*

Proof. The necessary condition has been proved in Lemma 4. We now prove the converse implication. if $f : [0,1]_{\mathbb{T}} \to X$ is measurable by semi-norm function such that for every $p \in \rho(X)$, the real valued function $p(f(x))$ is SMcS-integrable. The $p(f(x))$ is integrable. So by Theorem 3.2, the function f is Pettis integrable. It also follows from the assertion in ([19], Theorem 2.6). This completes the proof. □

Theorem 3.5. . *Let X be an (F)-space. Then Sequential-$\diamond$-Henstock Lemma holds true for each SMcS-integrable(or SH-integrable) function $f : [0,1]_{\mathbb{T}} \to X$ if and if X and for each $p \in \rho(X)$, the real valued function $p(f(X))$ is nuclear. SMcS-integrable.*

Proof. Assume that X is a nuclear (F)-space and let $P_{n}{}_{n=1}^{\infty}$ be a sequence of semi-norms determining the topology of X. For each n, X_n is the completion of the quotient space $X/p_n^{-1}(0)$. Suppose the sequence $\{p_n\}_{n=1}^{\infty}$ is increasing, then, there is a natural continuous embedding

$$x_n : X_{n+1} \to X_n$$

Since X is nuclear, for each n, π_n is an absolutely summing operator. that is, there is a positive constant C_n such that for arbitrary $x_1, x_2, ..., x_t \in X$, we have

$$\sum_{i=1}^{t} p_n(\pi_n(x_i)) \leq C_n p_{n+1}(\sum_{i=1}^{t} x_i).$$

Moreover, f is SMcS-integrable, so for a fixed $\varepsilon > 0$, there is a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]$ such that for all $\delta_{np}^{i}(x) - fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|| \sum_{i=1}^{m_n \in \mathbb{N}} pf(\zeta_{i_n}).\Diamond I - E\Diamond I| < \frac{\varepsilon}{C_{n-1}}$$

Since π_n is absolutely summing, we obtain

$$\begin{aligned} \sum_{i=1}^{n} p_{n-1}|f\Diamond I - F\Diamond I| &= \sum_{i=1}^{n} |p_{n-1}(\pi_{n-1})F\Diamond I \\ &\quad -F\Diamond I| \\ &\leq \sum_{i=1}^{n} p_{n-1}|F\Diamond I \\ &\quad -F\Diamond I| \\ &< \varepsilon. \end{aligned}$$

This is true for any n. Therefore, Sequential Henstock Lemma holds.
Conversely, suppose Sequential Henstock Lemma holds but the space X is not

nuclear. By the Grothendieck version of Dvoretzky Rogers' theorem in (F)-spaces([11], Theorem 7.3.2]) and definition of the function $f(t)$ in a Cantor set in $[0,1]_{\mathbb{T}}$,of Theorem 4 in [12], we want to show that f defined by its' primitive

$$f(t) = \begin{cases} \varphi & \text{if } \zeta \in C \\ \frac{3^r}{2^r}(t - a_i^r)x_r & \text{if } t \in (a_i^r, d_i^r], i = 1, ..., 2^r, r = 0, 1 \\ -\frac{3^r}{2^r}(t - a_i^r)x_r & \text{if } t \in (a_i^r, d_i^r], i = 1, ..., 2^r, r = 0, 1 \end{cases}$$

is SMcS-integrable to φ. Fix $\varepsilon > 0$, and let $p \in \rho(X)$. By [12], Theorem 4], there is a natural number R such that

$$\sup_{c_{i=1 or 0}} \sum_{R+1}^{\infty} |\varepsilon_i x_i)| < \frac{\varepsilon}{4}$$

for any sequence of real numbers $(\theta_i)_{R+1}^{\infty}$ satisfying the condition $(\theta_i)_{R+1}^{\infty} \leq 1$ for $i = R+1, R+2, ...$ we have

$$\sum_{R+1}^{\infty} |\varepsilon_i x_i| < \frac{\varepsilon}{2}.$$

We define a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]$ such that for all $\delta_{np}(x)-$ $fine$ tagged partitions $P_n = \{(t_{(i-1)_n}, t_{i_n}), \zeta_{i_n}\}$ where $[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ we have

$$|| \sum_{i=1}^{m_n \in \mathbb{N}} f(\zeta_{i_n}).\Diamond I = |\sum_{i \in I} f(\zeta_{i_n}).\Diamond I| < \frac{\varepsilon}{C_{n-1}}$$

for $i = 1, ..., n$, each interval $(t_{(i-1)_n}, t_{i_n})$ is either entirely in or a disjoint from $U_R = \bigcup_{r=0}^{R} \bigcup_{i=1}^{2^r} U_i^r$ where $U_i^r = (a_i^r, b_i^r) \setminus d_i^r$, $U = \bigcup_{R=0}^{\infty} U_R$, d_i^r is the center of (a_i^r, b_i^r). Thus

$$|\sum_{\zeta_i \in I} f(\zeta_{i_n}).\Diamond I \leq \sum_{I_i \subset U_R} |f(\zeta_{i_n}).\Diamond I| + \sum_{I_i \bigcap U_R = \phi} |f(\zeta_{i_n}).\Diamond I| \tag{3.2}$$

