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Viscosity approximation methods for fixed point problems for ψ-contraction mapping on ϕ-convex subsets in Hilbert lattice spaces

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Abstract. In the article, the Almost Sure Convergence Theorem, which is related to Combettes and Hirstoaga's outcome, as well as Wittmann's outcome are demonstrated. The present article generalizes the S. Takahashia and W. Takahashi results to ϕ-convex mapping and ϕ-convex subsets of Hilbert spaces.

Keywords: Viscosity approximation methods, Equilibrium problem, ϕ-convex set, ϕ-convex mapping, ψ-contactin mapping

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1. Introduction

Suppose C be a non-empty closed convex proper subset of the real Hilbert set $\mathcal{H}$. Consider the bifunction $F : C \times C \to \mathbb{R}$, in which $\mathbb{R}$ is the space of real numbers. The problem to find $\alpha \in C$ where

$$F(\alpha, \beta) \geqslant 0; \quad \text{for every } \beta \in C. \tag{1.1}$$

is said the equilibrium problem. Solution sets (1.1) are denoted by $EP(F)$.
Suppose $T : C \to \mathcal{H}$ be a mapping. if $F(\alpha, \beta) = \langle T\alpha, \beta - \alpha \rangle$ then in the context of every $\alpha, \beta \in C$ the following implication holds;

$$\alpha \in EP(F) \text{ iff } \langle T\alpha, \beta - \alpha \rangle \geqslant 0 \text{ for all } \beta \in C.$$

There are many essential problems in science that are modeled as an equilibrium problem. A repetitious program was presented by Combettes and Hirstoaga to find the best approximation to the primary inputs whenever the set $EP(F)$ is non-empty as well as introduced the Almost Sure Convergence Theorem [2].
The fixed point issue is finding a point of C that satisfies in the following formula

$$T(\alpha) - \alpha = 0 \tag{1.2}$$

in which $T : C \to C$ is a self mapping on a subset C of Hilbert space H. The set of solutions of problem (1.2) is presented by $F(T)$. If C is a bounded, closed and convex set, and T is a non-expansive mapping, $\|Tx - Ty\| \leqslant \|x - y\|$ for all in C, then the fixed point set $F(T)$ is non-empty; see [4].
The following strong convergence theorem was demonstrated by Moudafi [5] .

Theorem 1.1. *(Moudafi* [5]*) Suppose C be a non-empty, closed and convex subset of a Hilbert space $\mathcal{H}$ also suppose $T : C \to C$ be a non-expansive self-mapping, where that $F(T)$ is nonempty. Suppose f be a compression of C into itself and suppose $\{\alpha_n\}$ be a series defined as follows: $\alpha_1 = \alpha \in C$ and*

$$\alpha_{n+1} = \frac{1}{1 + s_n} T\alpha_n + \frac{s_n}{1 + s_n} f(\alpha_n) \tag{1.3}$$

for every $n \in \mathbb{N}$, such that $\{s_n\} \subseteq (0, 1)$ satisfies

$$\lim_{n\to\infty} s_n = 0, \quad \sum_{n=1}^{\infty} s_n = \infty \quad \text{and} \quad \lim_{n\to\infty} | \frac{1}{s_{n+1}} - \frac{1}{s_n} | = 0.$$

Then, $\{\alpha_n\}$ converges strongly to $\gamma \in F(T)$, where $\gamma = P_{F(T)} f(\gamma)$ and $P_{F(T)}$ is the metric projection of $\mathcal{H}$ onto $F(T)$.

Approximation of a fixed point in this manner is called the viscosity approximation method. In 2007, Satoru Takahashi and Wataru Takahashi [8] by presenting a repetitive method by viscosity approximation procedure found a usual solution of problems (1.1) and (1.2).
In this paper we define ϕ-convex set, ϕ-convex mapping and ψ-contraction mapping and present a repetitious programming by the viscosity approximation method for finding a usual solution of challenges (1.1) and (1.2) of a ϕ-convex self-mapping.

2. Preliminaries

Definition 2.1. The arranged real vector span $(\mathbf{X}, \leqslant)$ is said lattice vector space if the least upper bound represented by $\alpha \bigvee \beta = \sup(\alpha, \beta)$ and the greatest lower bound represented by $\alpha \bigwedge \beta = \inf(\alpha, \beta)$ exist for each two elements $\alpha, \beta \in \mathbf{X}$ and for all $\alpha, \beta, \gamma \in \mathbf{X}$ and $t \in \mathbb{R}^+$ the following two axioms are satisfied:

(1) $\alpha \leqslant \beta$ implies $\alpha + \gamma \leqslant \beta + \gamma$;
(2) $0 \leqslant \alpha$ implies $0 \leqslant t\alpha$,

in which $\mathbb{R}^+$ is the non-negative real number set.

