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Numerical range of a sum of weighted composition operators on the weighted Bergman spaces

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Abstract. Suppose that φ is an analytic self-map of the open unit disk $\mathbb{D}$ and ψ is a bounded analytic function on $\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is the operator on the weighted Bergman spaces A^2_α given by $C_{\psi,\varphi}f = \psi f \circ \varphi$ for any $f \in A^2_\alpha$. If $\varphi_1, \varphi_2, ..., \varphi_n$ are linear fractional non-automorphisms and functions $\psi_1, \psi_2, ..., \psi_n$ are bounded analytic functions, under some conditions, we obtain a subset of the numerical range of $C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2} + ... + C_{\psi_n,\varphi_n}$ on A^2_α and determine when zero lies in the interior of its numerical range.

Keywords: Weighted Bergman spaces, weighted composition operator, numerical range.

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1. Introduction

The boundary of the open unit disk $\mathbb{D}$ is denoted by $\partial\mathbb{D}$. For $\alpha > -1$, the *weighted Bergman spaces* A^2_α are the Hilbert space consisting of all

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analytic functions f on $\mathbb{D}$ with

$$\|f\|^2 = \int_{\mathbb{D}} |f|^2 dA_\alpha < \infty,$$

where $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA$ with dA is the normalized area measure in $\mathbb{D}$. It is well known that this space is a reproducing kernel Hilbert space, so for any w in $\mathbb{D}$, there is a *reproducing kernel function* K_w such that $\langle f, K_w \rangle = f(w)$ for each f in A^2_α (see [4, p. 27]). For any w in $\mathbb{D}$, the reproducing kernel function is defined by

$$K_w(z) = \frac{1}{(1-\overline{w}z)^{\alpha+2}}.$$

Moreover,

$$\|K_w\|^2 = \frac{1}{(1-|w|^2)^{\alpha+2}}.$$

The space H^∞ consists of all bounded analytic functions on the unit disk with supremum norm $\|f\|_\infty = \sup_{|z|<1} |f(z)|$ for any $f \in H^\infty$.

A linear fractional self-map of $\mathbb{D}$ is a map of the form $\varphi(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We know that the form of automorphisms of the unit disk is given by $\varphi(z) = \frac{\lambda(a-z)}{1-\bar{a}z}$, where $|\lambda| = 1$ and $|a| < 1$.

For an analytic self-map φ, we say φ has a *finite angular derivative* at ζ on the unit circle if there is an η on the unit circle so that $(\varphi(z) - \eta)/(z-\zeta)$ has a finite nontangential limit as $z \to \zeta$. This limit is denoted by $\varphi'(\zeta)$ if it exists as a finite complex number. For $\zeta \in \partial\mathbb{D}$, we denote $\varphi(\zeta) := \lim_{r\to 1} \varphi(r\zeta)$. For an analytic self-map φ of $\mathbb{D}$, we say that φ does not make contact with $\partial\mathbb{D}$ if $\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) \in \partial\mathbb{D}\}$ is empty.

A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ with a unique fixed point $\zeta \in \partial\mathbb{D}$ is called *parabolic*. For $T(z) = \frac{\zeta+z}{\zeta-z}$, the map $\phi = T \circ \varphi \circ T^{-1}$ is a linear fractional transformation, which takes the right half plane onto itself with the only fixed point ∞. Thus, there exists a complex number t with Re $t \geq 0$ such that $\phi(z) = z + t$. It is not hard to see that

$$\varphi(z) = \frac{(2-t)z + t\zeta}{2 + t - t\bar{\zeta}z}.$$

The constant number t is called the translation number of φ. Note that if Re $t = 0$, then φ is an automorphism. When Ret is strictly positive, we call φ positive parabolic non-automorphism. Among linear fractional transformations with a fixed point $\zeta \in \partial\mathbb{D}$, just the parabolic ones are characterized by $\varphi'(\zeta) = 1$ (see [8]).

