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Three weak solutions for a class of mixed boundary $(p,q)-$Laplacian systems

T. Norouzi Ghara [1] and G.A. Afrouzi [2]
[1] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
[2] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.

Abstract. In the present work, we investigate the existence of three weak solutions for a class of mixed boundary $(p,q)-$Laplacian systems. Our approach is variational method and main tool is the Theorem 1.1 due to Bonanno.

Keywords: Boundary value problem, Three weak solutions, Laplacian systems.

2000 Mathematics subject classification: 35J45, 35J50, 35J60, 35J65.

1. Introduction

The aim of this article is to show the existence of three weak solutions for the following problem

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda F_u(x,u,v) & \text{in } (0,1),\\ -(|v'|^{q-2}v')' = \lambda F_v(x,u,v) & \text{in } (0,1),\\ u(0)=u'(1)=v(0)=v'(1)=0. \end{cases} \tag{1.1}$$

[1]Corresponding author: tahere.noruzi5@gmail.com
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where $q \geq p > 1$ and λ is a real positive parameter.

$$F : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

is a function such that $F(.,s,t)$ is continuous on $[0,1]$ for all $(s,t) \in \mathbb{R}^2$ and $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for every $x \in [0,1]$, and F_u, F_v denote the partial derivatives of F with respect to the second and third variable. Moreover, $F(.,s,t)$ satisfies the condition

$$\sup_{|s|\leq\theta,|t|\leq\theta} (|F_u(.,s,t)| + |F_v(.,s,t)|) \in L^1([0,1]), \text{ for all } \theta > 0. \tag{1.2}$$

In recent years, the study of problems involving mixed boundary $(p,q)-$Laplacian systems has been widely approached. For example, in [5] the authors established the existence of a nontrivial solution for the system (1.1). In [4] using variational methods and critical point theory, the existence of at least one positive solution for the following problem was discussed

$$\begin{cases} -\Delta_p u = \lambda[g(x)a(u) + f(v)] & \text{in } \Omega, \\ -\Delta_q v = \lambda[g(x)b(v) + h(u)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, Shivaji and Son in [6] established the existence of three positive solutions for the following problem

$$\begin{cases} -\Delta_p u = \lambda[u^{p-1-\alpha} + f(v)] & \text{in } \Omega, \\ -\Delta_q v = \lambda[v^{q-1-\beta} + g(u)] & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega. \end{cases}$$

In the last decade or so, many authors applied variational methods to study the existence of three solutions for elliptic problem; see, for example, [1, 2] and the references therein.
The aim of this article is using variational method to prove the existence of three weak solutions for the problem (1.1). Our approach is the variational method and main tool is Theorem 1.1 due to Bonanno that we recall here.

Theorem 1.1. *(*[3, Theorem 7.1]*) Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals which Φ is bounded from below. Assume that there is $r \in]inf_X \Phi, sup_X \Psi[$ such that*

$$\varphi(r) < \rho(r),$$

where

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)}, \tag{1.3}$$

and

$$\rho(r) := \sup_{v\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{\Phi(v) - r}. \tag{1.4}$$

and for each $\lambda \in]\frac{1}{\rho(r)}, \frac{1}{\varphi(r)}[$ *the function* $I_\lambda = \Phi - \lambda\Psi$ *is bounded from below and satisfies (PS)-condition.*
Then, for each $\lambda \in]\frac{1}{\rho(r)}, \frac{1}{\varphi(r)}[$ *the function* I_λ *admits at least three critical points.*

Remark 1.2. ([3]) If we assume that $\Phi(0) = \Psi(0) = 0$ and there are $r > 0$ and $\overline{u} \in X$, with $\Phi(\overline{u}) > r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})},$$

then one has $\varphi(r) < \rho(r)$ and, in addition,

$$]\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}[.$$

Proposition 1.3. *(*[3, Proposition 2.2]*) Let* X *be a reflexive real Banach space;* $\Phi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on* X^**, and* $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that the function* $\Phi - \Psi$ *is coercive.*
Then, for all $r_1, r_2 \in [-\infty, +\infty]$*, with* $r_1 < r_2$*, the function* $\Phi - \Psi$ *satisfies the* ${}^{[r_1]}(PS)^{[r_2]}$*-condition.*

The plan of this article is as follows. In section 2 we recall some information that we need. In section 3 we mention our main result.

