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On the stability analysis and the solitonic wave structures for the stochastic resonant nonlinear Schrödinger equation with spatio temporal and inter-modal dispersion under generalized Kudryashov's law non-linearity

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Abstract. This paper discusses the optical soliton solutions of the stochastic resonant nonlinear Schrödinger equation (SRNLSE). The equation has spatio-temporal dispersion, inter-modal dispersion, multiplicative white noise, and nonlinearity under generalized Kudryashov's law. Optical soliton solutions in terms of bright, dark, periodic, and singular solitons are obtained from this equation by using the $\frac{G\prime}{G^2}$ -expansion method and a new Kudryashov method. This work provides insight into soliton dynamics in nonlinear optical systems with stochastic effects, where complex dispersion interactions play a dominant role. Specifically, it shows how the interplay of spatio-temporal dispersion (SPD) and inter-modal dispersion (IMD), in the presence of multiplicative noise, determines the behavior of solitons. We also discuss the effects of multiplicative noise on the exact solutions of the nonlinear Schrödinger equation using the Maple software. The stability of critical points is discussed by linearizing the system around equilibrium solutions and graphically indicating the behavior of these solutions as well.

Keywords: White noise, Kudryashov's law non-linearity, $(\frac{G\prime}{G^2})$-expansion method, New Kudryashov's method, Optical Solitons.

1. Introduction

The study of nonlinear stochastic partial differential equations (NSPDEs) is one of the significant research areas with applications across various fields: modern physics, biology, superfluid

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dynamics, image processing, optical fiber communications, plasma physics, and finance [1-5]. That makes NSPDEs a very interesting and highly relevant subject in research.The most widely studied stochastic process that is both a martingale and a Markov process is the Wiener process, or Brownian motion [6]. The Wiener process is at the main concept of stochastic calculus and is essential for modeling stochastic processes. It is a continuous process with increments that follow a normal distribution over any time scale. This process is very extensively used in dispersive systems [7, 8]. There is also an intimate relationship between partial differential equations (PDEs) and stochastic processes. Nonlinear Schrödinger equations (NLSEs) are among the most widely used models in applied science due to their broad range of applications [9-12]. The study of soliton solutions of these equations is very important in nonlinear science, as it explains the underlying physical mechanisms behind complex natural phenomena. This area has evolved into one of the most compelling and dynamic fields of research [13-15]. Recently, several new solitary solutions have been introduced through innovative

Approaches to nonlinear equation models [16, 17]. Investigations of N-soliton solutions, from which one can recover lump and rogue wave solutions, have been performed for both modified Korteweg – De-Vries-type integrable equations and reduced integrable nonlinear Schrödinger-type equations.

The study of NLSEs in optical solitons with nonlinearities has become a growing focus in nonlinear photonics [18, 19]. In recent years, various types of nonlinearities, including parabolic, Kerr, power, polynomial, and saturable laws, have been explored [20].

Islam et al. examined the influence of wave dispersion and nonlinearity parameters on the solitonic KMNE properties, noting that optical wave propagation takes forms such as bell-shaped, bright, dark, periodic, kink, and singular, with dynamic features dependent on dispersion parameters [21]. The significant solitonic applications of the Gross-Pitaevskii (GP) equation in water waves and plasma physics, as a model for nonlinear unidirectional wave propagation, have also been theoretically explored [22]. It was found that soliton characteristics are influenced by free parameters and dispersion coefficients.

This paper will review recent advancements in statistical models based on NSPDEs. We will focus on the Wiener process, discussing the extension of NLSEs and highlighting their relevance in various contexts. Motivating applications will be considered, particularly the impact of the noise term on the behavior of the solution, other methods for obtaining exact soliton solutions [30-37].

1.1. **Principal Model.** This study introduces, for the first time, the stochastic resonant NLSE incorporating both STD and IMD multiplicative noise in the Ito sense, along with generalized Kudryashov's law nonlinearity [23, 24].

$$\begin{aligned} & iQ_t + \alpha Q_{xx} + \beta Q_{xt} + \gamma \left(\frac{|Q|_{xx}}{|Q|} \right) Q - i\delta Q_x \\ & + \chi \left(Q - i\beta Q_x \right) \frac{dW\left(t\right)}{dt} + (b_1|Q|^{2n} + b_2|Q|^{2n} + |Q|^{2n} \\ & + b_3|Q|^{3n} + b_4|Q|^{4n} + c_1|Q|^{-n} + c_2|Q|^{-2n} + c_3|Q|^{-3n} + c_4|Q|^{-4n})Q = 0, \end{aligned} \tag{1.1}$$

The wave profile is represented by the complex-valued function $\mathrm{Q} = Q(x,t)$ in this case. With $i^2 = -1$, the parameters α, β, γ, δ, and are real-valued constants. In Eq.(1.1), the linear temporal evolution is represented by the first term, while the chromatic dispersion (CD) and STD terms are denoted by α and , respectively. The parameter denotes the coefficient of IMD, δ, and the coefficient of resonant nonlinearity, γ .Then, b_s and $\mathrm{c_s}$, (s $=$ 1 $-$ 4) are the coefficients

of self-phase modulation, n is power nonlinearity parameter. Lastly, W(t) is the normal Wiener process, and χ is the coefficient of noise strength. The white noise is represented by dW(t)/dt . This research investigates various aspects of noise's impact on the new extension of the nonlinear Schrodinger equation (NLSE) in the It o sense through the Wiener process. This is a broad and captivating field with ongoing active research across various methodologies. We apply the $(\frac{G\prime}{G^2})$-expansion method and the new Kudryashov method to derive new stochastic solutions for the extended NLSE. Compared to most existing methods, the proposed approach offers several advantages, including avoiding tedious calculations and producing essential solution families. It is simple, reliable, and efficient. This method can serve as a universal solver for various natural science systems. Additionally, it includes rational solutions, which are crucial for describing wave behavior at critical points.
The stochastic solutions presented for Equation (1.1) highlight a range of significant physical phenomena, such as the behavior of erbium atoms, fiber-optic communications, oceanic rogue waves,and the bending of light beams.

This article is organized as follows: Section 2 covers the mathematical analysis for Eq.(1.1).Section 3 introduces the extended $(\frac{G\prime}{G^2})$ -expansion method. Section 4 discusses the new Kudryashov's method with application. Section 5 gives stability analysis and finally, Section 6 presents the conclusions.

2. Mathematical model

Using a wave transformation including the Wiener process $W(t)$ and the noise coefficient χ, we can solve the stochastic Eq.(1.1), as follows:

$$Q(x,t) = E(y)\, e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}, \tag{2.1}$$

and

$$y = x - ct, \tag{2.2}$$

Here real constants κ, ω and c are employed. The pulse shape can be represented by the real function $E = E(y)$, where κ, ω and c stand for the wave number, soliton frequency, and soliton velocity, respectively. By putting (2.1) and (2.2) into Eq.(1.1), we can conclude:

$$\begin{aligned}\Re : (\gamma - c\beta + \alpha) E'' + \left[(\beta\kappa - 1)\left(\omega - \chi^2\right) - \delta\kappa - \alpha\kappa^2\right] E + b_1 E^{n+1} \\ + b_2 E^{2n+1} + 3E^{3n+1} + b_4 E^{4n+1} + c_1 E^{1-n} \\ + c_2 E^{1-2n} + c_3 E^{1-3n} + c_4 E^{1-4n} = 0,\end{aligned} \tag{2.3}$$

and

$$\Im : \left[(\beta\kappa - 1) c - 2\alpha\kappa - \delta + \beta\left(\omega - \chi^2\right)\right] E'(\tau) = 0 \tag{2.4}$$

From Eq.(2.4),

$$c = \frac{2\alpha\kappa + \delta - \beta\left(\omega - \chi^2\right)}{\beta\kappa - 1}, \quad \textit{provided} \quad \beta\kappa \neq 1 \tag{2.5}$$

By balancing E'' and E^{4n+1} in Eq.(2.3), we derive the balance $M = \frac{1}{2n}$. Since M is not an integer, we proceed by taking:

$$E(y) = [\Phi(y)]^{\frac{1}{2n}}, \quad \textit{provided} \quad \Phi(y) > 0. \tag{2.6}$$

Inserting (2.6) into Eq.(2.3) yields:

$$\begin{aligned} &2n\pi_0\Phi\Phi'' + (1-2n)\pi_0\Phi'^2 + 4n^2c_4 + 4n^2c_2\Phi \\ &+ 4n^2\pi_1\Phi^2 + 4n^2b_2\Phi^3 + 4n^2b_4\Phi^4 \\ &+ 4n^2c_3\Phi^{\frac{1}{2}} + 4n^2c_1\Phi^{\frac{3}{2}} + 4n^2b_1\Phi^{\frac{5}{2}} + 4n^2b_3\Phi^{\frac{7}{2}} = 0. \end{aligned} \tag{2.7}$$

Where

$$\pi_0 = \gamma - c\beta + \alpha, \quad \pi_1 = (\beta\kappa - 1)\left(\omega - \chi^2\right) - \delta\kappa - \alpha\kappa^2. \tag{2.8}$$