For an analytic self-map φ of $\mathbb{D}$ and positive integer n, we write $\varphi_1 := \varphi$ and $\varphi_{n+1} := \varphi \circ \varphi_n$. If φ is neither an elliptic automorphism nor the identity map, then there is a unique point w in the closed unit disk so that for all $z \in \mathbb{D}$, $\varphi_n(z) \to w$ as $n \to \infty$. This limit point is called the *Denjoy–Wolff point* of φ (see [4, Theorem 2.51]). We have

1) $\varphi(w) = w$ and $|\varphi'(w)| < 1$ when $|w| < 1$.
2) $\varphi(w) = w$ and $0 < \varphi'(w) \leq 1$ when $|w| = 1$.

In this paper, we use $\mathrm{co}(A)$ for the convex hull of a set A, which is the intersection of all convex sets containing A.

Let T be a bounded linear operator on Hilbert space H. The *numerical range* of T is denoted by $W(T)$ and is defined as

$$W(T) = \{\langle Tf, f\rangle : f \in H, \|f\| = 1\}.$$

The numerical ranges of C_φ on the Hardy space induced by holomorphic self-maps φ were investigated by Bourdon and Shapiro in [2] and [3]. Moreover, the shape of the numerical range for composition operators C_φ when φ is an automorphism was discussed in [3]. In [6], the numerical range of $C_{\psi,rz}$ with $|r| < 1$ and hermitian weighted composition operator on the Hardy space were found. The spectrum and the essential spectrum of the weighted composition operators on A^2_α have been recently studied (see [5]). In this paper, for a bounded operator T on the weighted Bergman spaces, the spectrum of T and the essential spectrum of T are denoted by $\sigma(T)$ and $\sigma_e(T)$.

2. Numerical range

Through this paper, we assume that $\psi, \psi_1, ..., \psi_n$ belong to H^∞. First, we state the following well-known lemma which will be used in the sequel.

Lemma 2.1. *Let $w \in \mathbb{D}$. Suppose that $C_{\psi,\varphi}$ is a bounded operator on A^2_α. Then*

$$C^*_{\psi,\varphi}(K_w) = \overline{\psi(w)} K_{\varphi(w)}.$$

Proof. One can see that

$$\langle f, C^*_{\psi,\varphi} K_w\rangle = \langle C_{\psi,\varphi} f, K_w\rangle = \psi(w) f(\varphi(w)) = \langle f, \overline{\psi(w)} K_{\varphi(w)}\rangle,$$

for each arbitrary function $f \in A^2_\alpha$. Then the result follows. □

In the case that φ is an analytic self-map of $\mathbb{D}$ which is not identity map, the closure of $W(C_\varphi)$ contains 0 on the Hardy space (see [2, Theorem 3.1]). In the proof of the following proposition, the out line of the proof of [2, Theorem 3.1] is followed to show that 0 belongs to the closure of numerical range of the sum of finite number of weighted composition operators on A^2_α.

Proposition 2.2. *Assume that $\varphi_1, ..., \varphi_n$ are not identity maps. Suppose that $\psi_1, ..., \psi_n \in H^\infty$. Then 0 is an element of the closure of $W(\sum_{i=1}^n C_{\psi_i,\varphi_i})$.*

Proof. For $w \in \mathbb{D}$,

$$\left\langle (C_{\psi_1,\varphi_1} + ... + C_{\psi_n,\varphi_n})^* \frac{K_w}{\|K_w\|}, \frac{K_w}{\|K_w\|} \right\rangle \in W\left(\sum_{i=1}^n C_{\psi_i,\varphi_i}\right).$$

We infer from Lemma 2.1 that

$$\begin{aligned} \left\langle (C_{\psi_1,\varphi_1} + ... + C_{\psi_n,\varphi_n})^* \frac{K_w}{\|K_w\|}, \frac{K_w}{\|K_w\|} \right\rangle &= \overline{\psi_1(w)} \left(\frac{1-|w|^2}{1-\overline{\varphi_1(w)}w} \right)^{\alpha+2} \\ &+ ... \\ &+ \overline{\psi_n(w)} \left(\frac{1-|w|^2}{1-\overline{\varphi_n(w)}w} \right)^{\alpha+2}. \end{aligned}$$

