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Geometrization of Heat Flow on Volumetrically Isothermal Manifolds via the Ricci Flow

S. Kumar [1]
[1] Department of Mathematics
Govt. P.G. Degree College, New Tehri, Tehri Garhwal, Uttarakhand, Zip-code: 249 001, India

Abstract. The present article serves the purpose of pursuing Geometrization of heat flow on volumetrically isothermal manifold by means of RF approach. In this article, we have analyzed the evolution of heat equation in a 3-dimensional smooth isothermal manifold bearing characteristics of Riemannian manifold and fundamental properties of thermodynamic systems. By making use of the notions of various curvatures, we have discussed different types of heat diffusion equation for our volumetrically isothermal manifold and its isothermal surfaces. Finally, we have delineated a heat diffusion model for such isothermal manifold and by decomposing it into isothermal surfaces we have developed equation for heat diffusion.

Keywords: Isothermal; Volumetric manifold; Ricci Flow (RF); Heat diffusion; Laplace-Beltrami; Riemannian

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1. Introduction

Ricci flow (RF), named in the admiration of Gregorio Ricci Curbastor has been having a large influence in the world of multi scale differential

[1] Corresponding author:sandeep_2297@rediffmail.com; drsandeepkumar-math@hotmail.com
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geometry and topology. It is often delineated as heat diffusion equation. In 1982 Sir Hamilton has precisely introduced the concept of RF to study compact three-manifolds with positive Ricci curvature and he showed that RF evolves the Riemannian metric by its Ricci curvature as the heat diffusion equation for metrics. The RF specially became popular in the study of brain imaging and diffusion tensor MRI, wherein it has been used as a fundamental tool to solve the problems concerning abnormal changes in brain cortical surfaces checking discreteness between different regions of brain and solving problems in shape analysis. Also RF provides a best method to find a best metric on a manifold by means of natural evolution expression described by a vector filed on the manifold of metrics.

In recent time, the term Ricci flow (here and hereafter will be pronounced as RF) becomes admirable due to the fact that it has been implemented for the demonstration of *Poincaré conjecture* on three dimensional manifolds [6], [3] & [2]. Richard Hamilton, 1982, was the pioneer, who at the very first introduced the RF for Riemannian manifolds of any dimension in his decisive work [7], [5] & [4]. Naturally, a RF on a surface (i.e., a manifold of dimension 2) is the procedure to deform the Riemannian metric of the surface, provided the Ricci curvature tensor filed of such surface does not vanish. Moreover, such deformation is proportional to the Gaussian curvatures of the underlying surface and such that the curvature develops itself like the heat diffusion. Ricci flow has been considered as a powerful tool for computing the conformal Riemannian metrics with prearranged Gaussian curvature. Furthermore, for many engineering applications, it is also frequently desirable to compute Riemannian metrics on the surfaces with prearranged Gaussian curvatures, such as parameterization of graphics, spline construction in geometric modeling, conformal brain mapping in medical image analysis and so on. The mechanism of RF could be well comprehended with the following general way: Let us imagine a room where a heater is placed at one of its corner and a window is opened at the other corner. Now, by heat transfer law, heat will be diffused (via heat bearer molecules present in the room climate) from the room and exhausted out through the window. The process of heat diffusion will be continued till the temperature difference exists between room and its surrounding, i.e., the diffusion of heat ceases when temperature of the room becomes same everywhere. With RF, the same occurs in curvature. Under RF, a geometric object that is distorted and rough will be morphed and changed as necessary so that all curvatures become smooth (i.e., even). The greatest consequence of RF came when Grigori Poincaré in 2002 ([7], [5]) applied the RF to pursue Poincaré conjecture. Additionally, the RF relation to the heat equation may pave the way for research in fluid dynamics, thermal Physics and even in the Einstein's well known theory of relativity [9].

2. The Ricci Flow (RF)

Ricci flow is a mean by which one can take an arbitrary Riemannian manifold and smooth out the geometry of that manifold to make it look more systematic and symmetric. It has been proven to be very useful tool in delineating the topology and geometric structure of such manifold. Thus RF can be described for Riemannian manifolds of any dimension, but for the sake of convenience, here we restrict ourselves to three dimensional manifolds, so that the geometric exposition of RF in such manifold can be well analyzed. One can also describe RF informally as the process of stretching the metric g in the direction of negative Ricci curvature and contracting the same in the direction of positive Ricci curvature. The stronger the curvature, the faster the contracting or stretching the metric and thus we can say that RF is a process which when applied to some manifold, increases or decreases the intimated metric i.e., it increases or decreases the distance between points of the manifold along negative or positive Ricci's curvatures direction. By altering the notion of distance, one can also affects the notions of angle and volume (even though it turns out in 2-dimensions that RF is conformal, which means that the notion of angle and its sense remains unaffected by the flow; this fact is closely related to the reason that in 1-dimensional manifold the Ricci curvature is same in all directions) as RF refers to conformally distort (stretch or contract) the Riemannian metric of a manifold by its Gaussian curvature such that the Gaussian curvature evolves according to heat diffusion process.