For $\alpha \in \mathbf{X}$ let

$$\alpha^+ := \alpha \vee 0, \quad \alpha^- := (-\alpha) \vee 0, \quad |\alpha| := \alpha \vee (-\alpha)$$

be the positive section, the negative section, and the absolute value of α, repectively.

Definition 2.2. A norm $\|\cdot\|$ on a vector lattice $\mathbf{X}$ is called a lattice norm if $|\alpha| \leqslant |\beta|$ then $\|\alpha\| \leqslant \|\beta\|$ for any $\alpha, \beta \in \mathbf{X}$.

Definition 2.3. A Banach (Hilbert) lattice is a real Banach (Hilbert) space $\mathbf{X}$ endowed with an ordering $\leqslant$ such that $(\mathbf{X}, \leqslant)$ is a vector lattice and the norm on $\mathbf{X}$ is a lattice norm.

Remember that a subset C of an ordered set X is said to be upward set, if $\zeta \in C$, $\alpha \in \mathbf{X}$ and $\zeta \leqslant \alpha$ we have $\alpha \in C$.

In the article, $\mathcal{H}$ is a Hilbert lattice space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let $\{\alpha_n\}$ be a sequence in $\mathcal{H}$, we denote the expression $\{\alpha_n\}$ converges weakly to α by $\alpha_n \rightharpoonup \alpha$, and moreover, the expression $\{\alpha_n\}$ converges strongly to α by $\alpha_n \to \alpha$. In a real Hilbert space $\mathcal{H}$, we have

$$\|t\alpha + (1-t)\beta\|^2 = \lambda^2\|\alpha\|^2 + (1-t)^2\|\beta\|^2 - 2t(1-t)\langle \alpha, \beta \rangle$$

Definition 2.4. A non-empty subset C of $\mathcal{H}$ is called:

(1) complete if every Cauchy sequence in C converges to a point in C;

(2) approximately compact if, for each $\alpha \in \mathcal{H}$, and any sequence $\{\alpha_n\} \subseteq C$ with $\|\alpha_n - \alpha\| \to d(\alpha, C) := \inf_{\zeta \in C} \|\alpha - \zeta\|$, consists a subsequence convergent to an element in C.

The metric projection on C is denoted by $P_C(\alpha)$ and is expounded as follow:

$$P_C(\alpha) := \{\zeta \in C \ \mid \ \|\alpha - \zeta\| = d(\alpha, C)\}.$$

This defines a set-valued mapping P_C from $\mathcal{H}$ into the subsets of C. Let $\mathcal{H}^+$ be as follows:

$$\mathcal{H}^+ = \{\eta \in \mathcal{H} \ \mid \ \eta \geqslant 0\}.$$

Definition 2.5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert lattice spaces. A function $\phi : \mathcal{H}_1^+ \to \mathcal{H}_2^+$ is called a comparison if ϕ is increasing, furthermore for each $\eta \in \mathcal{H}_1$, $\lim_{n\to\infty} \|\phi^n(\eta)\| = 0$.

Lemma 2.6. *Assume that $\phi : \mathcal{H}^+ \to \mathcal{H}^+$ is a comparison. Then for every $\eta > 0$, $\phi(\eta) < \eta$.*

Proof. Suppose that for some $\eta_0 > 0$ we have $\phi(\eta_0) \geqslant \eta_0$. Hence, by the monotonously of ϕ, $\phi^n(\eta_0) \geqslant \eta_0$ for all $n \in \mathbb{N}$. Since $\phi^n(\eta_0)$ and η_0 belongs to $\mathcal{H}^+$, implies that $|\phi^n(\eta_0)| \geqslant |\eta_0|$. since $\mathcal{H}$ is a norm lattice space, we have

$$\|\phi^n(\eta_0)\| \geqslant \|\eta_0\|. \tag{2.1}$$

Taking $n \to +\infty$ in (2.1), we have

$$\| \ \eta_0\| \leqslant 0$$

and so $\eta_0 = 0$. This is a contradiction therefore the assertion holds. □

Definition 2.7. Let $\mathcal{H}$ be Hilbert lattice space and ϕ be a comparison on $\mathcal{H}^+$. A subset C of $\mathcal{H}$ is ϕ-convex if for every $\alpha, \beta \in C$ with $\alpha \geqslant \beta$ and for any $t \in [0, 1]$ the following implication holds:

$$\beta + t\phi(\alpha - \beta) \in C,$$

Definition 2.8. Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert lattice spaces and ϕ a comparison on $\mathcal{H}_2^+$. The mapping $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called ϕ-convex if for every $\alpha, \beta \in C$ and $t \in [0, 1]$ the following relation holds:

$$T(t\alpha + (1 - t)\beta) \leqslant T\beta + t\phi(T\alpha - T\beta)$$

Definition 2.9. The self-mapping $T : \mathcal{H} \to \mathcal{H}$ is called ϕ-contraction if for every $\alpha, \beta \in C$ the following inference holds:

$$\|T(\alpha) - T(\beta)\| \leqslant \phi(\|\alpha - \beta\|),$$

in which $\phi : [0, +\infty) \to [0, 1]$ is a comparison.