Since $\varphi_1, ..., \varphi_n$ are not the identity maps, there is $\zeta \in \partial\mathbb{D}$ such that $\varphi_i(\zeta) \neq \zeta$ for each $i = 1, ..., n$. The previous equation implies that

$$\lim_{w\to\zeta} \left\langle \left(\sum_{i=1}^n C_{\psi_i,\varphi_i}\right)^* \frac{K_w}{\|K_w\|}, \frac{K_w}{\|K_w\|} \right\rangle = 0.$$

Therefore, the closure of $W(\sum_{i=1}^n C_{\psi_i,\varphi_i})$ contains 0. Thus the conclusion follows. □

In the following theorem, we assume that for each i, the function φ_i is a non-automorphism linear fractional self-map of $\mathbb{D}$. Thus for each i, the set $\{\zeta : \ \varphi_i(\zeta) \in \partial\mathbb{D}\}$ has at most on element in $\partial\mathbb{D}$.

Theorem 2.3. *Assume that $\varphi_1, ..., \varphi_n$ are linear fractional self-map of $\mathbb{D}$ which are not automorphisms. Let m be an integer such that $m < n$. Suppose that $\eta_i, \zeta_i \in \partial\mathbb{D}$ and $\varphi_i(\zeta_i) = \eta_i$ when $1 \leq i \leq m$. Moreover, for $m < i \leq n$, φ_i is not parabolic and $\varphi_i(\zeta_i) = \zeta_i$ for $\zeta_i \in \partial\mathbb{D}$. Assume that for each i, ψ_i is continuous at ζ_i. Suppose that $\zeta_i \neq \zeta_j$, $\zeta_i \neq \eta_j$ when $i \neq j$. Then for each $m < i \leq n$,*

$$\left\{ z : |z| < \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\} \subseteq W\left(\sum_{i=1}^n C_{\psi_i,\varphi_i}\right).$$

In particular, if $\psi_i(\zeta_i) \neq 0$ for some $m < i \leq n$. Then zero is an interior point of $W(\sum_{i=1}^n C_{\psi_i,\varphi_i})$.

Proof. For each $i \neq j$, the closure of $\varphi_i(\varphi_j(\mathbb{D}))$ is a subset of $\mathbb{D}$, so by [4, page 129], the operator $C_{\varphi_i \circ \varphi_j}$ is compact. Therefore, for $i \neq j$, $C_{\psi_j,\varphi_j} C_{\psi_i,\varphi_i}$ is compact. We infer from [1, Proposition 3.3] that

$$\sigma_e\left(\sum_{i=1}^n C_{\psi_i,\varphi_i}\right) \setminus \{0\} = \bigcup_{i=1}^n \sigma_e(C_{\psi_i,\varphi_i}) \setminus \{0\}.$$

Since for $m < i \leq n$, $\varphi_i(\zeta_i) = \zeta_i$ and φ_i is not parabolic, we break the solution in two cases.

a) Suppose that the Denjoy–Wolff point of φ_i belongs to $\mathbb{D}$ for some i, $m < i \leq n$. By [5, Theorem 2.11],

$$\sigma_e(C_{\psi_i,\varphi_i}) = \left\{ z : |z| < \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\}.$$

Thus for $m < i \leq n$

$$\left\{ z : |z| < \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\} \setminus \{0\} \subseteq \sigma_e\left(\sum_{i=1}^{n} C_{\psi_i,\varphi_i} \right) \setminus \{0\}.$$

We get from [7, Theorem 1.2-1] that for $m < i \leq n$,

$$\left\{ z : |z| \leq \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\} \setminus \{0\}$$

is the subset of the closure of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$. Since $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$ is convex, we conclude that

$$\left\{ z : |z| < \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\} \subseteq W\left(\sum_{i=1}^{n} C_{\psi_i,\varphi_i} \right).$$

b) Assume that ζ_i is the Denjoy–Wolff point of φ_i for some i, $m < i < n$. Since φ_i is not parabolic, by [5, Theorem 2.9],

$$\sigma_e(C_{\psi_i,\varphi_i}) = \left\{ z : |z| < \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\}.$$