For zintegr ability, one must choose:

$$b_1 = b_3 = c_1 = c_3 = 0. \tag{2.9}$$

Consequently, Eq.(1.1) can be rewritten as:

$$\begin{aligned} &iQ_t + \alpha Q_{xx} + \beta Q_{xt} + \gamma\left(\frac{|Q|_{xx}}{|Q|}\right)Q \\ &- i\delta Q_x + \chi\left(Q - i\beta Q_x\right)\frac{dW\left(t\right)}{dt} \\ &+ \Big(b_2\left|Q\right|^{2n} + b_4\left|Q\right|^{4n} + c_2\left|Q\right|^{-2n} + c_4\left|Q\right|^{-4n}\Big)Q = 0. \end{aligned} \tag{2.10}$$

As a result, Eq.(2.7) is replaced by:

$$\begin{aligned} &2n\pi_0\Phi\Phi'' + (1-2n)\pi_0\Phi'^2 + 4n^2c_4 \\ &+ 4n^2c_2\Phi + 4n^2\pi_1\Phi^2 + 4n^2b_2\Phi^3 + 4n^2b_4\Phi^4 = 0. \end{aligned} \tag{2.11}$$

In the following sections, we will solve Eq.(2.11) using the following method.
We suppose that the Eq.(1.1) has a form of solution as mention below:

$$\Phi\left(y\right) = l_0 + l_1\left(\frac{G\prime}{G^2}\right). \tag{2.12}$$

3. The extended $\left(\frac{G\prime}{G^2}\right)$ - expansion method

Assume that the solution of Eq.(1.1) takes the following form [25]:

$$Q\left(y\right) = l_0 + \sum_{i=1}^{m}\left[l_i(\frac{G\prime}{G^2}) + l^i\left(\frac{G\prime}{G^2}\right)^{-1}\right], \tag{3.1}$$

Where

$$\left(\frac{G\prime}{G^2}\right)' = z + s\left(\frac{G\prime}{G^2}\right)^2, \tag{3.2}$$

While z and s are real constants.
The general solutions of Eq.(3.2) With respect to parameters z and s are given below.

■ $zs > 0$

$$\left(\frac{G\prime}{G^2}\right) = \pm\sqrt{\frac{z}{s}}\left[\frac{r_1cos\left(\sqrt{zs}\ y\right) + r_2sin\left(\sqrt{zs}\ y\right)}{r_2cos\left(\sqrt{zs}\ y\right) - r_1sin\left(\sqrt{zs}\ y\right)}\right]. \tag{3.3}$$

■ $zs < 0$

$$\left(\frac{G\prime}{G^2}\right) = -\frac{\sqrt{|zs|}}{s}\left[\frac{r_1sinh\left(2\sqrt{|zs|}\ y\right) + r_1cosh\left(2\sqrt{|zs|}\ y\right) + r_2}{r_1sinh\left(2\sqrt{|zs|}\ y\right) + r_1cosh\left(2\sqrt{|zs|}\ y\right) - r_2}\right]. \tag{3.4}$$

■ $z=0,\ s\neq 0$

$$\left(\frac{G\prime}{G^2}\right)=-\left[\frac{r_1}{s\left(r_1y+r_2\right)}\right]. \tag{3.5}$$

By plugging Eq.(2.12) in Eq.(2.11) and take sum all of terms with same W^i equalize where $i\ =0,1,2$ and correlate the coefficient of distinct term of W^i later a collection of total algebraic equation is available as:

$$\left(\frac{G\prime}{G^2}\right)^0 : 4n^2c_4+4n^2c_2l_0+4n^2\pi_1l_0^2+\pi_0l_1^2z^2+4n^2b_2l_0^3-2n\pi_0l_1^2z^2+4n^2b_4l_0^4=0,$$
$$\left(\frac{G\prime}{G^2}\right)^1 : 4n^2c_2l_1+8n^2\pi_1l_1\ l_0+12n^2b_2l_0^2l_1+4\pi_0nszl_0l_1+16n^2b_4l_1l_0^3=0,$$
$$\left(\frac{G\prime}{G^2}\right)^2 : 4n^2\pi_1l_1^2+2\pi_0l_1^2zs+12n^2b_2l_0l_1^2+24n^2b_4l_1^2l_0^2=0,$$
$$\left(\frac{G\prime}{G^2}\right)^3 : 4n^2b_2l_1^3+4\pi_0ns^2l_0l_1+16n^2b_4l_1^3l_0=0,$$
$$\left(\frac{G\prime}{G^2}\right)^4 : \pi_0s^2l_1^2+2\pi_0ns^2l_1^2+4n^2b_4l_1^4=0.$$

To solve the above equations with the support of Maple to get the successive solutions:
Result 1:

$$\begin{aligned}
l_1&=\pm\sqrt{-\frac{3\pi_0}{sz\pi_0+2n^2\pi_1}}\ sa_0,\\
l_0&=l_0,\\
c_2&=-\frac{2a_0\left(szn\pi_0-n^3\pi_1-sz\pi_0+n^2\pi_1\right)}{3n^2}, && (3.6)\\
c_4&=-\frac{a_0^2(s^2z^2\pi_0^2-2szn^2\pi_0\pi_1+n^4\pi_1^2)(2n-1)}{3\left(sz\pi_0+2n^2\pi_1\right)n^2},\\
b_2&=-\frac{(n+1)\left(sz\pi_0+2n^2\pi_1\right)}{3a_0n^2},\\
b_4&=\frac{(2n+1)\left(sz\pi_0+2n^2\pi_1\right)}{12a_0^2n^2}. && (3.7)
\end{aligned}$$

Where

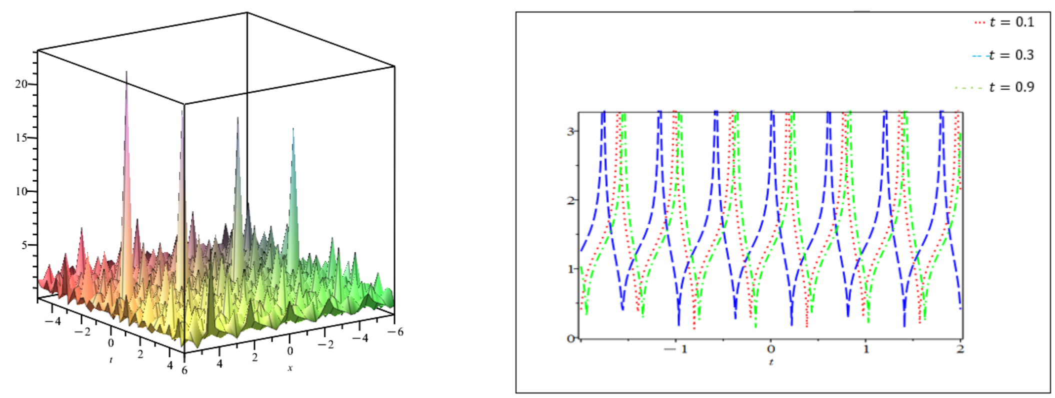
$$\begin{aligned}
\pi_0&=\gamma-c\beta+\alpha,\\
\pi_1&=(\beta\kappa-1)\left(\omega-\chi^2\right)-\delta\kappa-\alpha\kappa^2,\\
c&=\frac{2\alpha\kappa+\delta-\beta\left(\omega-\chi^2\right)}{\beta\kappa-1},\\
y&=x-ct.
\end{aligned}$$

3.1. Case 1

■ $zs>0$

Enter the Eq.(3.5) in the Eq.(2.11) and find the following solution

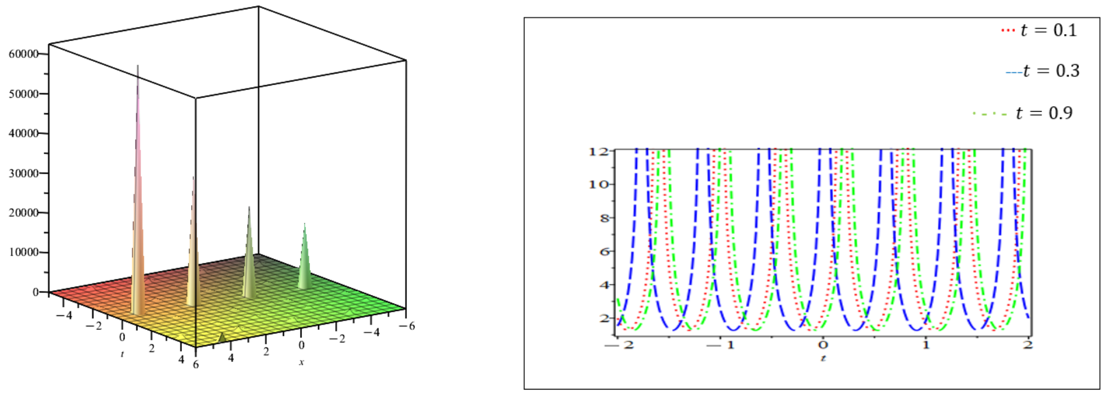
$$Q_1(x,t) = \left[l_0 \pm \sqrt{-\frac{3\pi_0}{sz\pi_0 + 2n^2\pi_1}} sa_0 \left(\pm\sqrt{\frac{z}{s}} \left(\frac{r_1 cos\left(\sqrt{zs}\,(x-ct)\right) + r_2 sin\left(\sqrt{zs}\,(x-ct)\right)}{r_2 cos\left(\sqrt{zs}\,(x-ct)\right) - r_1 sin\left(\sqrt{zs}\,(x-ct)\right)} \right) \right) \right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)} \quad (3.8)$$



(A) (B)

FIGURE 1. Periodic and solitary wave behavior of the $2D$ and $3D$ structure under the solution of Biswas Eq.(3.8) with the parameters $(x\ ,\ t)$ .The 2D graph along $x = 0.1$, 0.3 , 0.9 where, $z=1$, $s = 2$, $n = 3/2$, $\alpha = 0.2$, $\beta = 0.3$, $\delta = 0.5$, $\omega = 0.35$, $\gamma = 0.5$, $W(t) = 3t$, $\chi = 1$, $\kappa = 2$, $a_0 = 1.22$, $r_1 = 0.1$, $r_2 = 0.5$.