2.1. **Theoretical Background of Ricci Flow Equation.**

Concisely and lucidly, the RF can be delineated by the equation:

$$\frac{d}{dt}g = -2\,\mathrm{Ric} \equiv \partial_t g_{ij} = -2R_{ij}. \tag{2.1}$$

In principle, one can perform RF on a manifold for as long a time period as one wishes. Further, it is possible (particularly in the presence of positive curvature) for the RF to cause a manifold to generate singularities (points where one ceases to look like a manifold or those points on which manifold does cease itself to behave like a standard manifold). For instance, if one starts with a perfect ellipsoid with positive Ricci curvature and performs RF, what will happen? The ellipsoid will shrink or contract at a steady rate till it becomes an oval shaped point beyond which RF can no more be performed, that is to say the ellipsoid (the 3-maifold) will be no longer a three dimensional manifold. Moreover, in three dimensional manifold, many complicated singularities are possible; for example one can have a neck-pinch, in which a cylinder like 'neck' of the manifold shrinks under RF until at one or more places along the neck, the cylinder has narrowed down up to a point. Various kinds of possible singularity formations for three dimensional manifolds via RF have been classified by Poincaré [4].

During some previous years, Hamilton prepared the elementary annotations on RF and shown that RF is an outstanding device for simplifying the geometric as well as topological structures of the manifold under consideration and generally speaking, RF is able to compress all the positive curvature parts (most often including Gaussian and Ricci curvatures) of the underlying manifold into emptiness until they become quite homogeneous i.e., by implementing RF on the manifold, it begins to look much the same, it doesn't matter which one vantage point in the manifold one selects. Of course, the flow seems to isolate manifold into tremendously symmetric components. Here is an example of two dimensional manifold in which the RF always ends up intimating the manifold with a metric of constant curvature, which would be positive (as in the ellipse or sphere etc.), and the same becomes zero as in case of cylinder, or negative as in case of hyperbolic manifolds. Thus the fact that such a constant curvature with RF induced metric can always found is called the *"Uniformisation Theorem"* and has a vital significance in the theory of manifolds. It is also of great interest that in case of higher dimensional manifolds, the RF can most probably build up singularities before attaining perfect symmetry, but it is also possible to remove singularities by performing surgeries on the singularities so that the manifold could again turned into smooth form and one could again restart the RF process. It is also noteworthy that surgeries can however deform the topology of the underlying manifold, for instance they might convert an arbitrary connected manifold into two disconnected varieties.

2.2. **Volumetric Isothermal Manifolds.** In order to target our aim of pursuing Geometrization of heat flow on volumetrically isothermal manifold via RF, let us first concisely introduce some major concepts of differential geometry and their impacts over isothermal volumetrically manifold.

The core of both differential geometry and modern geometrical dynamics represents the concept of manifold. A manifold is an abstract mathematical space, which locally (i.e., in a closeup view) resembles the spaces described by Euclidean geometry, but which globally (i.e., when viewed as a whole) may have a more complicated structure.
For example, the surface of Earth is a manifold; locally it seems to be flat, but viewed as a whole from the outer space (globally) it is actually

round. Moreover, a manifold can be constructed by gluing separate Euclidean spaces together; for example, a world map can be made by gluing many maps of local regions together, and accounting for the resulting distortions. Another example of a manifold is a circle. A small piece of a circle appears to be like a slightlybent part of a straight line segment, but overall the circle and the segment are different 1-dmensional manifolds. A circle can be formed by bending a straight line segment and gluing the ends together. The surfaces of a sphere and a torus are examples of 2-dimensional manifolds. Thereby, manifolds are important objects in Mathematics, Physics and control theory, because they allow more complicated structures to be expressed and understood in terms of the wellunderstood properties of simpler Euclidean spaces.

Now as far as the present study concerns with the Geometrization of heat flow on volumetrically isothermal manifolds using RF techniques, we fabricate such underlying manifold with the help of thermal Physics as well as Differential Geometry as follows:
Here, we seek for an arbitrary closed and smooth continuum of whatever shape filled with some gas, having its boundaries made up of some conducting material (with some arbitrary conductivity $\mathbf{K}$), so that when it is becomes heated, the heat can diffuse and exchanged with its surrounding (however the choice of surrounding is quite arbitrary). In addition, we suppose that the temperature $\mathbf{T}$ of such closed continuum remains same due to isothermal heat conduction process exists between the continuum and its surrounding. In this way we create a hypothesis of isothermal manifold or continuum which is smooth and closed with the property that heat can be exchanged with its surrounding, so that the manifold always remains at constant temperature. The notion of volumetric manifolds and its specialization for the case of non-degenerate metric has also been introduced by [13]. However, we in our case, conceptually delineate the volumetrically isothermal manifold as an assemblage of gas molecules having the isothermal co-ordinates $(u_i, \forall i = \alpha, \beta, \gamma)$ such that there exists a metric g which remains invariant for each free path of molecule's Brownian motion due to their internal thermal energy. More precisely, the molecular motion in volumetric manifold bearing isothermal characteristics is assumed to be isometric.