We conclude that

$$\left\{ z : |z| < \frac{|\psi_i(\zeta_i)|}{|\varphi_i'(\zeta_i)|^{1+\frac{\alpha}{2}}} \right\} \subseteq \sigma_e\left(\sum_{i=1}^{n} C_{\psi_i,\varphi_i} \right) \setminus \{0\}.$$

By the similar idea used in Part (a), the result follows. □

In the previous theorem, we assumed that for each $m < i \leq n$, the function φ_i is not parabolic. In the next theorem, we suppose that for some i, the linear fractional self-maps φ_i are parabolic, but not positive parabolic.

Theorem 2.4. *Assume that $\varphi_1, ..., \varphi_n$ are linear fractional self-map of $\mathbb{D}$ that are not automorphisms. Let m be an integer such that $m < n$. Suppose that $\eta_i, \zeta_i \in \partial\mathbb{D}$ such that $\varphi_i(\zeta_i) = \eta_i$ for $1 \leq i \leq m$. Moreover, for $m < i \leq n$, $\zeta_i \in \partial\mathbb{D}$ and $\varphi(\zeta_i) = \zeta_i$. Let for each $i \neq j$, $\zeta_i \neq \zeta_j$ and $\zeta_i \neq \eta_j$. Suppose that there exists i_0 with $m < i_0 \leq n$ such that φ_{i_0} is* *parabolic but not positive parabolic and the translation number of* φ_{i_0} *is* t_{i_0}. *If* ψ_{i_0} *is continuous at* ζ_{i_0} *and* $\psi_{i_0}(\zeta_{i_0}) \neq 0$, *then*

$$co\left(\{\psi_{i_0}(\zeta_{i_0})e^{-\beta t_{i_0}} :\ 0 \leq \beta < \infty\}\right)$$

is the subset of the closure of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$ *and* 0 *belongs to the interior of* $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$.

Proof. We infer from [5, Theorem 2.8] that for i_0

$$\sigma_e(C_{\psi_{i_0},\varphi_{i_0}}) = \left(\{\psi_{i_0}(\zeta_{i_0})e^{-\beta t_{i_0}} :\ 0 \leq \beta < \infty\}\right) \bigcup \{0\},$$

which is a logarithmic spiral (see [4, p. 303]). Since for each $i \neq j$, $C_{\psi_j,\varphi_j} C_{\psi_i,\varphi_i}$ is compact, by [1, Proposition 3.3], we obtain

$$\sigma_e\left(\sum_{i=1}^{n} C_{\psi_i,\varphi_i}\right) \setminus \{0\} = \bigcup_{i=1}^{n} \sigma_e(C_{\psi_i,\varphi_i}) \setminus \{0\}.$$

Therefore,

$$\left(\{\psi_{i_0}(\zeta_{i_0})e^{-\beta t_{i_0}} :\ 0 \leq \beta < \infty\}\right) \setminus \{0\} \subseteq \sigma_e\left(\sum_{i=1}^{n} C_{\psi_i,\varphi_i}\right) \setminus \{0\}.$$

By the similar method used in the proof of Theorem 2.3, we conclude that

$$\mathrm{co}\left(\{\psi_{i_0}(\zeta_{i_0})e^{-\beta t_{i_0}} :\ 0 \leq \beta < \infty\}\right)$$

is the subset of the closure of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$. Since

$$\{\psi_{i_0}(\zeta_{i_0})e^{-\beta t_{i_0}} :\ 0 \leq \beta < \infty\}$$

is a logarithmic spiral (see [4, p. 303]), then 0 is an interior point of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$. □

In the following theorem, we assume that for some i, the function φ_i does not make contact with $\partial\mathbb{D}$ and we prove that zero is an interior point of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$.