Solution of Eq.(3.8) is a soliton with a dark and bright structure with parameters $n = \frac{3}{2}, z = 1$, $s = 2$,$r_1 = 0.1$ and $r_2 = 0.5$ of unlimited wings and conspiracy within Fig.1. The $3D$ plot reported along with gap. $2 < x < 2$ and $-2 < t < 2$ The $2D$ plot reported along with gap $-2 < t < 2$ as variation is about $x = 0.1, 0.3, 0.9$.



(A) (B)

FIGURE 2. Periodic and solitary wave behavior of the $2D$ and $3D$ structure under the solution of Biswas Eq.(3.8) with the parameters $(x\ ,\ t)$ .The 2D graph along $x = 0.1$, 0.3 , 0.9 where, $z = 1$, $s = 2$, $n = 3/7$, $\alpha = 0.2$, $\beta = 0.3$, $\delta = 0.5$, $\omega = 0.35$, $\gamma = 0.5$, $W(t) = 3t$, $\chi = 1$, $\kappa = 2$, $a_0 = 1.22$, $r_1 = 0.1$, $r_2 = 0.5$.

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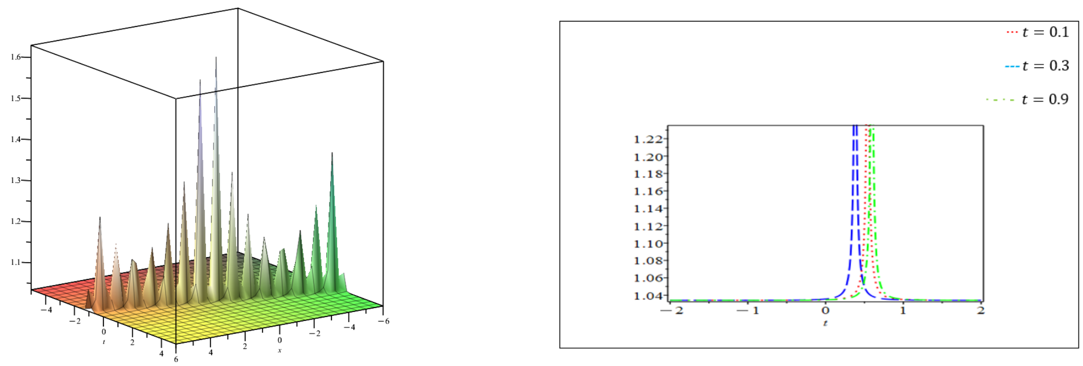
Solution of Eq.(3.8) is a soliton with a dark and bright structure with parameters $n = \frac{3}{7}$, $z = 1$,$s = 2$, $r_1 = 0.1$ and $r_2 = 0.5$ of unlimited wings and conspiracy within Fig. 2. The 3D plot reported along with gap. $-2 < x < 2$ and $-2 < t < 2$The 2D plot reported along with gap $-2 < t < 2$ as variation is about $x = 0.1, b\ \ 0.3,\ \ 0.9$.
By preceding the same path as previously, we also arrive at the subsequent solutions.
■ $zs < 0$

$$Q_2(x,t) = \left[l_0 \pm \sqrt{-\frac{3\pi_0}{sz\pi_0 + 2n^2\pi_1}} sa_0 \left(-\frac{\sqrt{|zs|}}{s}\left(\frac{r_1 sinh\left(2\sqrt{|zs|}(x-ct)\right) + r_1 cosh\left(2\sqrt{|zs|}(x-ct)\right) + r_2}{r_1 sinh\left(2\sqrt{|zs|}\ (x-ct)\right) + r_1 cosh\left(2\sqrt{|zs|}(x-ct)\right) - r_2}\right)\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}, \tag{3.9}$$

Solution of Eq.(3.9) is a soliton with a dark and bright structure with parameters $n = 3$, $z = 0.6$,$s = -0.2$, $r_1 = 0.1$ and $r_2 = 0.5$ of unlimited wings and conspiracy within Fig 3 The 3D plot reported along with gap $-2 < x < 2$ *and* $-2 < t < 2$. The 2D plot reported along with gap $-2 < t < 2$ as variation is about $x = 0.1, 0.3, 0.9$.
■ $z = 0, s \neq 0$

$$Q_3(x,t) = \left[l_0 \pm \sqrt{-\frac{3\pi_0}{sz\pi_0 + 2n^2\pi_1}} sa_0 \left(-\left(\frac{r_1}{s\left(r_1(x-ct) + r_2\right)}\right)\right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x + \omega t + \chi W(t) - \chi^2 t\right)} \tag{3.10}$$

Where



(A) (B)

FIGURE 3. Periodic and solitary wave behavior of $2D$ and $3D$ structure under the solution of Biswas Eq. (21) with the parameters $(x\ ,\ t)$ .The 2D graph along $x\ =\ 0.1,\ \ 0.3,\ \ 0.9$, where, $z = 0.6,\quad s\ = -0.2\ ,\ n\ = 3\ ,\ \alpha = 0.2\ ,\ \beta = 0.3\ ,\ \delta = 0.5\ ,\ \omega = 0.35\ ,\ \gamma = 0.5\ ,\ W\,(t) = 3t\ ,\ \chi = 1\ ,\ \kappa = 2,\ a_0 = 1.22\ ,\ r_1 = 0.1,\ \ r_2 = 0.5.$

$$\pi_0 = \gamma - c\beta + \alpha, \quad \pi_1 = (\beta\kappa - 1)\left(\omega - \chi^2\right) - \delta\kappa - \alpha\kappa^2,$$
$$c = \frac{2\alpha\kappa + \delta - \beta\left(\omega - \chi^2\right)}{\beta\kappa - 1}.$$

4. Description of the new Kudryashov approach

The Kudryashov approach employs a distinctive method, characterized by its reliance on the following function [26]:

$$W\,(y) = \frac{2\varsigma H}{(H^2 \mp l)\cosh(\sigma y)\ + (H^2 \mp l)\sinh(\sigma y)}, \tag{4.1}$$

And the exponential form is given as,

$$W\,(y) = \frac{2\varsigma H}{(H^2 e^{\varsigma\sigma y} \mp l\ e^{-\varsigma\sigma y})}. \tag{4.2}$$

Where σ, H, l are real numbers, $\varsigma = \mp 1$, and , $W(y)$ adheres to this relation:

$$\left(\frac{dW\,(y)}{dy}\right)^2 - \sigma^2 W^2\,(y)\,(1 \pm lW\,(y)) = 0. \tag{4.3}$$

Let us assume the solution as:

$$\Phi\,(y) = \sum_{j=0}^{n} h_j (W\,(y))^j, \quad h_n \neq 0 \tag{4.4}$$

Where, the unknown constants are $h_0, h_1, h_2, \cdots$
Additionally, Let us take the following solution for the Bernoulli's equation approach:

$$\Phi\,(y) = \sum_{j=0}^{n} b_j (W\,(y))^j, \quad b_n \neq 0, \tag{4.5}$$

Where, the unknown constants are $b_0, b_1, b_2, \cdots$ and n is homogeneous balancing constants. The function $W(y)$ satisfies the following Bernoulli's equation,

$$W'(y) = rW(y) - W^2(y). \tag{4.6}$$

The solution of Bernoulli's equation is given as,

$$W(y) = \frac{r}{2} + \frac{r}{2} tanh\left(\frac{r}{2}y\right). \tag{4.7}$$

Here, the homogeneous balance principle between $\Phi\Phi''$ and Φ^4 gives $M = 1$. Thus, the solution (4.4) is reduced as follows:

$$\Phi(y) = h_0 + h_1 W(y). \tag{4.8}$$

Now, inserting the above solution into (2.11) and by evaluating the coefficients of different powers of $W(y)$, we obtain the following system of algebraic equations.