Consequently, we capsulate our notion of volumetrically isothermal manifold primarily by introducing the basic assumption of a volume structure, which should be considered as independent from any metric. This is just a non-negative 3-form density ω and makes the manifold a volume manifold. Secondly, introducing the isothermal co-ordinates system say $(u_i, \forall\, i = 1, 2, 3)$ for each constituent molecules of the manifold. And finally, we would call for a metric structure g. However, it does not need to be friendly with the volume structure. Such incompatibility can be programmed by means of volume scalar ϕ by the relation:

$$\omega = \omega e^{-\phi}, \tag{2.2}$$

where $\omega = \iiint |\mathrm{Det}\ g|^{1/2} du_1 du_2 du_3$ is the Riemannian volume element density.
Furthermore, for the metric derivative along any component of isothermal co-ordinate vector say u_1, we have

$$\nabla_{u_1}\omega = u_1^j \nabla j\omega = u_1^j \nabla_j \omega e^{-\phi} = -(u_1.\partial\phi)\omega, \tag{2.3}$$

where the dot denotes inner product as a measure of incompatibility. Therefore, we define such a manifold endowed with an independent volume, isothermal co-ordinate system and a metric structure as a *volumetrically isothermal manifold* and denote it in abbreviated form as V_{iso}.

3. Heat Flow on Volumetrically Isothermal Manifolds via the Ricci Flow

In this section, we start our discussion regarding Geometrization of heat flow on volumetrically isothermal manifold via the RF device, and we explore some more complicated analysis regarding RF.

We suppose that V_{iso} is an arbitrary closed and smooth volumetrically isothermal 3-dimensional manifold which is orientable and V_{sur} is the arbitrary surrounding of this manifold. Now, to understand and classify the 3-dimensional closed smooth manifold V_{iso}, we consider its geometries corresponding to 3-dimensional Riemannian geometry by introducing a geometric structure on V_{iso}. Such a geometric structure is assumed as a complete and locally homogeneous Riemannian metric g. Moreover, in this way using [14], the V_{iso} manifold can be expressed as the quotient $\Gamma/G/H$, where G is the isometric group of the universal covering $(\tilde{V}_{iso}, g)$ (which in our case is an arbitrary surrounding V_{sur}) and Γ and H are discrete and compact subgroups of the Lie group G respectively. [14] also showed that in this way there would be eight such geometries G/H in 3-dimensional manifold which admit compact quotients.
As our manifold is volumetrically isothermal, we assume that it is composed of a finite number of arbitrarily closed isothermal curves (curves along which the temperature remains identical) along which heat flow exists and we then establish a geometric structure (i.e., a Riemannian metric tensor) for such curves. If g is such a geometric structure for an arbitrary isotherm curve $x = x(t)$ say, (where the parameter t is time parameter), then g will be the function of isothermal co-ordinates u_i, for which the metric g will be conformal to the Euclidean manifold, i.e.,

g will have the form:

$$g = e^{\phi}(u_i^2) \equiv e^{\phi}(u_1^2 + u_2^2 + u_3^2), \tag{3.1}$$

where ϕ is some function defining the volume scalar for the V_{iso} manifold. Besides this, if the aforementioned isothermal curve has the Riemannian curvature R_{ijkl}, then the evolution of metric (3.1) turned out as a non linear heat equation for the Riemann curvature under the RF (2.1) which is comprehended as below:

Theorem 3.1. *For an arbitrary isothermal curve $x = x(t)$, having the Riemannian curvature R_{ijkl}, the evolution of metric Eq. (3.1) turns out as a non linear hear equation of Riemann curvature under RF (2.1) and is given by*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = \nabla_g^2 R_{ijkl} + 2(D_{ijkl} - D_{ijlk} - D_{iljk} + D_{ikjl}) -\\ - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}),\end{aligned} \tag{3.2}$$

where $D_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and ∇_g^2 is the Laplacian with respect to the evolving metric (3.1) and is defined by $\nabla_g^2 \overset{def}{=} g^{ij}\nabla_i\nabla_j = g^{ij}\left(\frac{\partial^2}{\partial u_i \partial u_j} - \Gamma_{ij}^k \frac{\partial}{\partial u_k}\right)$.

The demonstration of the foregoing statement is as follows:

Proof. Let us choose $\{u_1, u_2, u_3\} \equiv \{u_m\}$ to be instantaneous isothermal co-ordinate system for a gas molecule at some specific instant t on its trajectory $x = x(t)$ in V_{iso} manifold. Then at such point we compute the RF for the well known Christoffel bracket as below:

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_{jl}^h &= \frac{1}{2}\left[\frac{\partial}{\partial u_j}(\frac{\partial}{\partial t}g_{lm}) + \frac{\partial}{\partial u_l}(\frac{\partial}{\partial t}g_{jm}) - \frac{\partial}{\partial u_m}(\frac{\partial}{\partial t}g_{jl})\right]\\ &= \frac{1}{2}g^{hm}\left[\nabla_j(-2R_{lm}) + \nabla_l(-2R_{jm}) - \nabla_m(-2R_{jl})\right].\end{aligned}$$

Moreover, computing RF for the mixed Riemannian curvature tensr as

$$\frac{\partial}{\partial t}R_{jkl}^h = \frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial t}\Gamma_{jl}^h\right) - \frac{\partial}{\partial u_j}\left(\frac{\partial}{\partial t}\Gamma_{il}^h\right),$$

while the RF for purely covariant Riemannian curvature tensor is computed by taking the inner product of g_{hk} with the above equation so that

$$\frac{\partial}{\partial t}R_{ijkl} = g_{hk}\frac{\partial}{\partial t}R_{ijl}^h + \frac{\partial g_{hk}}{\partial t}R_{ijl}^h = g_{hk}\left[\frac{\partial}{\partial u_i}(\frac{\partial}{\partial t}\Gamma_{jl}^h) - \frac{\partial}{\partial u_j}(\frac{\partial}{\partial t}\Gamma_{il}^h)\right] - 2R_{hk}R_{ijl}^h.$$