Theorem 2.10. [7] *Let (X, d) be a complete metric space and $T : X \to X$ a ϕ-contraction. Then T has a unique established point in X.*

Theorem 2.11. *Let C be a upward and ϕ-convex subset of $\mathcal{H}$. So for each $\alpha \in \mathcal{H}$, $P_C(\alpha)$ has at most one element.*

Proof. Let $h \in \mathcal{H}$ and suppose x_1 and x_2 are in $P_C(h)$. Without loss of generality, we can assume that $x_1 < x_2$. Then by ϕ-convexity, $x_2 + \frac{1}{2}\phi(x_2 - x_1) \in C$. Since ϕ is a comparison we can get

$$\phi(x_2 - x_1) \leqslant x_2 - x_1,$$

so

$$x_1 + \frac{1}{2}\phi(x_2 - x_1) \leqslant x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_2 + x_1).$$

Since C is upward, we have

$$\frac{1}{2}(x_2 + x_1) \in C.$$

Therefore

$$\begin{aligned} d(h, C) &\leqslant \|h - \frac{1}{2}(x_2 + x_1)\| \\ &\leqslant \frac{1}{2}\|h - x_2\| + \frac{1}{2}\|h - x_1\| \\ &= \frac{1}{2}d(h, C) + \frac{1}{2}d(h, C) \\ &= d(h, C). \end{aligned}$$

Hence all the above values are equal. By the equality in the triangle inequality, $\|h - x_2\| = \rho\|h - x_1\|$ for some $\rho \geqslant 0$. In addition, $\|h - x_2\| = d(h, C) = \|h - x_1\|$ implies $\rho = 1$ and hence $x_1 = x_2$. □

Lemma 2.12. [3] *Let K be a approximately compact subset of $\mathcal{H}$. Then for each $h \in \mathcal{H}$, $P_K(x)$ is non-empty.*

Theorem 2.13. *Let C be a upward and closed ϕ-convex subset of $\mathcal{H}$. Thus for any $x \in \mathcal{H}$, $P_C(x)$ is singleton.*

Proof. Let C be an upward and closed ϕ-convex set and fix any $h \in \mathcal{H}$. We assume that $\{y_n\}$ is a sequence such that

$$\|h - y_n\| \to d(h, C). \tag{2.2}$$

Let for each n and m, only one of the following is true

$$y_m > y_n \quad \text{or} \quad y_m < y_n$$

If $y_m > y_n$ then by ϕ-convexity of C we have

$$y_n + \frac{1}{2}\phi(y_m - y_n) \in C$$

Since ϕ is a comparison we have $\phi(y_m - y_n) \leqslant y_m - y_n$, so

$$\begin{aligned} y_n + \frac{1}{2}\phi(y_m - y_n) &\leqslant y_n + \frac{1}{2}(y_m - y_n) \\ &= \frac{1}{2}(y_m + y_n). \end{aligned}$$

Since C is an upward, we have

$$\frac{1}{2}(y_m + y_n) \in C.$$

Similarly, If $y_m < y_n$, we have $\frac{1}{2}(y_n + y_m) \in C$. Then by the parallelogram law,

$$\begin{aligned} \|y_n - y_m\|^2 &= \|(h - y_m) - (h - y_n)\|^2 \\ &= 2(\|h - y_n\|^2 + \|h - y_m\|^2) - \|2h - (y_m + y_n)\|^2 \\ &= 2(\|h - y_n\|^2 + \|h - y_m\|^2) - 4\|h - \frac{1}{2}(y_m + y_n)\|^2. \end{aligned}$$

Since $\frac{1}{2}(y_m + y_n) \in C$, we have

$$\|y_n - y_m\|^2 \leqslant 2(\|h - y_n\|^2 + \|h - y_m\|^2) - 4d(h, C)^2. \tag{2.3}$$

The sequence $\{y_n\}$ is a Cauchy sequence by (2.2), (2.3) and the fact that the right side of the latter gravitates to zero as n and m gravitates to infinity. $\{y_n\}$ converges to several points $y \in C$, since C is closed. This means that C is approximately compact and hence the result holds by Theorem 2.11 and Lemma 2.12. □

Theorem 2.14. *Let C be a ϕ-convex subset of the inner product space $\mathcal{H}$, $h \in \mathcal{H}$, and $c_0 \in C$. Thus $c_0 \in P_C(h)$ iff*

$$\langle h - c_0, c - c_0 \rangle \leqslant 0 \quad \text{for all } c \in C. \tag{2.4}$$