$$\begin{aligned}
&(W(y))^0 : 4n^2c_4 + 4n^2c_2h_0 + 4n^2\pi_1h_0^2 + 4n^2b_2h_0^3 + 4n^2b_4h_0^4 = 0,\\
&(W(y))^1 : 4n^2c_2h_1 + 8n^2\pi_1h_1h_0 + 2n\pi_0\sigma^2h_0h_1 + 12n^2b_2h_0^2h_1 + 16\\
&\qquad n^2b_4h_1h_0^3 = 0,\\
&(W(y))^2 : \pi_0h_1^2\sigma^2 + 4n^2\pi_1h_1^2 + 12n^2b_2h_0h_1^2 + 24n^2b_4h_1^2h_0^2 = 0,\\
&(W(y))^3 : 4n^2b_2h_1^3 + 16n^2b_4h_1^3h_0 + 4n\pi_0lh_0h_1\sigma^2 = 0,\\
&(W(y))^4 : \pi_0\sigma^2h_1^2l + 4n^2b_4h_1^4 + 2\pi_0n\sigma^2h_1^2l = 0.
\end{aligned}$$

Solving the system using Mathematica or Maple yields these results:
Result 1:

$$\begin{aligned}
h_0 &= \pm\frac{\sqrt{-\frac{4n^2\pi_1+\sigma^2\pi_0}{6l\pi_0}}h_1}{\sigma}, \quad h_1 = h_1\\
b_2 &= \pm\frac{l\sigma\sqrt{-\frac{4n^2\pi_1+\sigma^2\pi_0}{6l\pi_0}}\ \pi_0(n+1)}{h_1n^2},\\
b_4 &= -\frac{l\sigma^2\pi_0(2n+1)}{4n^2h_1^2}\\
c_2 &= \pm\frac{\left(2n^3\ \pi_1 - n\sigma^2\pi_0 - 2n^2\pi_1 + \sigma^2\pi_0\right)\sqrt{-\frac{4n^2\pi_1+\sigma^2\pi_0}{6l\pi_0}}\ h_1}{3\sigma n^2}\\
c_4 &= \frac{1}{144l\sigma^2\ \pi_0n^2}(h_1^2(32n^5\pi_1^2 - 32n^3\sigma^2\pi_0\pi_1 - 10n\sigma^4\pi_0^2 - 16n^4\pi_1^2 + 16n^2\sigma^2\pi_0\pi_1 + 5\sigma^4\pi_0^2\ )),
\end{aligned} \tag{4.9}$$

Where

$$\begin{aligned}
&\pi_0 = \gamma - c\beta + \alpha, \quad \pi_1 = (\beta\kappa - 1)\left(\omega - \chi^2\right) - \delta\kappa - \alpha\kappa^2,\\
&c = \frac{2\alpha\kappa + \delta - \beta\left(\omega - \chi^2\right)}{\beta\kappa - 1}.
\end{aligned}$$

Utilizing equations (2.1), (4.2), (4.8), and (4.9), the following solutions are acquired:

$$Q_4(x,t) = \left[\pm\frac{\sqrt{-\frac{4n^2\pi_1+\sigma^2\pi_0}{6l\pi_0}}\ h_1}{\sigma} + h_1\left(\frac{2\varsigma H}{(H^2e^{\varsigma\sigma y} \mp l\ e^{-\varsigma\sigma y})}\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)} \tag{4.10}$$

Where

$$\pi_0 = \gamma - c\beta + \alpha, \quad \pi_1 = (\beta\kappa - 1)\left(\omega - \chi^2\right) - \delta\kappa - \alpha\kappa^2, \tag{4.11}$$

$$c = \frac{2\alpha\kappa + \delta - \beta\left(\omega - \chi^2\right)}{\beta\kappa - 1}, \quad y = x - ct. \tag{4.12}$$

Where $l < 0$.Utilizing equations (2.1), (4.1), (4.8), and (4.9), the acquisition of the subsequent hyperbolic solutions:

$$\begin{aligned} Q_5(x,t) = \Bigg[&\pm \frac{\sqrt{-\frac{4n^2\pi_1 + \sigma^2\pi_0}{6l\pi_0}}h_1}{\sigma} \\ &+ h_1\left(\frac{2\varsigma H}{(H^2 - l)\,cosh\left(\sigma\left(x - ct\right)\right) + (H^2 + l)\,sinh\left(\sigma\left(x - ct\right)\right)}\right)\Bigg]^{\frac{1}{2n}} \\ &\times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}. \end{aligned} \tag{4.13}$$

In case, we insert $l = -H^2$ into Eq(4.13), the following bright solitary wave solitons are acquired:

$$\begin{aligned} Q_6\left(x,t\right) = &\left[\pm\frac{\sqrt{-\frac{4n^2\pi_1 + \sigma^2\pi_0}{6l\pi_0}}h_1}{\sigma} + h_1\left(\frac{\varsigma \operatorname{sech}\left(\sigma\left(x - ct\right)\right)}{H}\right)\right]^{\frac{1}{2n}} \\ &\times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}. \end{aligned} \tag{4.14}$$

In case, we insert $l = H^2$ into Eq.(4.13), the following singular solutions are acquired:

$$\begin{aligned} Q_7\left(x,t\right) = &\left[\pm\frac{\sqrt{-\frac{4n^2\pi_1 + \sigma^2\pi_0}{6l\pi_0}}h_1}{\sigma} + h_1\left(\frac{\varsigma csch(\sigma(x - c)t))}{H}\right)\right]^{\frac{1}{2n}} \\ &\times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}. \end{aligned} \tag{4.15}$$

Result 2:

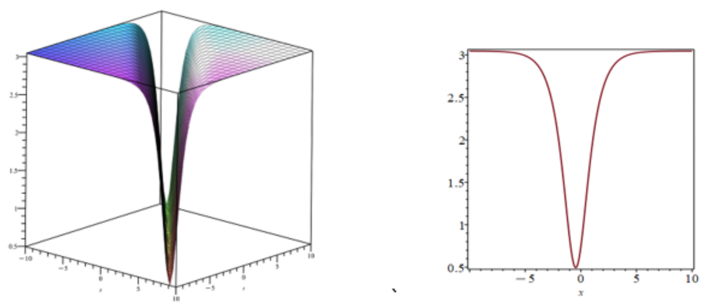
$$\begin{aligned} h_0 &= \pm\frac{\sqrt{-\frac{-8n^3\pi_1 - 2n\pi_0\sigma^2 - 4n^2\pi_1 - \pi_0\sigma^2}{24b_4}}}{n}, \\ h_1 &= \pm\frac{\sqrt{-\frac{2n\pi_0 l + l\pi_0}{4b_4}}\sigma}{n}, \\ b_2 &= \mp\frac{4\sqrt{-\frac{-8n^3\pi_1 - 2n\sigma^2\pi_0 - 4n^2\pi_1 - \pi_0\sigma^2}{24b_4}}b_4\left(n+1\right)}{n(2n+1)}, \\ c_4 &= -\frac{1}{576b_4n^4}(64n^6\pi_1^2 - 64n^4\sigma^2\pi_0\pi_1 - 20n^2\sigma^4\pi_0^2 - 16n^4\pi_1^2 \\ &\quad + 16n^2\sigma^2\pi_0\pi_1 + 5\sigma^4\pi_0^2), \\ c_2 &= \pm\frac{1}{3n^3}\left(\left(-\frac{-8n^3\pi_1 - 2n\pi_0\sigma^2 - 4n^2\pi_1 - \pi_0\sigma^2}{24b_4}\right)^{\frac{1}{2}}(2n^3 - n\pi_0 + \sigma^2 - 2n^2\pi_1 + \pi_0\sigma^2)\right). \end{aligned} \tag{4.16}$$

Utilizing equations (2.1), (4.2), (4.8) and (4.16), the following solutions are acquired:

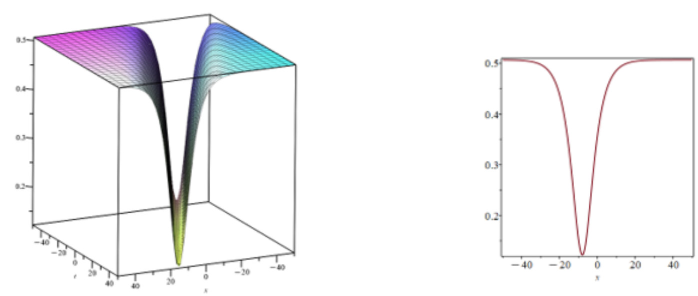
$$Q_8\left(x,t\right)=\left[\pm\frac{\sqrt{-\frac{-8n^3\pi_1-2n\pi_0\sigma^2-4n^2\pi_1-\pi_0\sigma^2}{24b_4}}}{n},\right.$$

$$\left.\pm\frac{\sqrt{-\frac{2n\pi_0 l+l\pi_0}{4b_4}}\ \sigma}{n}\left(\frac{2\varsigma H}{\left(H^2e^{\varsigma\sigma(\ x-ct)}\mp l\ e^{-\varsigma\sigma(\ x-ct)}\right)}\right)\right]^{\frac{1}{2n}}$$