From the definition of f and d_p^{ξ} given by

$$\delta_p(\xi) = \begin{cases} min\{|\xi - a_i^r|, |\xi - b_i^r|, |\xi - d_i^r|\}, & \text{if } \xi \in U_i^r, i = 1, ..., 2^r, r = 0, 1 \\ \varrho, & \text{if } \xi \in V. \end{cases}$$

where $V = C\bigcup(\bigcup_{r=0}^{R}\bigcup_{i=1}^{2^r} d_i^r)$. For $\xi = U_i^r$, there are numbers $\theta_i^r, 0 \leq r \leq R, i = 1, ..., 2^r$, such that $|\theta_i^r| < 2\varrho$ we have

$$\begin{aligned}
|\sum_{I\in U_R} f(\zeta_{i_n}).\Diamond I| &\leq \sum_{r=0}^{R}\sum_{i=1}^{2^r}\frac{3^r}{2^r}|(x_r)|\theta_i^r \\
&\leq \sum_{r=0}^{R}\frac{3^r}{2^r}|(x_r)|\sum_{i=1}^{2^r}\theta_i^r \\
&\leq \sum_{r=0}^{R}\frac{3^r}{2^r}|(x_r)|\sum_{i=1}^{2^r}2\varrho \\
&= 2\varrho\sum_{r=0}^{R}\frac{3^r}{2^r}|(x_r)| \\
&\leq 2\varrho K.\frac{3^{R+1}-1}{2} \\
&= K\varrho(3^{R+1}) < \frac{\varepsilon}{2}. \qquad (3.3)
\end{aligned}$$

where $k = max\{|(x_1|, ..., |(x_R))|\}$. For $r \in R$, we can find numbers θ for which $|\theta_r| < 1$ and

$$\sum_{I_i\bigcap U_R=\phi}|(f(\zeta_{i_n})\Diamond I| < p(\sum_{r=R+1}^{\infty}x_i\theta_r) < \frac{\varepsilon}{2} \qquad (3.4)$$

From (3.2), (3.3), (3.4), we obtain

$$\sum_{t_i\in U}|f(\zeta_{i_n}).\Diamond I| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Therefore, f is SMcS-integrable to φ
let

$$f(t) = \begin{cases} \varphi & \text{if } t\in C \\ \frac{3^r}{2^r}(t-a_i^r)x_r & \text{if } t\in(a_i^r, d_i^r], i=1,...2^r, r=0,1 \\ -\frac{3^r}{2^r}(t-a_i^r)x_r & \text{if } t\in(a_i^r, d_i^r], i=1,...2^r, r=0,1 \end{cases}$$

be the primitive of f. Since the series $\sum_{i=0}^{\infty}p(x_{i_n})$ is not absolutely convergent, ther is ε_0 such that for all N there are $m, n > N$ satisfying

$$\sum_{i=0}^{\infty}p(x_{i_n}) \geq 6\varepsilon_0. \qquad (3.5)$$

Define a sequence of gauges $\{\delta_{np}(x)\}_{n=1}^{\infty}$ on $[0,1]_{\mathbb{T}}$ such that for all $\delta_{np}^i(x) - fine$ tagged partitions $P_n = \{d_i^r, (a_i^r, b_i^r), r = m, ..., n; i = 1, ..., 2^r\}$ where

$[t_{(i-1)_n}, t_{i_n}] \in [0,1]_{\mathbb{T}}$ and $t_{(i-1)_n} \leq \zeta_{(i)_n} \leq t_{i_n}$ relative to ε_0, Let $\mathbb{N}$ be a natural number for which $\frac{1}{2}.3^{-N} < \varrho$. Then for $n, m > \mathbb{N}$. we have

$$\begin{aligned} \sum_{r=m}^{n}\sum_{i=1}^{2^r}|f(\zeta_{i_n}).\Diamond I - F\Diamond I| &= \sum_{r=m}^{n}\sum_{i=1}^{2^r}|F\Diamond I| \\ &= \sum_{r=m}^{n}\sum_{i=1}^{2^r}\frac{3^r}{2^r}\frac{1}{2.3^{r+1}|x_{i_n}|} \\ &= \frac{1}{6}\sum_{i=0}^{\infty}|x_{i_n}| \\ &\geq \varepsilon_0, \end{aligned}$$

is a contradiction that the Sequential Henstock Lemma holds. Therefore the claim holds. This completes the proof. □

4. Conclusion

⋄-Henstock integral on time scale is a special kind of integral that is equipped with the topology inherited from the standard topology on $\mathbb{R}$. It discusses the locally convex valued space functions defined on classical interval $[a,b]_{\mathbb{T}}$ as a basis for defining its concepts and integrability properties. Measurable by seminorm functions are shown to be Henstock, McShane and Pettis integrable with their theorems proved through the use of sequence approach and application of the Sequential-⋄ lemma. Sequential ⋄-Henstock integral theorems connotes that the integrability of a real valued functions is preserved by extending the guage function and the Riemann sums by applying the sequence approach.