Proof. If (2.4) holds and $c \in C$, then

$$\begin{aligned} \|h - c_0\|^2 &= \langle h - c_0, h - c_0 \rangle \\ &= \langle h - c_0, h - c \rangle + \langle h - c_0, c - c_0 \rangle \\ &\leqslant \langle h - c_0, h - c \rangle \\ &\leqslant \|h - c_0\| \|h - c\|, \end{aligned}$$

by Schwarz's inequality. Hence $\|h - c_0\| \leqslant \|h - c\|$, and so $c_0 = P_C(\mathcal{H})$. Conversely, suppose (2.4) fails. Then $\langle h - c_0, c - c_0 \rangle > 0$ for some $c \in C$. For each $0 < \lambda < 1$, the element $c_\lambda := c_0 + \lambda\phi(c - c_0)$ is in C by

ϕ-convexity and

$$\begin{aligned}\|h-c_\lambda\|^2 &= \langle h-c_\lambda, h-c_\lambda\rangle \\ &= \langle h-c_0-\lambda\phi(c-c_0), h-c_0-\lambda\phi(c-c_0)\rangle \\ &= \|h-c_0\|^2 - 2\lambda\langle h-c_0, \phi(c-c_0)\rangle + \lambda^2\|\phi(c-c_0)\|^2 \\ &= \|h-c_0\|^2 - \lambda\big[2\langle h-c_0, \phi(c-c_0)\rangle - \lambda\|\phi(c-c_0)\|^2\big]\end{aligned}$$

For $\lambda > 0$ which is small enough, the phrase which is located in bracket is positive. Therefore $\|h-c_\lambda\|^2 < \|h-c_0\|^2$. Hence $c_0 \neq P_C(x)$. □

It is known that for each sequence $\{h_n\} \subseteq \mathcal{H}$ with $h_n \longrightarrow h$, the inequality

$$\liminf_{n\to\infty} \|h_n - h_0\| < \liminf_{n\to\infty} \|h_n - h\|$$

holds for all $h \in \mathcal{H}$ with $h_0 \neq h$ ([6, 10]). For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us suppose that F satisfies the following situations:

- $(F1)$ $F(x,x) = 0$ for all $x \in C$;
- $(F2)$ F is monotone, i.e., $F(x,y) + F(y,x) \leqslant 0$ for every single one of the $x, y \in C$;
- $(F3)$ for any $x, y, z \in C$,
$$\lim_{t\downarrow 0} F(tx + (1-t)z, y) \leqslant F(x,y);$$
- $(F4)$ for every $x \in C$, $y \mapsto F(x,y)$ is convex and lesser semi-continuous.

Lemma 2.15. [1] *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(F1)-(F4)$. Let $r > 0$ and $x \in \mathcal{H}$. In addition, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geqslant 0$$

for every $y \in C$.

Lemma 2.16. [2] *We suppose that $F : C\times C \to \mathbb{R}$ satisfies $(F1)-(F4)$. For $r > 0$ together with $x \in \mathcal{H}$, spell out a mapping $T_r : \mathcal{H} \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geqslant 0, \forall y \in C\}$$

for each $z \in \mathcal{H}$. Then, the following holds:

1. *T_r is single-valued;*
2. *T_r is fixedly non-expansive, i.e., for any $x, y \in \mathcal{H}$,*
$$\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x - y\rangle;$$
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

3. Main Results

Lemma 3.1. [8] *Suppose* $\{a_n\} \subset [0,\infty)$, $\{b_n\} \subset [0,\infty)$ *and* $\{c_n\} \subset [0,1)$ *be arrangement of real numbers suchlike*

$$a_{n+1} \leqslant (1-c_n)a_n + b_n \ \text{ for all } \ n \in \mathbb{N},$$

$$\sum_{n=1}^{\infty} c_n = \infty \ \text{ and } \ \sum_{n=1}^{\infty} a_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Theorem 3.2. *Suppose* C *be a non-empty, closed* ϕ*-convex subset of Hilbert lattice space* $\mathcal{H}$. *Assume that* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(F1)-(F4)$ *and suppose* S *be a non-expansive mapping of* C *into* $\mathcal{H}$ *such that* $F(S) \cap EP(F) \neq \emptyset$. *Suppose* f *be a* ψ*-contraction of* $\mathcal{H}$ *into itself as well as suppose* $\{x_n\}$ *and* $\{u_n\}$ *be sequences multiplied by* $x_1 \in \mathcal{H}$ *and*

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geqslant 0, \quad \text{for all} \quad y \in C, \\ x_{n+1} = Su_n + \lambda_n \phi(f(x_n) - Su_n), \\ u_n = T_{r_n} x_n \end{cases}$$

for any $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subseteq [0,1]$ *moreover* $\{r_n\} \subseteq (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \lambda_n = 0; \quad \sum_{n=1}^{\infty} \lambda_n = \infty; \quad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty;$$

$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty.$$

Additionally, $\{x_n\}$ *together with* $\{u_n\}$ *converge firmly to* $z \in F(S) \cap EP(F)$, *wherever* $z = P_{F(S)\cap EP(F)} f(z)$.