Theorem 2.5. *Let* $\varphi_1, ..., \varphi_n$ *be linear fractional non-automorphisms. Let* m *be an integer with* $m < n$. *Suppose that for* $1 \leq i \leq m$, *the function* φ_i *does not make contact with* $\partial\mathbb{D}$ *and for* $m < i \leq n$, φ_i *has a fixed point at* $\zeta_i \in \partial\mathbb{D}$. *Assume that for* $i \neq j$ *that* $m < i, j \leq n$, $\zeta_i \neq \zeta_j$. *If there exists* i_0 *with* $m < i_0 \leq n$, *such that* ψ_{i_0} *is continuous at* ζ_{i_0}, $\psi_{i_0}(\zeta_{i_0}) \neq 0$ *and* φ_{i_0} *is not positive parabolic, then* 0 *is in the interior of* $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$.

Proof. We obtain from [4, p. 129] that for each $1 \leq i \leq m$, the operator C_{ψ_i,φ_i} is compact. Therefore,

$$\sigma_e\left(\sum_{i=1}^{n} C_{\psi_i,\varphi_i}\right) = \sigma_e\left(\sum_{i=m+1}^{n} C_{\psi_i,\varphi_i}\right).$$

Again we can easily see that for $i \neq j$ that $m < i, j \leq n$, the operator $C_{\psi_i,\varphi_i} C_{\psi_j,\varphi_i}$ is compact. Then by the similar argument stated in the proof of the previous two theorems, we have $\mathrm{co}(\sigma_e(C_{\psi_{i_0},\varphi_{i_0}}))$ is the closure of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$. From [5, Theorem 2.8, Theorem 2.9 and Theorem 2.11] and the fact that $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$ is convex, we easily conclude that zero is an interior point of $W(\sum_{i=1}^{n} C_{\psi_i,\varphi_i})$. □

In the following, some examples for the previous theorems are provided.

Example 2.6. (a) Let $\varphi_1(z) = \frac{-iz}{2+z}$, $\varphi_2(z) = \frac{z}{2+z}$ and $\varphi_3(z) = \frac{z}{2-z}$. We can see that $\varphi_1(-1) = i$, $\varphi_2(-1) = -1$ and $\varphi_3(1) = 1$. Note that φ_2 and φ_3 are not parabolic. Suppose that $\psi_1(z) = \frac{1}{3+z}$, $\psi_2(z) = \frac{z}{2z+1}$ and $\psi_3(z) = \frac{1}{2}z$. Theorem 2.3 implies that $\{z : |z| < 1\}$ is contained in $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2} + C_{\psi_3,\varphi_3})$.

(b) Let $\varphi_1(z) = \frac{i-z}{2i}$, $\varphi_2(z) = \frac{-z}{2+z}$ and $\varphi_3(z) = \frac{(1-i)z+(i-1)}{3+i-(i-1)z}$. We can easily see that $\varphi_1(-i) = 1$, $\varphi_2(-1) = 1$ and $\varphi_3(i) = i$. Put $\psi_1(z) = \frac{z^3}{2z+i}$, $\psi_2(z) = (2z^2+1)e^z$ and $\psi_3(z) = (z^4+1)e^{-z}$. We infer from Theorem 2.4 that $\mathrm{co}\left(\{2e^{-i}e^{-\beta(1+i)} : \beta \geq 0\}\right)$ is the subset of the closure of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2} + C_{\psi_3,\varphi_3})$ and zero is the interior point of this numerical range.

(c) Assume that $\varphi_1(z) = \frac{z+i}{5}$, $\varphi_2(z) = \frac{(1-i)z+(i-1)}{3+i-(i-1)z}$ and $\varphi_3(z) = \frac{i+z}{2}$. It is obvious that φ_2 and φ_3 are not positive parabolic and $\varphi_2(1) = 1$ and $\varphi_3(i) = i$. Define $\psi_1(z) = z^3$, $\psi_2(z) = 2e^{-iz}$ and $\psi_3(z) = \frac{1}{2z+i}$. We obtain from Theorem 2.5, that zero lies in the interior of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2} + C_{\psi_3,\varphi_3})$.

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