Up until this research work, however, ⋄-Henstock integral on time scale did not include definitions and theorems based on sequence and it is in our viewpoint that the Sequential-⋄-Henstock integral can be used to renew the interest of integration theorists and researchers on Henstock integral. In line with this, the results of this research can now be extended to studies in more abstract spaces and applications arising from this as well as to the conclusion of Sequential-⋄-Henstock integral in introductory Calculus can be assessed for possible pedagogical benefits. In conclusion, Sequential-⋄-Henstock integral hold for classes of functions, such as step functions, measuration functions, absolutely integrable functions? It is of the view of the authors that these problems could be considered for further research:

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References

[1] D. A. Afariogun and A. A. Mogbademu, *Henstock-Kurzweil-Stieltjes-◊-Double integral for Gronwall-Bellman's Type Lemma on time scales*, Int. J. Nonlinear Anal. Appl. **14** (2023) 1, 833-841.

[2] D. A. Afariogun, A. A. Mogbademu,and B. O. Fagbemigun, *On l_p-Valued Functions for Henstock-Kurzweil-Stieltjes-⋄-Double Integrals on Time Scales*, Ajayi Crowther J. Pure Appl. Sci. **2(2)**(2023), pp. 79-86.

[3] S. Avsec, B. Bannish, B. Johnson, and S. Meckler, *The Henstock-Kurzweil delta integral on unbounded time scales*, Panamer. Math. J. No. 3, **16** (2006), 77-98.

[4] C. Blondia, *Integration in locally convex space*, Simon Stevin **55**, (1981), 81-102.

[5] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhauser Boston, MA.(2001).

[6] R. Henstock, *Theory of Integration*. Butterworths, London, 1968.

[7] R. Henstock, *The General Theory of Integration*, Oxford University Press, Oxford, UK, (1991), 20-25.

[8] S. Hilger, *Ein MaB kettenkalkul mit Anwendung auf Zentrumannigfahigkeiten*, Ph.D. Thesis, Universtat Wurzburg, (1988).

[9] V. O. Iluebe and A. A. Mogbademu, *On Sequential Henstock Stieltjes Integral For $L^p[0,1]$-Interval Valued Functions*, Bull. Int. Math. Virtual Inst., **12**(2022), 369 - 378.

[10] V. O. Iluebe and A. A Mogbademu, *General Sequential Topological Henstock-Type Integral*, Acta Universitatis Apulences. **74**(2023), pp. 85-96.

[11] V. Marraffa, The Riemann-Type integrals for functions taking values in a locally convex space, Czechoslovak Math J. 56, (2006), 475-489.

[12] C. W. McArthur and J.R Rutherford, *Some Applications of an inequality in locally convex space*. Trans. Amer. Math. Soc. **137**, (1969), 115-123.

[13] S. Nakanishi, *Henstock integral for function in Locally convex space*, Math. Japanica **39**, (1994), 309-335.

[14] L. A. Paxton, *Sequential Approach to the Henstock Integral*, Washington State University,arXiv:1609.05454v1 [maths.CA], (2016), 1-10.

[15] A. Peterson and B. Thompson, *Henstock-Kurzweil delta and nabla integrals*, J. Math. Anal. Appl., No. 1, **323** (2006), 162-178.

[16] L. D. Piazza and K. Musial, A characterization of variationally McShane integrable Banach space-valued functions, Illinois. J. Math. **45**, (2000). 279-289.

[17] A. Pietsch, Nuclear locally convex spaces, Springer Berlin, Heldelberg. **39**, (1995), 39-67.

[18] K. Sakurada and S. Nakamishi, *Equivalence of the McShane and Bochner integrals for functions with values in Hilbertian (UCs-N) spaces endowed with nuclearity*, Math. Japanica. **47**, (1998), 261-272.

[19] R. Sergio, Jr. Canon, *On Equivalence of the ϕ-Integral and the Henstock Integral for TVS-valued Functions*, Mathematical Analysis. Vol. **8**, (2014), 625-632.

[20] V. A. Skvortsov and A.P. Solodov, *A variational integral for Banach-valued functions*, R.A.E. **24**, (1998/9), 799-806.

[21] B. Thompson, *Henstock-Kurzweil integrals on time scales*, Panamer. Math. J. No. 1, **18** (2008), 1-19.