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The Lotka-Volterra Predator-Prey Equations

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ABSTRACT. One may find out the application of mathematics in the areas of ecology, biology, environmental sciences etc. Mathematics is particulary used in the problem of predator-prey known as lotka-Volterra predator-prey equations. Indeed, differential equations is employed very much in many areas of other sciences. However, most of natural problems involve some unknown functions. In this paper, an environmental case containing two related populations of prey and predator species is studied. As the classic Lotka-Volterra assumptions are unrealistic, it is assumed that there is logistic behavior for both existing species. We see that two populations influence the size of each other.

Keywords: Lotka-Volterra Model; Prey-Predator; Growth Rate.

2000 Mathematics subject classification: 34A36, 34K10.

1. INTRODUCTION

Vito Volterra, the Italian mathematician, proposed a differential equations model to describe the population dynamics of two interacting species of a predator and its prey in the 1920s. He hoped to explain the increasing in predator fish (and so,decreasing in prey fish) in the Adriatic Sea during World War I. These equations are studied independently by Alfred Lotka to describe a hypothetical chemical reaction in which the chemical concentrations oscillate [1,2], in the United States. There

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seemed to be a periodic behavior on a two species population system. Volterra developed a model based on a few assumptions or observations. Prey in the absence of predation grows unboundedly. Predator population in the absence of prey decreases exponentially. Predator reduces the prey growth rate, proportional to both the predator and prey populations. Prey will increase the predator growth rate somehow proportional to the prey and predator populations.

2. Main Results

This section includes a mathematical model and solutions analysis. Consider the following system:

$$\begin{cases} \frac{dx}{dt} = ax - bxy\\ \frac{dy}{dt} = cxy - dy \end{cases}$$
(2.1)

The function x(t) represents the populations of prey at time t, and also the function y(t) represents the populations of predator at time t. All of the parameters a, b, c, d are non-negative constants. The parameter a represents the natural growth rate of the prey in the absence of predator. The parameter b represents the effect of predator in the prey population. Moreover if b is the only decreasing factor for the prey population, then preys be eaten by predators. The parameter crepresents the effect of prey in the predator population, moreover if c is the only increasing factor for the predator population, then a population growth proportional to the food available. The parameter d represents the natural death rate of the predator in the absence of prey. If both parameters b and c are zero, then (2.1) leads into the following system:

$$\begin{cases} \frac{dx}{dt} = ax\\ \frac{dy}{dt} = -dy \end{cases}$$
(2.2)

which represents a uncoupled system.



Fig. 1 Population changes in uncoupled sys-

tems (2.2)

First step: By variables changing:

$$u(\tau) = \frac{cx(t)}{d} \qquad v(\tau) = \frac{by(t)}{a} \qquad \tau = at \qquad \alpha = \frac{d}{a}$$

the system (2.1) leads:

$$\begin{cases} \frac{du}{d\tau} = u(1-v) \\ \frac{dv}{d\tau} = \alpha v(u-1) \end{cases}$$
(2.3)

This is a system of equations having only one parameter. The parameter a is ratio between the decreasing and increasing factors of both existing species. It is an autonomous system. Because the independent variable does not appear in the right term in an explicit way.

Second step: To find out the trajectory equation, divide the second equation of (2.3) by the first. And so we have

$$\frac{dv}{du} = \frac{\alpha v(u-1)}{u(1-v)} \tag{2.4}$$

It may be calculated, using the integrating factor $(\frac{1}{uv})$ in the equation (2.4).

$$H(u,v) = \alpha u + v - \ln(u^{\alpha}v)$$

Result 1: The existence of a first integral H implies that the solution (u, v) belongs to the contour of H. The graph of H is a closed curve.

Fig. 2 The graph of
$$H$$

The following two questions will appear as described in the next example. What is exactly a trajectory equation? What does "eliminate the time" means?

Example: Consider the harmonic oscillator $\frac{d^2x}{dt^2} + \omega^2 x = 0$. Its solution can be found as follows:

$$\begin{cases} x(t) = A\cos(\omega t + \theta) \\ x'(t) = -\omega A\sin(\omega t + \theta) \end{cases}$$
(2.5)

If we "eliminate the time" from the representation, we obtain the trajectory equation. In this case H(u, v) = H(x, x') which is Hamiltonian system $H(x, x') = \frac{1}{2}kx^2 + \frac{1}{2}mx'^2$. This equation represents equation of a circle or ellipse.

Result 2: If the trajectory H is closed curves, then the system has periodic solutions. If we take polar coordinates, H is an increasing function for every θ (direction). Neither u nor v can approach to infinity, because it causes H to diverge.



Fig. 2 Population of predator against prey

It is clear that the above model is not realistic, and so we bring a realistic model.

The classic Lotka-Volterra assumptions are unrealistic. Specially the first one, where it is assumed that the prey population would increase indefinitely in the absence of predators. One of the possible solutions is to assume logistic behavior in the prey and predator population:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - xy\\ \frac{dy}{dt} = y(1-\frac{y}{x}) \end{cases}$$
(2.6)

There is stable and attractive fixed point which is shown in fig. 4 as A. Also, the system (2.6) does not long oscillations.



Fig. 3 : Population changes in realistic model (2.6)

3. Conclusion

Lotka-Volterra equations form an ideal model, indeed the said equations describe well some experimental results and observations. The Lotka-Volterra system dynamics indeed shows some very unique features. In fact, it presents stable oscillations (except to the fixed point, and for zero initial values of either the prey or the predator). And also, There is not extinction for both prey or predator species if they are presented in the initial conditions. There are plenty of more realistic models with different behavior, usually without the oscillatory dynamics.

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