2. PRELIMINARIES

Let X denote the Cartesian product of two Sobolev spaces

$$X_1 = \{u \in W^{1,p}([0,1]); u(0) = 0\},$$

and

$$X_2 = \{v \in W^{1,q}([0,1]); v(0) = 0\}.$$

The space X will be endowed with the norm

$$\|(u,v)\| = \|u\|_p + \|v\|_q,$$

where

$$\|u\|_p = (\int_0^1 |u'(x)|^p dx)^{\frac{1}{p}},$$

and

$$\|v\|_q = (\int_0^1 |v'(x)|^q dx)^{\frac{1}{q}}.$$

Since $p > 1$ and $q > 1$, the Rellich−Kondrachov theorem ensures that $(X_1, \| . \|_p) \hookrightarrow (C^0([0,1]), \| . \|_\infty)$ and $(X_2, \| . \|_q) \hookrightarrow (C^0([0,1]), \| . \|_\infty)$ are compact, therefore for all $(u, v) \in X$ we have [7, 8]

$$\|u\|_\infty < \|u\|_p, \qquad \|v\|_\infty < \|v\|_q. \tag{2.1}$$

Definition 2.1. We say that $(u, v) \in X$ is a weak solution of problem (1.1) if

$$\int_0^1 |u'|^{p-2}u'\varphi'dx+\int_0^1 |v'|^{q-2}v'\psi'dx-\lambda\int_0^1 F_u(x,u,v)\varphi dx-\lambda\int_0^1 F_v(x,u,v)\psi dx = 0,$$

for all $(\varphi, \psi) \in X$.

We see that weak solutions of system (1.1) are critical points of the functional $I_\lambda : X \longrightarrow \mathbb{R}$, given by

$$I_\lambda(u, v) = \Phi(u, v) - \lambda\Psi(u, v),$$

for all $(u, v) \in X$, where

$$\Phi(u, v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q \tag{2.2}$$

and

$$\Psi(u, v) = \int_0^1 F(x, u, v)dx. \tag{2.3}$$

Since X is compactly embedded in $C^0([0,1]) \times C^0([0,1])$, it is well known that Φ and Ψ are well defined and Gâteaux differentiable functionals whose Gâteaux derivatives at $(u, v) \in X$ are given by

$$< \Phi'(u, v), (\varphi, \psi) >= \int_0^1 |u'|^{p-2}u'\varphi'dx + \int_0^1 |v'|^{q-2}v'\psi'dx,$$

$$< \Psi'(u, v), (\varphi, \psi) >= \int_0^1 F_u(x, u, v)\varphi dx + \int_0^1 F_v(x, u, v)\psi dx,$$

for all $(\varphi, \psi) \in X$. Moreover, by the weakly lower semicontinuity of norm, we see that Φ is sequentially weak lower semicontinuous. Thanks to $p, q > 1$ and condition (1.2), Ψ has a compact derivative, it follows that Ψ is sequentially weakly continuous.
Now, put

$$Q(r) = \{\theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \ ; \ \frac{|\theta_1|^p}{p} + \frac{|\theta_2|^q}{q} \leq r\},$$

and

$$\beta^* = \max\{\frac{2^{p-1}}{p}, \frac{2^{q-1}}{q}\}, \qquad \beta_* = \min\{\frac{2^{p-1}}{p}, \frac{2^{q-1}}{q}\}.$$

3. Main results

Now, we state main result to find three weak solution for the problem (1.1).