Combining the above three identities, we obtain

$$\frac{\partial}{\partial t}R_{ijkl} = g_{hk}[\{\frac{1}{2}\nabla_i[g^{hm}(\nabla_j(-2R_{lm}) + \nabla_l(-2R_{jm}) - \nabla_m(-2R_{jl}))]\}-$$
$$-\{\frac{1}{2}\nabla_j[g^{hm}(\nabla_i(-2R_{lm}) + \nabla_l(-2R_{im}) - \nabla_m(-2R_{il}))]\}] - 2R_{hk}R^h_{ijl}$$
$$= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} - R_{ijlp}g^{pq}R_{qk} - R_{ijkp}g^{pq}R_{ql} - 2R_{ijpl}g^{pq}Rqk$$
$$= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} - g^{pq}(R_{ijkp}R_{ql} + R_{ijpl}R_{qk}).$$

Here, we have used the fact that $R_{ijkl} = -R_{jikl}$.
Now we proceed to check for the following identity, which is almost analogous to Simon's identity in extrinsic differential geometry.

$$\nabla^2_g R_{ijkl} + 2(D_{ijkl} - D_{ijlk} - D_{iljk} + D_{ikjl}) = \nabla_i\nabla_k R_{jl} - \nabla_j\nabla_k R_{il} - \nabla_i\nabla_l R_{jk} + \nabla_j\nabla_l R_{ik} + $$
$$+ g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}). \quad (3.3)$$

Naturally, if we introduce the second Bianchi's identity, which is given by

$$\nabla_m R_{ijkl} + \nabla_i R_{jmkl} + \nabla_j R_{mikl} = 0, \quad (3.4)$$

we obtain

$$\nabla^2_g R_{ijkl} = g^{pq}\nabla_p\nabla_q R_{ijkl} = g^{pq}\nabla_p\nabla_i R_{qjkl} - g^{pq}\nabla_p\nabla_j R_{qikl}.$$

Using the *swap formula*

$$\begin{cases}\nabla_i\nabla_j T^l - \nabla_j\nabla_i T^l = R^l_{ikl}T^k \\ \nabla_i\nabla_j T_k - \nabla_j\nabla_i T_k = R_{ijkl}g^{lm}T_m\end{cases}, \quad (A)$$

and the second Bianchi's identity (3.4), we acquire,

$$g^{pq}\nabla_p\nabla_i R_{qjkl} - g^{pq}\nabla_i\nabla_p R_{qijl} = g^{pq}g^{mn}(R_{piqm}R_{nikl} + R_{pijm}R_{qnkl} + R_{pikm}R_{qnjl} + R_{pilm}R_{qjkn}) =$$
$$= R_{im}g^{mn}R_{njkl} + g^{pq}g^{mn}R_{pimj}(R_{qkln} + R_{qlnk}) + g_{pq}g^{mn}R_{pikm}R_{qjnl} + g^{pq}g^{mn}R_{pilm}R_{qjnk} =$$
$$= R_{im}g^{mn}R_{njkl} - D_{ijkl} + D_{ijlk} - D_{ikjl} + D_{iljk},$$

whereas using the contracted second Bianchi's identity which is

$$g^{pq}\nabla_p R_{qjkl} = \nabla_k R_{jl} - \nabla_l R_{jk}, \quad (3.5)$$

we have

$$g^{pq}\nabla_i\nabla_p R_{qjkl} = \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk}.$$

Thus we have

$$g^{pq}\nabla_p\nabla_i R_{qjkl} = \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - (D_{ijkl} - D_{ijlk} + D_{ikjl} - D_{iljk}) + R_{qi}g^{pq}R_{pjkl}.$$

Hence, in view of all the foregoing details, we obtain the desired relation as follows:

$$\nabla_g^2 R_{ijkl} = g^{pq}\nabla_p\nabla_i R_{qikl}$$
$$= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - (D_{ijkl} - D_{ijlk} + D_{ikjl} - D_{iljk}) + R_{qi}g^{pq}R_{pjkl}$$
$$-\nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} + (D_{ijkl} - D_{jilk} - D_{jlik} + D_{jkil}) - R_{qj}g^{pq}R_{pikl}$$
$$= \nabla_i\nabla_k R_{jl} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} + g^{pq}\left(R_{pjkl}R_{qi} + R_{ipkl}R_{qj}\right) -$$
$$-2\left(D_{ijkl} - D_{ijlk} + D_{ikjl} - D_{iljk}\right),$$

where in the last step, we have used the symmetric relation:

$$D_{ijkl} = D_{klij} = D_{jilk}.$$

□

Let us now check for the heat diffusion equation of gas molecule at some specific instant t while its trajectory in the V_{iso} manifold has some positive contracted curvature tensor (i.e., Ricci curvature tensor). As the Ricci tensor can be obtained by the contraction of Riemannian curvature tensor via two ways. One process leads the Riemannian tensor to be a zero tensor and the other lead to a well known Ricci tensor. Hence, it is evident that under the first process of contraction, equation (3.2) goes to be zero and thus the RF or the heat equation along any arbitrary isothermal curve vanishes identically over V_{iso} manifold. From the stand point of second process, we have the following fact:
begintheorem