$$\times\, e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}. \tag{4.17}$$



(A) $\delta=1$



(B) $\delta=0.3$

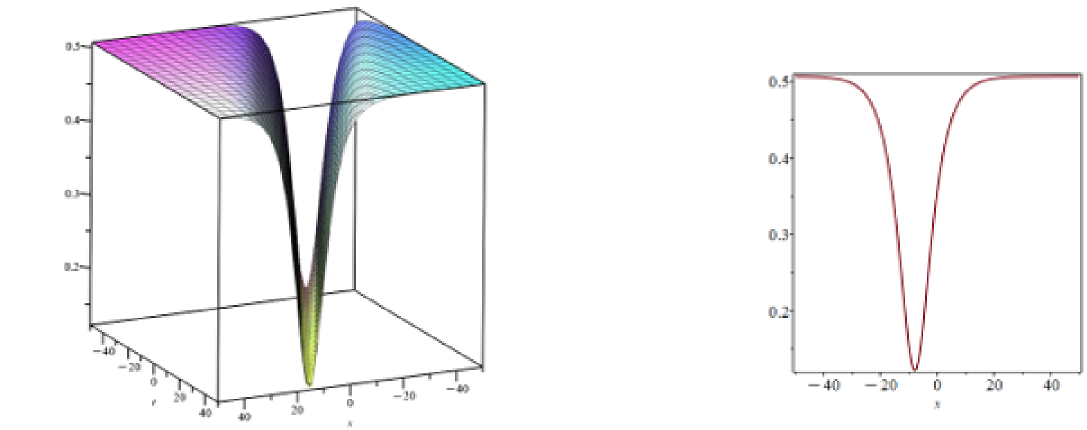
FIGURE 4. The comparison of $Q_8(x,t)$ where $l=-1,\quad H=1,\quad n=\frac{1}{2},\ \alpha=-0.3,\quad \beta=0.9,\quad \omega=0.35,\quad \gamma=0.55, W(t)=3t,\quad \chi=1,\quad \sigma=-1,\quad \kappa=-3,\quad b_4=0.33,\quad \varsigma=1.$

Utilizing equations (2.1), (4.1), (4.8),and (4.16), the acquisition of the subsequent hyperbolic solutions:

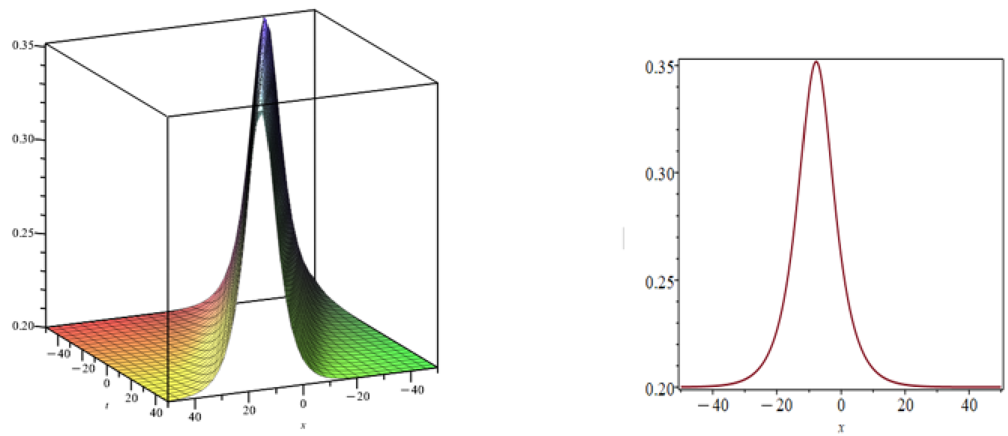
$$Q_9\left(x,t\right)=\left[\pm\frac{\sqrt{-\frac{-8n^3\pi_1-2n\pi_0\sigma^2-4n^2\pi_1-\pi_0\sigma^2}{24b_4}}}{n}\right.$$

$$\left.\pm\frac{\sqrt{-\frac{2n\pi_0 l+l\pi_0}{4b_4}}\ \sigma}{n}\left(\frac{2\varsigma H}{(H^2-l)cosh\left(\sigma\left(x-ct\right)\right)+\left(H^2+l\right)sinh\left(\sigma(x-ct)\right)}\right)\right]^{\frac{1}{2n}}$$

$$\times\, e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}. \tag{4.18}$$



(A)

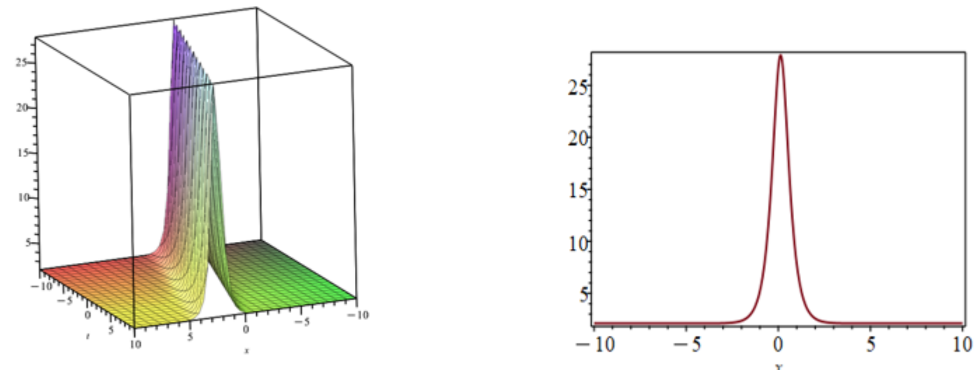


(B)

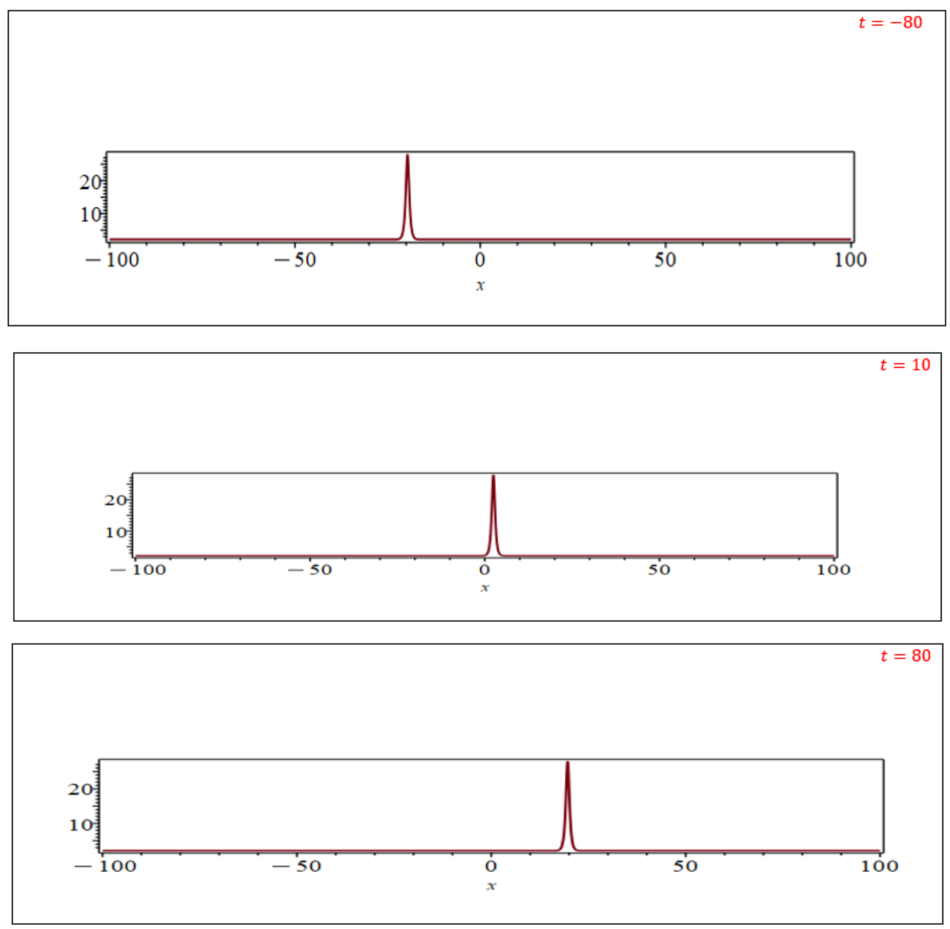
FIGURE 5. The dark and bell-shaped plots of $Q_9\ (x,\ t)$ and $Q_{13}\ (x,t)$ where $l=-1,\quad H=0.2\ ,\quad n=\frac{1}{2}\ ,\quad \alpha=0.3\ ,\quad \beta=0.4\ ,\quad \delta=0.5\ ,\quad \omega=0.35\ ,\quad \gamma=0.5\ ,\quad W(t)=3t\ ,\quad \chi=1\ ,\quad \sigma=-0.2\ ,\quad \kappa=-3,\quad b_4=0.33\ ,\varsigma=1,\quad h_0=0.2$.

In case, we insert $l=-H^2$ into Eq.(4.18), the following bright solitary wave solitons are acquired:

$$Q_{10}(x,t)=\left[\pm\frac{\sqrt{-\frac{-8n^3\pi_1-2n\pi_0\sigma^2-4n^2\pi_1-\pi_0\sigma^2}{24b_4}}}{n}\pm\frac{\sqrt{-\frac{2n\pi_0 l+l\pi_0}{4b_4}}\sigma}{n}\left(\frac{\varsigma sech\left(\sigma\left(x-ct\right)\right)}{H}\right)\right]^{\frac{1}{2n}},$$
$$\times\, e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}, \tag{4.19}$$



(A)



(B)

FIGURE 6. The bright soliton graphs of $Q_{10}(x,t)$ where $l = -1$, $H = 0.2$, $n = \frac{1}{2}$, $\alpha = 0.3$, $\beta = 0.4$, $\delta = 1$, $\omega = 0.35$, $\gamma = 0.5$, $W(t) = 3\,t$, $\chi = 1$, $\sigma = -2.5$, $\kappa = -3$, $b_4 = 0.33, \varsigma = 1$.