Proof. Suppose $Q = P_{F(S)\cap EP(F)}$. Then Qf is a ψ- contraction of $\mathcal{H}$ into itself. In point of fact, $\|f(x) - f(y)\| \leqslant \psi(\|x-y\|)$ for each $x, y \in \mathcal{H}$. So, we get

$$\|Qf(x) - Qf(y)\| \leqslant \|f(x) - f(y)\| \leqslant \psi(\|x-y\|)$$

for all of $x, y \in \mathcal{H}$. So, Qf is a ψ-contraction on $\mathcal{H}$. Because $\mathcal{H}$ is complete, there exists an special component $z \in \mathcal{H}$ suchlike $z = Qf(z)$. This $z \in H$ is an element of C. Suppose $v \in F(S) \cap EP(F)$. Then from $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leqslant \|x_n - v\|$$

for each one of the $n \in \mathbb{N}$. Put $M = \max\{\|x_1 - v\|, \frac{1}{1-a}\|f(v) - v\|\}$. It is clear that $\|x_1 - v\| \leqslant M$. Since ϕ is a comparison the following will obtain

$$\begin{aligned}\|Su_n + \lambda_n\phi(f(x_n) - Su_n) - v\| &\leqslant \|Su_n + \lambda_n(f(x_n) - Su_n) - v\| \\ &\leqslant \lambda_n\|f(x_n) - v\| + (1 - \lambda_n)\|Su_n - v\|.\end{aligned}$$

Suppose $\|x_n - v\| \leqslant M$. Then, we will have

$$\begin{aligned}\|x_{n+1} - v\| &= \|Su_n + \lambda_n\phi(f(x_n) - Su_n) - v\| \\ &\leqslant \lambda_n\|f(x_n) - v\| + (1 - \lambda_n)\|Su_n - v\| \\ &\leqslant \lambda_n(\|f(x_n) - f(v)\| + \|f(v) - v\|) + (1 - \lambda_n)\|u_n - v\| \\ &\leqslant \lambda_n(\psi(\|x_n - v\|) + \|f(v) - v\|) + (1 - \lambda_n)\|u_n - v\| \\ &\leqslant \lambda_n(\|x_n - v\| + \|f(v) - v\|) + (1 - \lambda_n)\|u_n - v\| \\ &\leqslant \lambda_n\|f(v) - v\| + \|x_n - v\|.\end{aligned}$$

Since λ_n is increasing as well as, by above inequality we get

$$\lambda_n\|f(v) - v\| + \|x_n - v\| \leqslant \lambda_m\|f(v) - v\| + \|x_n - v\|; \quad \text{for any} \quad m > n.$$

So for every $m > n$ we get

$$\|x_{n+1} - v\| \leqslant \lambda_m\|f(v) - v\| + \|u_n - v\|.$$

By taking $n \to +\infty$ in above inequality we obtain

$$\|x_{n+1} - v\| \leqslant \|u_n - v\| \leqslant M.$$

Hence for each $n \in \mathbb{N}$, the inequality $\|x_n - v\| \leqslant M$ will be obtained and so $\{x_n\}$ is finite. Boundedness of the $\{u_n\}$, $\{Sx_n\}$ and $\{f(x_n)\}$ are concluded.
Next, we can show that $\|x_{n+1} - x_n\| \to 0$. In fact,

$$\begin{aligned}\|x_{n+1} - x_n\| =& \|Su_n + \lambda_n\phi(f(x_n) - Su_n) - Su_{n-1} - \lambda_{n-1}\phi(f(x_{n-1}) - Su_{n-1})\| \\ \leqslant& \|Su_n + \lambda_n(f(x_n) - Su_n) - Su_{n-1} - \lambda_{n-1}(f(x_{n-1}) - Su_{n-1})\| \\ =& \|\lambda_n f(x_n) - \lambda_n f(x_{n-1}) + \lambda_n f(x_{n-1}) - \lambda_{n-1} f(x_{n-1}) \\ &+ (1 - \lambda_n)Su_n - (1 - \lambda_n)Su_{n-1} + (1 - \lambda_n)Su_{n-1} - (1 - \lambda_n)Su_{n-1}\| \\ \leqslant& \lambda_n\psi(\|x_n - x_{n-1}\|) + |\lambda_n - \lambda_{n-1}|K + (1 - \lambda_n)\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|K\end{aligned} \tag{3.1}$$

wherever $K = \sup\{\|f(x_n)\| + \|Su_n\| : n \in \mathbb{N}\}$.
However, $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, therefore we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0 \quad \text{for every} \quad y \in C \tag{3.2}$$

and

$$f(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\geqslant 0 \quad \text{for each one of the} \quad y\in C \tag{3.3}$$