Theorem 3.2. *Under the RF (2.1), an instantaneous gas molecule along an arbitrary isothermal trajectory $x = x(t)$ of a V_{iso} manifold having some positive Ricci curvature R_{ik}, will have the heat diffusion equation delineated as:*

$$\frac{\partial}{\partial t}R_{ik} = \nabla_g^2 R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}. \tag{3.6}$$

Proof. In order to demonstrate the validity of above expression, we let that $\{u_1, u_2, u_3\} \equiv \{u_m\}$ to be instantaneous isothermal co-ordinate system for a gas molecule at some specific instant t on its trajectory $x = x(t)$ in V_{iso} manifold. Now taking usual RF for the Ricci tensor R_{ik} as below:

$$\frac{\partial}{\partial t}R_{ik} = \frac{\partial}{\partial t}\left(g^{jl}R_{ijkl}\right) = \left(\frac{\partial}{\partial t}g^{jl}\right)R_{ijkl} + g^{jl}\frac{\partial}{\partial t}R_{ijkl}. \tag{3.7}$$

Now, in view of (3.2), we have

$$\frac{\partial}{\partial t}R_{ik} = g^{jl}[\nabla_g^2 R_{ijkl} + 2(D_{ijkl} - D_{ijlk} - D_{iljk} + D_{ikjl}) - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + \\ + R_{ijkp}R_{ql})] - g^{jp}\left(\frac{\partial}{\partial t}g_{pq}\right)g^{ql}R_{ijkl} \\ = \nabla_g^2 R_{ik} + 2g^{jl}(D_{ijkl} - 2D_{ijlk}) + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pk}R_{qi}.$$

We now assert that the term $g^{jl}(D_{ijkl} - 2D_{ijlk})$ involved in the last expression vanishes. Of course! As by using the first Bianchi identity which is;

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0, \tag{3.8}$$

we have

$$g^{jl}D_{ijkl} = g^{jl}g^{pr}g^{qs}R_{piqj}R_{rksl} = g^{jl}g^{pr}g^{qs}R_{pqij}R_{rskl} = g^{jl}g^{pr}g^{qs}(R_{piqj} - R_{pjqi})(R_{rksl} - R_{rlsk}) \\ = 2g^{jl}(D_{ijkl} - D_{ijlk}),$$

as expected. Therefore, we obtain the RF equation in the form:

$$\frac{\partial}{\partial t}R_{ik} = \nabla_g^2 R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}.$$

□

From the heat diffusion equation (3.6), there arises an special case, which is as follows:

Theorem 3.3. *Under the RF (2.1), an instantaneous gas molecule along arbitrary isothermal trajectory $x = x(t)$ of a V_{iso} manifold having some scalar curvature R, will have the evolution equation outlined as:*

$$\frac{\partial R}{\partial t} = \nabla_g^2 R + 2|Ric|^2. \tag{3.9}$$

Proof. To evolve the heat diffusion equation (3.9), let us take the RF for scalar curvature of arbitrary isothermal trajectory $x = x(t)$ of an instantaneous gas molecule as in usual way, we get

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial t}\left(g^{ik}R_{ik}\right) = g^{ik}\frac{\partial}{\partial t}R_{ik} + R_{ik}\left(\frac{\partial}{\partial t}g^{ik}\right) = g^{ik}\frac{\partial}{\partial t}R_{ik} + \left(-g^{ip}\frac{\partial g_{pq}}{\partial t}g^{qk}\right)R_{ik}.$$

Making use of (2.1) and (3.6), the last expression yields

$$\frac{\partial R}{\partial t} = g^{ik}(\nabla_g^2 R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}) + 2R_{pq}R_{ik}g^{ip}g^{qk} = \nabla_g^2 R + 2|\mathrm{Ric}|^2.$$

□

It is of great interest that all the above expressions are valid for any non-orthonormal frame of reference used for V_{iso} manifold. In case if one desires to obtain all the above heat diffusion equations in an orthonormal frame of reference, then one must consider such a frame of reference in the V_{iso} manifold such that the frame remains all time orthonormal. For, we assume an abstract vector bundle V over V_{iso} isomorphic to its tangent bundle $T_{V_{iso}}$ and define a local frame $P = \{P_1, P_2, ., P_a, .P_n\}$ on V, such that $P_a = P^i_a \frac{\partial}{\partial u_i}$ with the isomorphism $\{P^i_a\}$. Choosing $\{P^i_a\}$ at some specific time $t = 0$ so that this local frame is an orthonormal frame at some time $t = 0$. Now in view of these notions, we evolve the isomorphism $\{P^i_a\}$ by the equation;

$$\frac{\partial}{\partial t} P^i_a = g^{ij} R_{jk} P^k_a.$$

Under the above expression, the assumed frame will remain orthonormal for all time, since the pull back metric on V given by

$$H_{ab} = g_{ij} P^i_a P^j_b,$$

remains invariant for all time. Therefore, in orthonormal frame we have the following:

$$R_{abcd} = P^i_a P^j_b P^k_c P^l_d R_{ijkl},$$

$$\Gamma^a_{jb} = P^a_i \frac{\partial P^i_b}{\partial u_j} + \Gamma^i_{jk} P^a_i P^k_b,$$

where $P^a_i = \left(P^i_a\right)^{-1}$,

$$\nabla_i V^a = \frac{\partial}{\partial u_j} V^a + \Gamma^a_{ib} V^b,$$

and

$$\nabla_b V^a = P^i_b \nabla_i V^a,$$

where Γ^a_{jb} is the well known affine connection of the vector bundle V with the metric H_{ab}. Naturally, the covariant derivative for the isomorphic tensor P^i_b with respect to some isothermal co-ordinate say u_i produces

$$\nabla_i P^j_b = \frac{\partial P^j_b}{\partial u_i} + P^k_b \Gamma^j_{ik} - P^j_c \Gamma^c_{ib} = \frac{\partial P^j_b}{\partial u_i} + P^k_b \Gamma^j_{ik} - P^j_c \left(P^c_k \frac{\partial P^k_b}{\partial u_i} + \Gamma^l_{ik} P^c_l P^k_b \right) = 0.$$

Moreover, the covariant derivative of metric H_{ab} with respect to some isothermal co-ordinate u_i gives

$$\nabla_i H_{ab} = \nabla_i \left(g_{ij} P^i_a P^j_b \right) = 0.$$

So for a covariant vector V_b in the isothermal manifold V_{iso}, we have

$$\nabla_a V_b = P^i_a P^j_b \nabla_i V^j,$$

and the Laplacian for the covariant Riemannian curvature is

$$\nabla_g^2 R_{abcd} = \nabla_l \nabla_l R_{abcd} = g^{ij} \nabla_i \nabla_j R_{abcd} = g^{ij} P_a^k P_b^l P_c^m P_d^n \nabla_i \nabla_j R_{klmn}.$$

Thus from all the foregoing data, in an orthonormal frame of reference $P = \{P_1, P_2, \cdots P_a, \cdots P_n\}$, we have the following reaction diffusion equations evolved for different kinds of curvatures under usual RF (2.1):

$$\frac{\partial R_{abcd}}{\partial t} = \nabla_g^2 R_{abcd} + 2(D_{abcd} - D_{abdc} - D_{adbc} + D_{acbd}). \tag{3.10}$$

$$\frac{\partial R_{ab}}{\partial t} = \nabla_g^2 R_{ab} + 2 R_{abcd} R_{cd}. \tag{3.11}$$

$$\frac{\partial R}{\partial t} = \nabla_g^2 R + 2|\mathrm{Ric}|^2, \tag{3.12}$$

where $D_{abcd} = R_{aebf} R_{cedf}$.

Here, equation (3.10) will be called isothermal reaction diffusion equation evolved under RF in an orthonormal frame of V_{iso}. While the equation (3.11) will be called Ricci reaction diffusion equation and the equation (3.12) will be known as invariant heat diffusion equation.

In the foregoing analysis, we have supposed that the volumetrically isothermal manifold is composed of a finite number of arbitrarily closed isothermal curves along which heat flow exists. To make the study of RF more interesting, let us now make use of some fabulous decomposition techniques like; sphere or prime decomposition and torus decomposition [10] and assume that under such decomposition techniques our arbitrary closed manifold V_{iso} can be decomposed or split into discrete pieces according to the structure of simplest surfaces embedded in V_{iso}, namely spheres and tori. From Topological stand point, the decomposition procedures for 3-manifolds can be classically accomplished as follows:

Sphere of Prime Decomposition. In this technique, if our V_{iso} manifold is closed and orientable then V_{iso} admits a finite connected sum splits

$$V_{iso} = (\pi_1 \# \pi_2 \# \cdots \# \pi_p) \, \# \, (\tau_1 \# \tau_2 \# \cdots \# \tau_q) \, \# \left(\# S^2 \times S^1 \right), \tag{3.13}$$

where π and τ factors are closed irreducible isothermal 3-manifolds, i.e., every embedded 2-sphere S^2 enclosed a 3-ball. The π factors have infinite fundamental sub-manifold and are isothermal spherical 3-manifolds while the τ factors have finite fundamental sub-manifold and have universal cover or surrounding $(\tilde{V}_{iso}, g) = V_{sur}$ which in this case is an isothermal 3-sphere.

Torus split technique. In this procedure, if the V_{iso} manifold is closed, irreducible, then there exists a finite collection, probably vacant, of disjoint incompressible tori in V_{iso} that separate V_{iso} into a finite collection of compact isothermal 3-manifolds (having toral boundary) each of which is torus irreducible or Seifert fibered.

With the help of these decomposition methods, we now pursue for RF on surfaces (i.e., 2-manifolds) of V_{iso} manifold with the assumption that such surfaces are isothermal Riemannian surfaces having some suitable complex structure. We also discuss Gaussian and harmonic mappings between such surface and V_{iso} manifold.