Theorem 3.1. *Assume that there exist constants c, δ with $\frac{c}{\beta_*} < (\delta^p + \delta^q)$ and a function $\eta \in L^1([0,1])$ such that*

- (a_1) $\int_0^{\frac{1}{2}} F(x,s,t) \geq 0 \qquad \forall\ (x,s,t) \in [0,1] \times [0,\delta] \times [0,\delta]$.
- (a_2) $F(x,s,t) \leq \eta(x)(1 + |s|^m + |t|^n)$, *where $m < p$, $n < q$ for almost every $x \in [0,1]$ and for every $(s,t) \in \mathbb{R}^2$.*
- (a_3)

$$\frac{\int_0^1 \max_{(s,t)\in Q(c)} F(x,s,t)dx}{c} < \frac{\int_{\frac{1}{2}}^1 F(x,\delta,\delta)dx}{\frac{2^{p-1}\delta^p}{p} + \frac{2^{q-1}\delta^q}{q}}.$$

Then, for each parameter λ belonging to

$$\left]\frac{\frac{2^{p-1}\delta^p}{p} + \frac{2^{q-1}\delta^q}{q}}{\int_{\frac{1}{2}}^1 F(x,\delta,\delta)dx}, \frac{c}{\int_0^1 \max_{(s,t)\in Q(c)} F(x,s,t)dx}\right[, \tag{3.1}$$

the problem (1.1) *possesses at least three distinct weak solutions in X.*

Proof. Our aim is to apply Theorem 1.1 to our problem. To this end, let Φ, Ψ be the functionals defined by (2.2), (2.3) respectively. Then $\Psi' : X \to X^*$ is a compact operator. On the other hand the fact that X is compactly embedded into $C^0([0,1])$ implies that the operator $\Psi' : X \to X^*$ is compact. Furthermore, Φ is bounded from below.
Set $r = c$ and define the function $\overline{u} \in X$ by putting

$$\overline{u}(x) := \begin{cases} 2\delta x & x \in [0, \frac{1}{2}[, \\ \delta & x \in [\frac{1}{2}, 1]. \end{cases}$$

clearly $(\overline{u}, \overline{u}) \in X$.
From $\frac{c}{\beta_*} < (\delta^p + \delta^q)$, and

$$\beta_*(\delta^p + \delta^q) \leq \Phi(\overline{u}, \overline{u}) \leq \beta^*(\delta^p + \delta^q), \tag{3.2}$$

one has

$$\Phi(\overline{u}, \overline{u}) > r > 0.$$

From (a_1), we have

$$\Psi(\overline{u},\overline{u}) = \int_0^1 F(x,\overline{u},\overline{u})dx \geq \int_{\frac{1}{2}}^1 F(x,\delta,\delta)dx. \qquad (3.3)$$

Therefore, one has

$$\frac{\Psi(\overline{u},\overline{u})}{\Phi(\overline{u},\overline{u})} \geq \frac{\int_{\frac{1}{2}}^1 F(x,\delta,\delta)dx}{\frac{2^{p-1}\delta^p}{p} + \frac{2^{q-1}\delta^q}{q}}. \qquad (3.4)$$

Here and in the sequel we have $\Phi(0,0) = \Psi(0,0) = 0$ and $\Phi(u,v) \geqslant 0$ for every $(u,v) \in X$. For all $(u,v) \in X$ such that $(u,v) \in \Phi^{-1}(]-\infty, r[)$ from (2.1) we have $\frac{|u|^p}{p} + \frac{|v|^q}{q} \leq c$ therefore

$$\Psi(u,v) = \int_0^1 F(x,u,v)dx \leq \int_0^1 \max_{(s,t)\in Q(c)} F(x,s,t)dx,$$

and

$$\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v) \leq \int_0^1 \max_{(s,t)\in Q(c)} F(x,s,t)dx. \qquad (3.5)$$

So, from (a_3)

$$\frac{\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v)}{c} \leq \frac{\Psi(\overline{u},\overline{u})}{\Phi(\overline{u},\overline{u})}.$$