Suppose $y=u_{n+1}$ in (3.2) and $y=u_n$ in (3.3). Then will achieve

$$f(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\geqslant 0$$
$$f(u_{n+1},u_n)+\frac{1}{r_n}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\geqslant 0.$$

So by property $(F2)$ we get

$$\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle\geqslant 0;$$

therefore

$$\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\geqslant 0$$

so

$$\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\rangle\geqslant 0,$$

therefore

$$\begin{aligned}\|u_{n+1}-u_n\|^2&\leqslant\langle u_{n+1}-u_n,x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\rangle\\&\leqslant\|u_{n+1}-u_n\|\Big\{\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|u_{n+1}-x_{n+1}\|\Big\}.\end{aligned}$$

Without loss of generality, let b be a real number in such manner $0<b<r_n$ for all $n\in\mathbb{N}$. So

$$\begin{aligned}\|u_{n+1}-u_n\|&\leqslant\|x_{n+1}-x_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|\\&\leqslant\|x_{n+1}-x_n\|+\frac{1}{b}|r_{n+1}-r_n|L,\end{aligned} \tag{3.4}$$

in which $L=\sup\{\|u_n-x_n\| \ : \ n\in\mathbb{N}\}$.
Hence, by (3.1) we will have

$$\begin{aligned}\|x_{n+1}-x_n\|&\leqslant\lambda_n\psi(\|x_n-x_{n-1}\|)+2|\lambda_n-\lambda_{n-1}|K+(1-\lambda_n)\|u_n-u_{n-1}\|\\&\leqslant\lambda_n\|x_n-x_{n-1}\|+2|\lambda_n-\lambda_{n-1}|K+(1-\lambda_n)(\|x_{n+1}-x_n\|+\frac{1}{b}|r_{n+1}-r_n|L)\\&=\|x_n-x_{n-1}\|+2|\lambda_n-\lambda_{n-1}|K+\frac{1-\lambda_n}{b}|r_{n+1}-r_n|L.\end{aligned}$$

By Lemma 3.1,

$$\|x_{n+1}-x_n\|\to 0.$$

By (3.4) and $|r_{n+1} - r_n| \to 0$, we will get

$$\|u_{n+1} - u_n\| \to 0.$$

As long as, $x_{n+1} = Su_n + \lambda_n \phi(f(x_n) - Su_n)$ and ϕ is a comparison, we conclude that

$$\begin{aligned}\|x_n - Su_n\| &\leqslant \|x_n - Su_{n-1}\| + \|Su_n - Su_{n-1}\| \\ &= \|\lambda_n \phi(f(x_n) - Su_n)\| + \|Su_n - Su_{n-1}\| \\ &\leqslant \lambda_n \|f(x_n) - Su_n\| + \|u_n - u_{n-1}\|.\end{aligned}$$

Since $\lambda_n \to 0$, $\|x_n - Su_n\| \to 0$. Suppose $v \in F(S) \cap EP(F)$, then we get

$$\begin{aligned}\|u_n - v\|^2 &= \|T_{r_n} x_n - T_{r_n} v\|^2 \\ &\leqslant \langle T_{r_n} x_n - T_{r_n} v, x_n - v\rangle \\ &= \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}\Big(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\Big)\end{aligned}$$

along with

$$\|u_n - v\|^2 \leqslant \|x_n - v\|^2 - \|x_n - u_n\|^2.$$

Since $\|.\|_2$ is a convex and ϕ is a comparison, we get

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|Su_n + \lambda_n \phi(f(x_n) - Su_n) - v\|^2 \\ &= \|Su_n + \lambda_n (f(x_n) - Su_n) - v\|^2 \\ &\leqslant \|\lambda_n f(x_n) + (1 - \lambda_n) Su_n - v\|^2 \\ &\leqslant \lambda_n \|f(x_n) - v\|^2 + (1 - \lambda_n)\|u_n - v\|^2 \\ &\leqslant \lambda_n \|f(x_n) - v\|^2 + (1 - \lambda_n)\big(\|x_n - v\|^2 - \|x_n - u_n\|^2\big) \\ &\leqslant \lambda_n \|f(x_n) - v\|^2 + \|x_n - v\|^2 - (1 - \lambda_n)\|x_n - u_n\|^2\end{aligned}$$

so

$$\begin{aligned}(1 - \lambda_n)\|x_n - u_n\|^2 &\leqslant \lambda_n \|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\ &\leqslant \lambda_n \|f(x_n) - v\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - v\| + \|x_{n+1} - v\|\big).\end{aligned}$$