Suppose, S_{iso} is one of the embedded in V_{iso} and $r =: S_{iso} \to V_{iso}$ is a function with at least C^2 continuity. Let $r_i = \frac{\partial r}{\partial u_i}, \forall i = 1, 2$ be the tangent vectors along the isothermal curves. Now, if $r_1 \times r_2 \neq 0$, then the mapping r defines a regular surface. The normal for this surface is defined as $n = \frac{r_1 \times r_2}{|r_1 \times r_2|}$. Moreover, if our surface according to the sphere decomposition behaves like a sphere, we define a Gaussian mapping $G : r(u_1, u_2) \to n(u_1, u_2) \in S^2_{iso}$, which maps the surface to the unit sphere S^2_{iso}. Also, the length of a general tangent vector $dr = r_1 du_1 + r_2 du_2$ can be computed as

$$ds^2 = \langle dr, dr \rangle = \begin{pmatrix} du_1 & du_2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \end{pmatrix},$$

where $\langle \, , \rangle$ is the inner product in V_{iso}, and $g_{ij} = \langle r_i, r_j \rangle$. Hence the matrix $g = (g_{ij})$ is the well known Riemannian metric tensor which defines the inner product $\langle \, , \rangle_g$ on the tangent planes of the isothermal surface. Since all Riemannian metrics and surfaces can be classified by the conformal equivalence relation then each conformal equivalent class would be called a conformal structure. Also, any Riemannian metric on the Isothermal surface is conformally equivalent to Euclidean flat metric and such flat metric can be developed by choosing a special parameters (in our case isothermal co-ordinates) such that any alteration in parameterization doesnt affect the metric. This metric for isothermal surface can be represented by

$$ds^2(u_i) = e^{2\lambda(u_i)}(du_i^2) \forall i = 1, 2 \; i.e, \; ds^2(u_1, u_2) = e^{2\lambda(u_1, u_2)}(du_1^2 + du_2^2), \tag{3.14}$$

where λ is some constant depending upon time t. Under the isothermal co-ordinates given by (3.14), the Gaussian curvature K of a S_{iso} patch say $\sigma(u_1, u_2)$ is given by the relation:

$$K = -e^{-2\lambda} \nabla_g^2 \lambda, \tag{3.15}$$

where $\nabla_g^2 \equiv \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}$ is the Laplacian operator in case of isothermal surface and the term $e^{-2\lambda} \nabla_g^2$ is the Laplace-Beltrami operator, which in the V_{iso} manifold will be defined by

$$\nabla_{LB} = e^{-2\lambda}\nabla_g^2 = \frac{1}{|g|^{1/2}}\frac{\partial}{\partial u_i}\left(|g|^{1/2}g^{ij}\frac{\partial}{\partial u_j}\right).$$

The total curvature of $S_{iso} = \sigma(u_1, u_2)$ can be calculated by the topology of the surface patch. In this concern, the Gauss-Bonnet theorem [12] explains the connection between total Gaussian curvature on the surface and the topology of S_{iso} as follows:

$$\int_{iso} K\, dA + \int_{\partial S_{iso}} k_e\, ds = 2\pi\chi(S_{iso}), \tag{3.16}$$

where $\chi(S_{iso})$ is the Euler number of S_{iso} and is defined by $\chi(S_{iso}) = [2 - 2e - b]$, e, b are genus and the number of boundaries of the surface respectively. After this, we briefly analyze conformal deformation in the metric of S_{iso} and harmonic energy for a real function defined in S_{iso}. We assume S_{iso} to be a surface embedded in V_{iso} and that S_{iso} has a Riemannian metric g induced from Euclidean metric of V_{iso} then it is clear from [15] that if $\lambda : S_{iso} \to V_{iso}$ is a scalar function defined on S_{iso}, we say that $\overline{g} = e^{2\lambda(u_i)}\, g$ is also a Riemannian metric on $S_{iso}$and angle $\theta_{\overline{g}}$ measured by $\overline{g}$ is equal to angle θ_g measured by g.Thus we say that $\overline{g}$ is a conformal deformation from g. Moreover under such conformal deformation, the Gaussian curvature of the surface will change according to the following well known Yamabi equation delineating the relation between conformal deformation and curvature alteration:

$$\overline{K} = e^{-2\lambda(u_i)}\left(K - \nabla_g^2\lambda\right),\ \overline{k} = e^{-2\lambda(u_i)}(k - \partial_n\lambda),$$

where k is the geodesic curvature of the boundary curve of surface and n is the exterior tangent vector perpendicular to the boundary of surface. Now, we introduce a real function $\varphi : S_{iso} \to V_{iso}$ on the S_{iso} having a metric g. Then the harmonic energy of φ for the S_{iso} is defined by

$$E(\varphi) = \int_{S_{iso}} |\nabla_g^2\varphi|^2\, dA_g,$$

where dA_g is the area element under the metric g.
If the real function φ is harmonic, then all of its components must be harmonic, satisfying the following Laplace equation:

$$\nabla_g^2\varphi = 0,\ \text{where}\ \nabla_g^2 = e^{-2\lambda(u_1,u_2)}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right).$$

With these notions, we describe the RF over the S_{iso} as the process of deforming the metric $g(t)$ by its induced Gaussian curvature $K(t)$, where t being the time parameter, such that the Gaussian curvature evolves as a heat diffusion equation:

$$\frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t).$$

Let us now fabricate a heat diffusion model for a volumetrically isothermal manifold V_{iso} as follows:

3.1. **Modeling of Heat flow, Heat equation for $\mathbf{V_{iso}}$ and $\mathbf{S_{iso}}$.** We know that a body heat will flow in the direction of decreasing temperature. Physical experiments explore that the rate of heat flow is proportional to the gradient of temperature. This implies that the velocity v_i of the heat flow in the body is of the form:

$$u_i = -K_{thermal} \text{ grad } U(u_i, t). \tag{3.17}$$

Here we have considered that $U(u_i, t)$ is the temperature of considered manifold and $K_{thermal}$ is the thermal conductivity of the boundary of V_{iso} manifold. It is also assumed that under ordinary physical circumstances $K_{thermal}$ is a constant. Using these information, we set up a mathematical model of heat diffusion in V_{iso}.