Now, we prove that the functional $\Phi - \lambda\Psi$ is coercive, for each $(u,v) \in X$, by using (a_2) one has

$$\Phi(u,v) - \lambda\Psi(u,v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \lambda\int_0^1 F(x,u,v)dx \geq$$

$$\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \lambda\int_0^1 \eta(x)(1+|s|^m+|t|^n)dx,$$

we have

$$\int_0^1 \eta(x)(1+|u|^m+|v|^n)dx \leq (1+\|u\|_\infty^m + \|v\|_\infty^n)\int_0^1 |\eta(x)|dx \leq$$

$$(1+\|u\|_p^m + \|v\|_q^n)\|\eta\|_{L^1([0,1])},$$

so

$$\Phi(u,v) - \lambda\Psi(u,v) \geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \lambda(1+\|u\|_p^m + \|v\|_q^n)\|\eta\|_{L^1([0,1])},$$

hence

$$\lim_{\|(u,v)\|\to+\infty} \Phi(u,v) - \lambda\Psi(u,v) = +\infty.$$

So the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below because it is coercive and weakly sequentially lower semicontinuous. Thus, the functionals Φ, Ψ satisfy all regularity assumptions requested in Theorem 1.1, (for (PS)-condition we apply Proposition 1.3).
Thus, all necessary conditions are verified. Since all the assumptions of Theorem 1.1 and Remark 1.2 are satisfied, then, for each

$$\lambda \in]\frac{\frac{2^{p-1}\delta^p}{p} + \frac{2^{q-1}\delta^q}{q}}{\int_{\frac{1}{2}}^1 F(x,\delta,\delta)dx}, \frac{c}{\int_0^1 \max_{(s,t)\in Q(c)} F(x,s,t)dx}[,$$

the functional I_λ has at least three distinct critical points that are weak solutions of the problem(1.1). The proof is complete. □

Corollary 3.2. *Assume that in system (1.1) the function F does not depend on $x \in [0,1]$. Then, from the theorem 3.1, for each*

$$\lambda \in]\frac{\frac{2^p\delta^p}{p} + \frac{2^q\delta^q}{q}}{F(\delta,\delta)}, \frac{c}{F(c,c)}[,$$

the following system admits three weak solutions

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda F_u(u,v) & in \;\; (0,1), \\ -(|v'|^{q-2}v')' = \lambda F_v(u,v) & in \;\; (0,1), \\ u(0) = u'(1) = v(0) = v'(1) = 0. \end{cases}$$

Example 3.3. Consider the following problem

$$\begin{cases} -(|u'|^3u')' = \lambda F_u(u,v) & \text{in} \;\; (0,1), \\ -(|v'|^4v')' = \lambda F_v(u,v) & \text{in} \;\; (0,1), \\ u(0) = u'(1) = v(0) = v'(1) = 0, \end{cases} \tag{3.6}$$

if we chosse $p = 5$, $q = 6$ and

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$F(u,v) = \begin{cases} u^2v^2(sin\ logu + 2)(sin\ logv + 2) & \text{if} \;\; u > 0, v > 0, \\ 0 \quad \text{otherwise}, \end{cases} \tag{3.7}$$

we have

$$F_u(u,v) = \begin{cases} 2uv^2(2sin\ logu + 4 + cos\ logu)(sin\ logv + 2) & \text{if} \;\; u > 0, v > 0, \\ 0 \quad \text{otherwise}, \end{cases} \tag{3.8}$$

$$F_v(u,v) = \begin{cases} 2u^2v(sin\ logu + 2)(cos\ logv + 2sin\ logu + 4) & \text{if} \;\; u > 0, v > 0, \\ 0 \quad \text{otherwise}. \end{cases} \tag{3.9}$$

For example, if we chosse $c = e^{-\frac{5}{2}\pi}$ and $\delta = e^{-\frac{\pi}{2}}$, then for each $\lambda \in (21.3e^{-\frac{\pi}{2}}, e^{7.5\pi})$ the system (3.6) admits at least three solutions.

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