In case, we insert $l = H^2$ into Eq.(4.18), the following singular solutions are acquired:

$$Q_{11}(x,t) = \left[\pm \frac{\sqrt{-\frac{-8n^3\pi_1 - 2n\pi_0\sigma^2 - 4n^2\pi_1 - \pi_0\sigma^2}{24b_4}}}{n} \pm \frac{\sqrt{-\frac{2n\pi_0 l + l\pi_0}{4b_4}}\sigma}{n} \left(\frac{\varsigma csch\left(\sigma\left(x-ct\right)\right)}{H} \right) \right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}, \quad (4.20)$$

Result 3:

$$\begin{aligned} h_1 &= \pm\sqrt{-\frac{6l\pi_0}{4n^2\pi_1 + \pi_0\sigma^2}} h_0\sigma, \\ b_4 &= \frac{(2n+1)\left(4n^2\pi_1 + \pi_0\sigma^2\right)}{24h_0^2 n^2} \\ c_2 &= \frac{h_0(2n^3\pi_1 - n\pi_0\sigma^2 - 2n^2\pi_1 + \pi_0\sigma^2)}{3n^2} \\ c_4 &= -\frac{h_0^2\left(8n^3\pi_1 - 10n\pi_0\sigma^2 - 4n^2\pi_1 + 5\pi_0\sigma^2\right)}{24n^2}, \\ b_2 &= -\frac{(n+1)\left(4n^2\pi_1 + \pi_0\sigma^2\right)}{6h_0 n^2}. \end{aligned} \quad (4.21)$$

Utilizing equations (2.1), (4.2), (4.8), and (4.21), the following solutions are acquired:

$$Q_{12}(x,t) = \left[h_0 \pm \sqrt{-\frac{6l\pi_0}{4n^2\pi_1 + \pi_0\sigma^2}} h_0\sigma \left(\frac{2\varsigma H}{\left(H^2 e^{\varsigma\sigma(x-ct)} \mp l e^{-\varsigma\sigma(x-ct)}\right)} \right) \right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}. \quad (4.22)$$

Utilizing equations (2.1), (4.1), (4.8), and (4.21),the acquisition of the subsequent hyperbolic solutions:

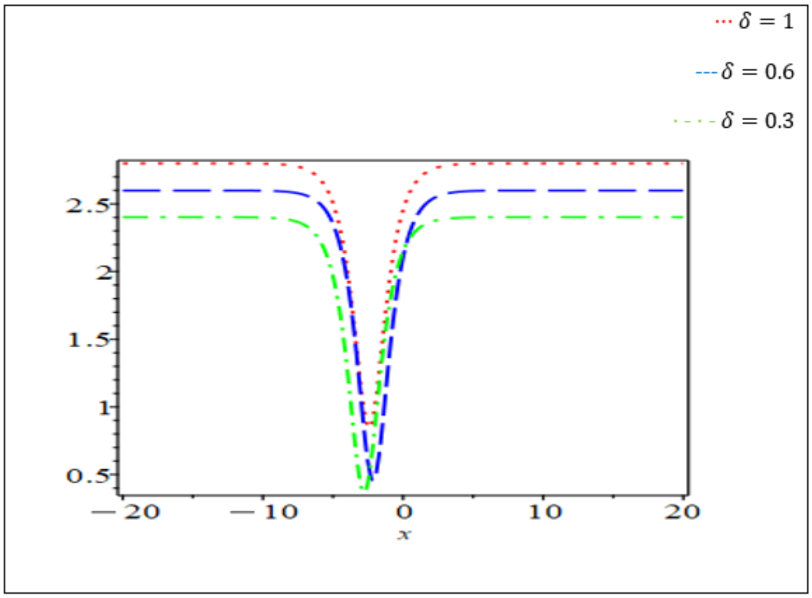
$$Q_{13}(x,t) = \left[h_0 \pm \sqrt{-\frac{6l\pi_0}{4n^2\pi_1 + \pi_0\sigma^2}} h_0\sigma \left(\frac{2\varsigma H}{(H^2 - l)\cosh\left(\sigma\left(x-ct\right)\right) + (H^2 + l)\sinh\left(\sigma(x-ct)\right)} \right) \right]^{\frac{1}{2n}}, \times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}. \quad (4.23)$$

In case, we insert $l = -H^2$ into Eq.(4.23), the following bright solitary wave solitons are acquired:

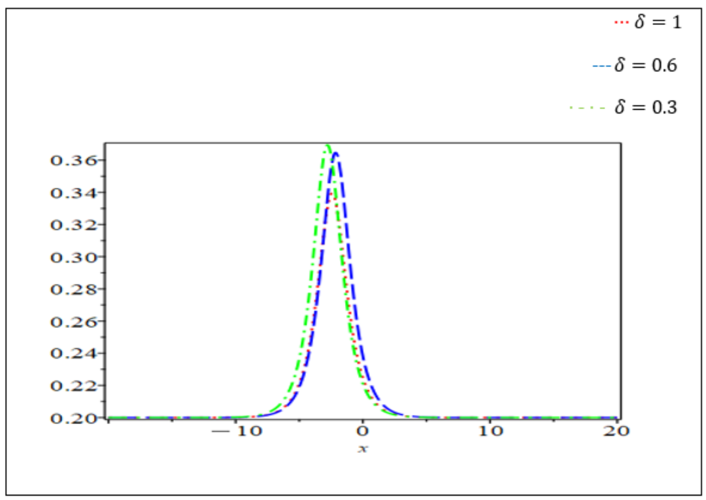
$$Q_{14}(x,t) = \left[h_0 \pm \sqrt{-\frac{6l\pi_0}{4n^2\pi_1 + \pi_0\sigma^2}} h_0\sigma \times \left(\frac{\varsigma sech\left(\sigma\left(x-ct\right)\right)}{H} \right) \right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)}. \quad (4.24)$$

In case, we insert $l = H^2$ into Eq.(4.23), the following singular solutions are acquired:

$$Q_{15}(x,t) = \left[h_0 \pm \sqrt{-\frac{6l\pi_0}{4n^2\pi_1 + \pi_0\sigma^2}} h_0\sigma \times \left(\frac{\varsigma csch\left(\sigma\left(x-ct\right)\right)}{H} \right) \right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \chi W(t) - \chi^2 t)} \quad (4.25)$$



(A)

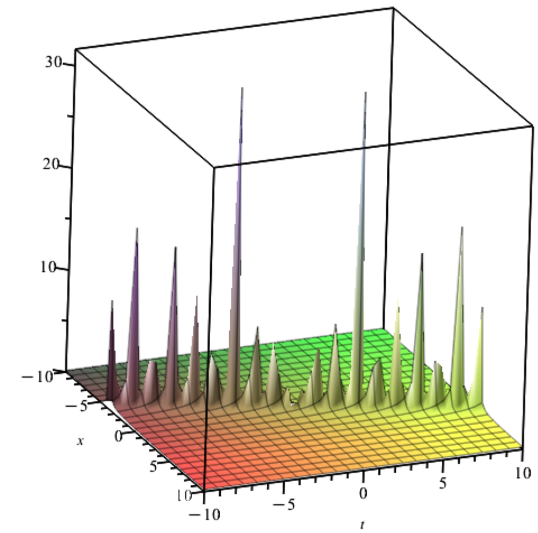


(B)

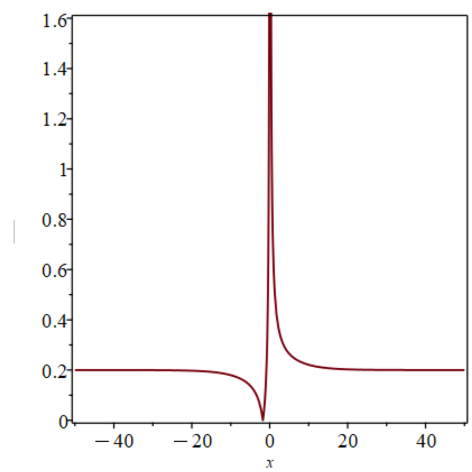
FIGURE 7. The dark plot of $Q_8\ (x,t)$ and the bright plot of $Q_{13}\ (x,t)$ where $l=-1,\ H\ =1\ ,\ \ n\ =\frac{1}{2},\ \ \alpha=-0.3\ ,\ \ \beta=0.4\ ,\ \ \omega=0.35\ ,\ \ \gamma=0.5\ ,\ \ W\,(t)=3t\ ,\ \ \chi=1\ \ ,\ \sigma=-1\ ,\ \ \kappa=-3,\ \ \ h_0=0.2\ ,\ \ b_4=0.33\ ,\ \varsigma=-1,t=2.$