In this way, obtains $\|x_n - u_n\| \to 0$. Since

$$\|Su_n - u_n\| \leqslant \|Su_n - x_n\| + \|x_n - u_n\|,$$

hence

$$\|Su_n - u_n\| \to 0.$$

We claim that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leqslant 0,$$

in which $z = P_{F(S)\cap EP(F)}f(z)$. Assume that $\{u_{n_i}\}$ is a subsequence of $\{u_n\}$ suchlike

$$\limsup_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle = \limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle.$$

Since $\{u_{n_i}\}$ is bounded, it has a subsequence which links up weakly to w. Without loss of generality, we can suppose $\{u_{n_i}\} \rightharpoonup w$. From $\|Su_n - u_n\| \to 0$, we conclude $Su_{n_i} \rightharpoonup w$. Now we show that $w \in EP(F)$. By $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \qquad \text{for all } \ y \in C.$$

By (F2), we get

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant F(u_n, y)$$

and so

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geqslant F(u_{n_i}, y).$$

Because of $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (F4)

$$F(y, w) \leqslant 0 \qquad \text{for all } \ y \in C.$$

Suppose $y_t = w + t\phi(y - t)$ where $y \in C$ and $w \in C$. So, by ϕ-convexity of C, $y_t \in C$ and consequently $F(y_t, w) \leqslant 0$. As a result by conditions (F1) and (F4) we can have

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\leqslant tF(y_t, y) + (1 - t)F(y_t, w) \\ &\leqslant F(y_t, y) \end{aligned}$$

and so $0 \leqslant F(y_t, y)$. By (F3), we conclude

$$F(y, w) \leqslant 0 \qquad \text{for all of the} \qquad y \in C$$

and hence $w \in EP(F)$.
Now we prove that $w \in F(S)$. Suppose $w \notin F(S)$. Because of $u_{n_i} \rightharpoonup w$ and $w \neq Sw$, By Opial's theorem [6], we have

$$\liminf_{i\to\infty}\|u_{n_i} - w\| < \liminf_{i\to\infty}\|u_{n_i} - Sw\| \leqslant \liminf_{i\to\infty}\{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\}.$$

However,

$$\liminf_{i\to\infty}\|u_{n_i}-w\|\leqslant\liminf_{i\to\infty}\|u_{n_i}-w\|,$$

since $\|Su_n-u_n\|\to 0$ and S is non-expansive.
This is a paradox. Hence, $w\in F(S)$ and so $w\in F(S)\cap EP(F)$. Since $z=P_{F(S)\cap EP(F)}f(z)$, we conclude

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle&=\lim_{i\to\infty}\langle f(z)-z,x_{n_i}-z\rangle\\&=\langle f(z)-z,w-z\rangle\leqslant 0.\end{aligned}\tag{3.5}$$

The following holds

$$\begin{aligned}\|x_{n+1}-z\|&=\|Su_n+\lambda_n\phi(f(x_n)-Su_n)-z\|\\&\leqslant\|Su_n+\lambda_n(f(x_n)-Su_n)-z\|\\&\leqslant\|\lambda_n(f(x_n)-z)\|+\|(1-\lambda_n)(Su_n-z)\|,\end{aligned}$$

since $x_{n+1}=Su_n+\lambda_n\phi(f(x_n)-Su_n)$ and ϕ is comparison. Therefore,

$$(1-\lambda_n)^2\|Su_n-z\|^2\geqslant\|x_{n+1}-z\|^2-2\lambda_n\langle f(x_n)-z,x_{n+1}-z\rangle.$$

Hence we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2&\leqslant(1-\lambda_n)^2\|Su_n-z\|^2+2\lambda_n\langle f(x_n)-z,x_{n+1}-z\rangle\\&\leqslant(1-\lambda_n)^2\|u_n-z\|^2+\lambda_n\langle f(x_n)-f(z),x_{n+1}-z\rangle+2\lambda_n\langle f(x_n)-z,x_{n+1}-z\rangle\\&\leqslant(1-\lambda_n)^2\|x_n-z\|^2+\lambda_n\psi\big(\|x_n-z\|\big)\|x_{n+1}-z\|+2\lambda_n\langle f(x_n)-z,x_{n+1}-z\rangle\\&\leqslant(1-\lambda_n)^2\|x_n-z\|^2+\lambda_n\|x_n-z\|\|x_{n+1}-z\|+2\lambda_n\langle f(x_n)-z,x_{n+1}-z\rangle\\&\leqslant(1-\lambda_n)^2\|x_n-z\|^2+\frac{\lambda_n}{2}\big(\|x_n-z\|^2+\|x_{n+1}-z\|^2\big)+2\lambda_n\langle f(x_n)-z,x_{n+1}-z\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1}-z\|^2&\leqslant\frac{(1-\lambda_n)^2+\frac{\lambda_n}{2}}{1-\frac{\lambda_n}{2}}\|x_n-z\|^2+\frac{2\lambda_n}{1-\frac{\lambda_n}{2}}\langle f(z)-z,x_{n+1}-z\rangle\\&=\frac{1-2\lambda_n-\frac{\lambda_n}{2}}{1-\frac{\lambda_n}{2}}\|x_n-z\|^2+\frac{\lambda_n^2}{1-\frac{\lambda_n}{2}}\|x_n-z\|^2+\frac{2\lambda_n}{1-\frac{\lambda_n}{2}}\langle f(z)-z,x_{n+1}-z\rangle\\&\leqslant\big(1-\frac{\lambda_n}{1-\frac{\lambda_n}{2}}\big)\|x_n-z\|^2+\frac{\lambda_n}{1-\frac{\lambda_n}{2}}\bigg(\lambda_n\Gamma+\frac{1}{2}\langle f(z)-z,x_{n+1}-z\rangle\bigg),\end{aligned}$$