Let R be the gaseous region of V_{iso} covered by a boundary surface S_{iso}. Then the amount of heat leaving R per unit of time is given by

$$\iint_{S_{iso}} v_i.n_i \, dA, \tag{3.18}$$

where n_i s the unit normal vector of S_{iso}.
From equation (3.17) and the Gauss divergence theorem, we obtain

$$\iint_{S_{iso}} v_i.n_i \, dA = -K_{thermal} \iiint_R \text{div}(\text{grad}U(u_i, t))du_i = -K_{thermal} \iiint_R \nabla_g^2 U(u_i, t)du_i$$
$$= -K_{thermal} \iiint_R g^{ij} \left(\frac{\partial^2}{\partial u_i \partial u_j} - \Gamma_{ij}^k \frac{\partial}{\partial u_k} \right) U(u_i, t)du_i. \tag{3.19}$$

On the other hand, the total amount of heat H_{total} in R is

$$H_{total} = \iiint_R \alpha\beta U(u_i, t)du_i, \tag{3.20}$$

where α is the specific heat of V_{iso} system and β is its density. When heat diffuses from the V_{iso} manifold to the V_{sur}, then time rate of decrease of H_{total} will be given by:

$$-\frac{\partial H_{total}}{\partial t} = -\iiint_R \alpha\beta \frac{\partial U(u_i, t)}{\partial t} du_i \tag{3.21}$$

and this must be equal to the amount of heat leaving given by (3.19), thus we have

$$-\iiint_R \alpha\beta \frac{\partial U(u_i, t)}{\partial t} du_i = -K_{thermal} \iiint_R \nabla_g^2 U(u_i, t)du_i \tag{3.22}$$

or

$$\iiint_R \left(\alpha\beta \frac{\partial U(u_i,t)}{\partial t} - K_{thermal}\nabla_g^2 U(u_i,t)\right) du_i = 0 \tag{3.23}$$

or

$$\iiint_R \left(\alpha\beta \frac{\partial U(u_i,t)}{\partial t} - K_{thermal} g^{ij}\left(\frac{\partial^2}{\partial u_i \partial u_j} - \Gamma_{ij}^k \frac{\partial}{\partial u_k}\right) U(u_i,t)\right) du_i = 0. \tag{3.24}$$

Since the heat diffusion equation holds for any region R in the V_{iso} manifold, then if the integrand is everywhere continuous, it must vanish, i.e.,

$$\frac{\partial U(u_i,t)}{\partial t} = c^2 \nabla_g^2 U(u_i,t), \tag{3.25}$$

where $c^2 = \dfrac{K_{thermal}}{\alpha\beta}$.

Here, c^2 is called the thermal diffusivity of the V_{iso} system and this partial differential equation is called heat diffusion equation.

Eventually, there is an another design of heat diffusion equation for a smooth S_{iso} surface embedded in V_{iso} that if $U(u_1,u_2,t)$ is a temperature filed on the S_{iso} surface, then according to thermal dynamics, the temperature field will be evolved under the RF in accordance with the following expression:

$$\begin{aligned}\frac{dU(u_1,u_2,t)}{dt} &= -\nabla_{LB} U(u_1,u_2,t) = e^{-2\lambda}\nabla_g^2 U(u_1,u_2,t) = \\ &= \frac{1}{|g|^{1/2}} \frac{\partial}{\partial u_i}\left(|g|^{1/2} g^{ij} \frac{\partial}{\partial u_j}\right) U(u_1,u_2,t).\end{aligned} \tag{3.26}$$

Concluding Remarks. Here is the brief discussion over some main outcomes obtained from the article written in favor of Geometrization of heat flow on V_{iso} manifolds via RF:

(1) In section (2.3), we have developed an independent notion of volumetrically isothermal manifold by developing a volume structure (2.2) and a metric derivative (2.3).

(2) In section (3), we have discussed Geometrization of heat flow in V_{iso} manifold by letting that the manifold is composed of a finite number of arbitrarily closed isothermal curves (curves along which the temperature remains the identical) along which heat flow exists and we have then established a geometric structure (i.e., a Riemannian metric tensor) given by (3.1) for such curves. By taking usual RF for Riemannian curvature, Ricci curvature, Christoffel bracket symbol and Scalar curvature, we have discussed various heat diffusion equation along arbitrary trajectory of gas molecule at some specific instant.

(3) Further, by using well known split techniques like; prime and torus techniques, we have developed a metric (3.14) for an arbitrary isothermal surface patch of the isothermal manifold and have described RF, harmonic maps and conformal deformation for such isothermal surfaces.

(4) Finally we have tried to fabricate a heat diffusion model for our V_{iso} manifold and the isothermal surface patch embedded in it.

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