Result 4:

$$
\begin{aligned}
h_0 &= \frac{3n^2c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)},\\
c_4 &= -\frac{3c_2^2n^2\left(8n^3\pi_1-10n\pi_0\sigma^2-4n^2\pi_1+5\pi_0\sigma^2\right)}{8(2n^2\pi_1-\pi_0\sigma^2)^2(n-1)^2},\\
b_2 &= -\frac{1}{18c_2n^4}((n+1)(8n^5\pi_1^2-2n^3\sigma^2\pi_0\pi_1-n\sigma^4\pi_0^2-8n^4\pi_1^2+2n^2\sigma^2\pi_0\pi_1+\sigma^4\pi_0^2)),\\
h_1 &= \pm\frac{6\sqrt{-\frac{3l\pi_0}{8n^2\pi_1+2\pi_0\sigma^2}}\ n^2\sigma c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)},\quad \sigma=\sigma\\
b_4 &= \frac{1}{216n^6c_2^2}((2n+1)\left(16n^8\pi_1^3-12n^6\sigma^2\pi_0\pi_1^2+n^2\sigma^6\pi_0^3-32n^7\pi_1^3\right.\\
&\left.+24n^5\sigma^2\pi_0\pi_1^2-2n\sigma^6\pi_0^3+16n^6\pi_1^3-12n^4\sigma^2\pi_0\pi_1^2+\sigma^6\pi_0^3\right)).
\end{aligned}
\tag{4.26}
$$



(A)



(B)

FIGURE 8. The singular solution's plots of $Q_{14}(x,t)$, where $l = -1$,$H = 1$,$n = \frac{1}{2}$,$\alpha = 0.2$,$\beta = -0.1$,$\omega = 0.35$,$\delta = 1$, $\gamma = 0.75$, $W(t) = 3t$,$\chi = 1$, $\sigma = -0.2$,$\kappa = -3$,$h_0 = 0.2$,$\xi = -1$

Utilizing equations (2.1), (4.2), (4.8), and (4.26),the following solutions are acquired:

$$Q_{16}\left(x,t\right) = \left[\frac{3n^2c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \pm \frac{6\sqrt{-\frac{3l\pi_0}{8n^2\pi_1+2\pi_0\sigma^2}}\; n^2\sigma c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \times \left(\frac{2\varsigma H}{(H^2e^{\varsigma\sigma(\ x-ct)} \mp l\ e^{-\varsigma\sigma(x-ct)})}\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)} \tag{4.27}$$

Utilizing equations (2.1), (4.1), (4.8) and (4.26) the acquisition of the subsequent hyperbolic solutions:

$$Q_{17}(x,t) = \left[\frac{3n^2c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \pm \frac{6\sqrt{-\frac{3l\pi_0}{8n^2\pi_1+2\pi_0\sigma^2}}n^2\sigma c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \left(\frac{2\varsigma H}{(H^2-l)cosh(\sigma(x-ct))+(H^2+l)sinh(\sigma(x-ct))}\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}. \tag{4.28}$$

In case, we insert $l = -H^2$ into Eq.(4.28), the following bright solitary wave solitons are acquired:

$$Q_{18}(x,t) = \left[\frac{3n^2c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \pm \frac{6\sqrt{-\frac{3l\pi_0}{8n^2\pi_1+2\pi_0\sigma^2}}n^2\sigma c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \times \left(\frac{\varsigma sech(\sigma(x-ct))}{H}\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}. \tag{4.29}$$

In case, we insert $l = H^2$ into Eq.(4.28), the following singular solutions are acquired:

$$Q_{19}(x,t) = \left[\frac{3n^2c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \pm \frac{6\sqrt{-\frac{3l\pi_0}{8n^2\pi_1+2\pi_0\sigma^2}}n^2\sigma c_2}{(n-1)(2n^2\pi_1-\pi_0\sigma^2)} \times \left(\frac{\varsigma csch(\sigma(x-ct))}{H}\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}. \tag{4.30}$$

Result 5:

$$\begin{aligned}
h_1 &= \pm\sqrt{-\frac{4ln\pi_1h_0-4l\pi_1h_0-6lc_2}{2n\pi_1h_0-2\pi_1h_0-c_2}}h_0,\\
b_2 &= -\frac{(n+1)(2n\pi_1h_0-2\pi_1h_0-c_2)}{2(n-1)h_0^2},\\
b_4 &= +\frac{(2n+1)(\ 2n\pi_1h_0-2\pi_1h_0-c_2)}{8h_0^3(n-1)},\\
\sigma &= \pm\sqrt{-\frac{-2n\pi_1h_0+2\pi_1h_0+3c_2}{n\pi_0h_0-\pi_0h_0}}n,\\
c_4 &= \frac{h_0(4n^2\pi_1h_0-6n\pi_1h_0-10nc_2+2\pi_1h_0+5c_2}{8(n-1)},
\end{aligned} \tag{4.31}$$

Utilizing equations (2.1), (4.2), (4.8) and (4.31), the following solutions are acquired:

$$Q_{20}(x,t) = \left[h_0 \pm \sqrt{-\frac{4ln\pi_1h_0-4l\pi_1h_0-6lc_2}{2n\pi_1h_0-2\pi_1h_0-c_2}}h_0\left(\frac{2\varsigma H}{(H^2e^{\varsigma\sigma(\ x-ct)} \mp l\ e^{-\varsigma\sigma(\ x-ct)})}\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}, \tag{4.32}$$

Utilizing equations (2.1), (4.1), (4.8) and (4.31) the acquisition of the subsequent hyperbolic solutions:

$$Q_{21}(x,t)=\Bigg[h_0\pm\sqrt{-\frac{4ln\pi_1h_0-4l\pi_1h_0-6lc_2}{2n\pi_1h_0-2\pi_1h_0-c_2}}h_0,$$
$$\times\left(\frac{2\varsigma H}{(H^2-l)\,cosh\,(\sigma\,(x-ct))+(H^2+l)\,sinh\,(\sigma(x-ct))}\right)\Bigg]^{\frac{1}{2n}}$$
$$\times\, e^{i(-\kappa x+\omega t+\chi W(t)-\chi^2 t)}. \tag{4.33}$$

In case, we insert $l=-H^2$ into Eq.(4.33), the following bright solitary wave solitons are acquired:

$$Q_{22}(x,t)=\left[h_0\pm\sqrt{-\frac{4ln\pi_1h_0-4l\pi_1h_0-6lc_2}{2n\pi_1h_0-2\pi_1h_0-c_2}}h_0\times\left(\frac{\varsigma sech(\sigma(x-c)t))}{H}\right)\right]^{\frac{1}{2n}}$$
$$\times\, e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}, \tag{4.34}$$

In case, we insert $l=H^2$ into Eq.(4.33), the following singular solutions are acquired:

$$Q_{23}(x,t)=\left[h_0\pm\sqrt{-\frac{4ln\pi_1h_0-4l\pi_1h_0-6lc_2}{2n\pi_1h_0-2\pi_1h_0-c_2}}h_0\times\left(\frac{\varsigma csch(\sigma(x-c)t))}{H}\right)\right]^{\frac{1}{2n}}$$
$$\times\, e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}, \tag{4.35}$$

Where

$$\pi_0=\gamma-c\beta+\alpha,\quad \pi_1=(\beta\kappa-1)(\omega-\chi^2)-\delta\kappa-\alpha\kappa^2,$$
$$c=\frac{2\alpha\kappa+\delta-\beta(\omega-\chi^2)}{\beta\kappa-1}$$

Now, in order to explore numerous solitary wave solutions for the present model using the Bernoulli's equation method,we employ Eq.(4.5).This equation is transformed into the subsequent form by applying the homogeneous balance principle, resulting in $M=1$:

$$\Phi(y)=b_0+b_1W(y), \tag{4.36}$$

Here, by incorporating the aforementioned solution into Eq.(2.11) and employing both Eq.(4.7) and its derivative, we derive a polynomial representing the function $W(y)$. The determination of the coefficients corresponding to various powers of $W(y)$ leads to the following system of algebraic equations:

$$(W(y))^0: 4n^2c_4+4n^2c_2b_0+4n^2\pi_1b_0^2+4n^2b_2b_0^3+4n^2b_4b_0^4=0,$$
$$(W(y))^1: 4n^2c_2b_1+8n^2\pi_1b_0b_1+2\pi_0nb_0b_1r^2+12n^2b_2b_0^2b_1+16$$
$$n^2b_4b_0^3b_1=0,$$
$$(W(y))^2: \pi_0b_1^2r^2-6n\pi_0b_0b_1r+4n^2\pi_1b_1^2+12n^2b_2b_0b_1^2+24n^2$$
$$b_4b_1^2b_0^2=0,$$
$$(W(y))^3: -2\pi_0b_1^2r+4n\pi_0b_0b_1-2n\pi_0b_1^2r+4n^2b_2b_1^3+16n^2,$$
$$b_4b_1^3b_0=0,$$
$$(W(y))^4: \pi_0b_1^2+2n\pi_0b_1^2+4n^2b_4b_1^4=0.$$