In which $\Gamma=\sup\{\|x_n-z\|^2\ :\ n\in\mathbb{N}\}$. We put $\beta_n=\frac{\lambda_n}{1-\frac{\lambda_n}{2}}$. Then $\sum_{n=1}^{\infty}\beta_n=\infty$ and $\lim_{n\to\infty}\beta_n=0$.

We now choose an arbitrary $\epsilon > 0$. Then by (3.5), there exists $m \in \mathbb{N}$ such that

$$\lambda_n \Gamma \leqslant \frac{\epsilon}{2} \qquad \text{and} \qquad \langle f(z) - z, x_{n+1} - z \rangle \leqslant \frac{\epsilon}{2}$$

for any $n \geqslant m$.
So

$$\|x_{n+1} - z\|^2 \leqslant (1 - \beta_n)\|x_n - z\|^2 + \big(1 - (1 - \beta_n)\big)\epsilon.$$

Therefore

$$\begin{aligned}
\|x_{n+2} - z\|^2 &\leqslant (1 - \beta_{n+1})\|x_{n+1} - z\|^2 + \big(1 - (1 - \beta_{n+1})\big)\epsilon \\
&\leqslant (1 - \beta_{n+1})\Big((1 - \beta_n)\|x_n - z\|^2 + \big(1 - (1 - \beta_n)\big)\epsilon\Big) + \big(1 - (1 - \beta_{n+1})\big)\epsilon \\
&= (1 - \beta_{n+1})(1 - \beta_n)\|x_n - z\|^2 + \Big((1 - \beta_n) + (1 - \beta_{n+1})(1 - \beta_n) + \big(1 - (1 - \beta_{n+1})\big)\Big)\epsilon \\
&= (1 - \beta_{n+1})(1 - \beta_n)\|x_n - z\|^2 + \Big(1 - (1 - \beta_{n+1})(1 - \beta_n)\Big)\epsilon.
\end{aligned}$$

Hence by induction:

$$\|x_{n+m} - z\|^2 \leqslant \prod_{k=m}^{n+m-1} (1 - \beta_k)\|x_m - z\|^2 + \big(1 - \prod_{k=m}^{n+m-1} (1 - \beta_k)\big)\epsilon.$$

$\prod_{k=m}^{\infty}(1 - \beta_k) = 0$ holds, since $\sum_{k=0}^{\infty} \beta_k = \infty$. As a consequence we can get the following:

$$\limsup_{n\to\infty} \|x_n - z\|^2 = \limsup_{n\to\infty} \|x_{n+m} - z\|^2 \leqslant \epsilon.$$

Therefore,

$$\limsup_{n\to\infty} \|x_n - z\|^2 \leqslant 0.$$

Hence, $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, wherever $z = P_{F(S)\cap EP(F)} f(z)$. □

References

[1] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63**, 123–145, 1994.

[2] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6**, 117–136, 2005.

[3] F. Deutsch, *Best approximation in inner product spaces*, Springer-Verlag, New York, 2001.

[4] S. D. Flam, A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Program, **78**, 29–41, 1997.

[5] A. Moudafi, *Viscosity approximation methods for fixed-point problems* , J. Math. Anal. Appl., **241**,46–55, 2000.

[6] Z. Opial, *Weak convergence of the sequence of successive approximation for non-expansive mappings*, Bull. Amer. Math. Soc., **73**, 561–597, 1967.
[7] I. A. Rus, *Generalized contractions*, Seminar on Fixed Point Theory, vol. 3, pp. 1–130, 1983.
[8] S. Takahashi, W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331**, 506-515, 2007.
[9] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000.
[10] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[11] R. Wittmann, *Approximation of fixed points of non-expansive mappings*, Arch. Math. 58 (1992) 486-491.