Solving the system using Mathematica or Maple yields these results:
Result 1:
Case1.1

$$
\begin{aligned}
b_0 &= \frac{(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})b_1}{6\pi_0},\\
b_2 &= \frac{\pi_0}{2b_1n^2}(\frac{(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})n}{3\pi_0}+nr\\
&+\frac{-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})}{3\pi_0}+r),\\
c_4 &= \frac{1}{144\pi_0n^2}(b_1^2(32n^5\pi_1^2+16n^3r^2\pi_0\pi_1+2nr^4\pi_0^2-16n^4\pi_1^2\\
&-8n^2r^2\pi_0\pi_1-r^4\pi_0^2)),\\
&b_1=b_1,\\
&r=r,\quad b_4=-\frac{\pi_0(2n+1)}{4n^2b_1^2},\\
c_2 &= \frac{1}{12n^2}(b_1(\frac{4(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})n^3\pi_1}{3\pi_0},\\
&+\frac{(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})nr^2}{3}+4n^3r\pi_1+nr^3\pi_0\\
&-\frac{4(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})n^2\pi_1}{3\pi_0}\\
&-\frac{(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})r^2}{3}-4n^2r\pi_1-r^3\pi_0)).
\end{aligned}
$$

Case 1.2

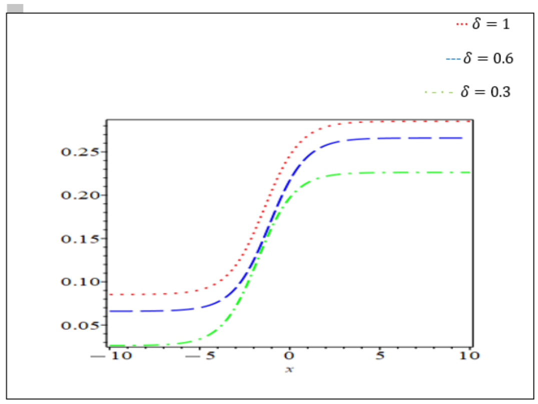
$$\begin{aligned}
b_0 &= -\frac{(3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})b_1}{6\pi_0},\\
b_2 &= \frac{\pi_0}{2b_1n^2}(-\frac{(3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})n}{3\pi_0}+nr\\
&\quad+\frac{3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})}{3\pi_0}+r),\\
c_4 &= \frac{1}{144\pi_0n^2}(b_1^2(32n^5\pi_1^2+16n^3r^2\pi_0\pi_1+2nr^4\pi_0^2-16n^4\pi_1^2\\
&\quad-8n^2r^2\pi_0\pi_1-r^4\pi_0^2)),\\
b_1 &= b_1,\\
r &= r,\quad b_4=-\frac{\pi_0(2n+1)}{4n^2b_1^2},
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
c_2 &= \frac{1}{12n^2}(b_1(-\frac{4(3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})n^3\pi_1}{3\pi_0}\\
&\quad-\frac{(3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})nr^2}{3}+4n^3r\pi_1+nr^3\pi_0\\
&\quad+\frac{4(3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})n^2\pi_1}{3\pi_0}\\
&\quad+\frac{(3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2})r^2}{3}-4n^2r\pi_1-r^3\pi_0)).
\end{aligned} \tag{4.38}$$

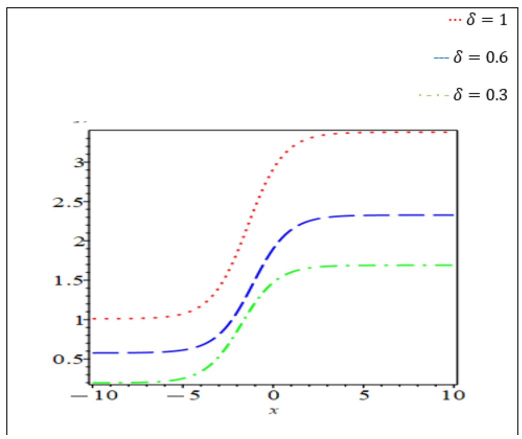
Utilizing (2.1), (4.6), (4.36), and (4.37), the following hyperbolic function solutions are achieved:

$$\begin{aligned}
Q_{24}(x,t) = &\left[\frac{\left(-3r\pi_0+\sqrt{-24n^2\pi_0\pi_1+3r^2\pi_0^2}\right)b_1}{6\pi_0}+b_1\left(\frac{r}{2}+\frac{r}{2}tanh\left(\frac{r}{2}(x-ct)\right)\right)\right]\\
&\times e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}.
\end{aligned} \tag{4.39}$$

Result 2:



(A)



(B)

FIGURE 9. The kink-type plots of $Q_{24}(x,t))$ and $Q_{26}(x,t)$, where, $r = 1$, $H = -0.2$, $n = \frac{1}{2}$, $\alpha = 0.2$, $\beta = -0.4$, $\omega = 0.35$, $\gamma = 0.75$, $W(t) = 3t$, $\chi = 1$, $\sigma = -1$, $\kappa = 3, b_2 = 1.7$, $b_1 = 0.2$, $t = 2$.

Case 2.1

$$
\begin{aligned}
b_0 &= \frac{(nb_2b_1 + \sqrt{3n^2b_1^2b_2^2 + 2n^2\pi_0\pi_1 + 4n\pi_0\pi_1 + 2\pi_0\pi_1})\, b_1 n}{(n+1)\pi_0}, \\
b_4 &= -\frac{\pi_0\,(2n+1)}{4n^2b_1^2}, \\
c_2 &= \frac{1}{\pi_0^2\,(n+1)^3}(2n^2b_2b_1^2\,(n^3b_1^2\,b_2^2 - n^2b_1^2\,b_2^2 + n^3\pi_0\pi_1 + n^2\pi_0\pi_1 \\
&\quad - n\pi_0\pi_1 - \pi_0\pi_1)), \\
c_4 &= \frac{1}{\pi_0^3(n+1)^4}(n^2b_1^2(2n^5b_1^4\,b_2^4 - n^4b_1^4\,b_2^4 + 4n^5\pi_0\pi_1b_1^2b_2^2 + 6n^4 \\
&\quad \pi_0\pi_1b_1^2\,b_2^2 + 2n^5\pi_0^2\pi_1^2 + 7n^4\pi_0^2\pi_1^2 - 2n^2\pi_0\pi_1b_1^2\,b_2^2 + 8n^3\pi_0^2\pi_1^2 \\
&\quad +2n^2\pi_0^2\pi_1^2 - 2n\pi_0^2\pi_1^2 - \pi_0^2\pi_1^2)), \\
r &= -\frac{2\sqrt{3n^2b_1^2\,b_2^2 + 2n^2\pi_0\pi_1 + 4n\pi_0\pi_1 + 2\pi_0\pi_1}\;n}{\pi_0\,(n+1)}.
\end{aligned}
\tag{4.40}
$$

Case 2.2

$$b_0 = \frac{(nb_2b_1 - \sqrt{3n^2b_1^2b_2^2 + 2n^2\pi_0\pi_1 + 4n\pi_0\pi_1 + 2\pi_0\pi_1})\ b_1n}{(n+1)\pi_0}\ ,$$

$$b_4 = -\frac{\pi_0\,(2n+1)}{4n^2b_1^2}\ ,$$

$$c_2 = \frac{1}{\pi_0^2\ (n+1)^3}(2n^2b_2b_1^2\ (n^3b_1^2\ b_2^2 - n^2b_1^2\ b_2^2 + n^3\pi_0\pi_1 + n^2\pi_0\pi_1 - n\pi_0\pi_1 - \pi_0\pi_1)),$$

$$c_4 = \frac{1}{\pi_0^3(n+1)^4}(n^2b_1^2\ (2n^5b_1^4\ b_2^4 - n^4b_1^4\ b_2^4 + 4n^5\pi_0\pi_1b_1^2\ b_2^2 + 6n^4 \pi_0\pi_1b_1^2\ b_2^2 + 2n^5\pi_0^2\pi_1^2 + 7n^4\pi_0^2\pi_1^2 - 2n^2\pi_0\pi_1b_1^2\ b_2^2 + 8n^3\pi_0^2\pi_1^2 + 2n^2\pi_0^2\pi_1^2 - 2n\pi_0^2\pi_1^2 - \pi_0^2\pi_1^2)),$$

$$r = \frac{2\sqrt{3n^2b_1^2b_2^2 + 2n^2\pi_0\pi_1 + 4n\pi_0\pi_1 + 2\pi_0\pi_1}n}{\pi_0\,(n+1)}, \tag{4.41}$$

Utilizing (2.1), (4.6), (4.40), and (4.41), the following hyperbolic function solutions are achieved:

$$Q_{25}\,(x,t) = \left[\frac{(nb_2b_1 + \sqrt{3n^2b_1^2b_2^2 + 2n^2\pi_0\pi_1 + 4n\pi_0\pi_1 + 2\pi_0\pi_1})\ b_1n}{(n+1)\pi_0} + b_1\left(\frac{r}{2} + \frac{r}{2}tanh\left(\frac{r}{2}(x-ct)\right)\ \right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)} \tag{4.42}$$

Result 3:

Case 3.1

$$b_0 = -\frac{8n^3\pi_1 - nr^2\pi_0 + 3r\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}n}{12n^2b_2} + \frac{8n^2\pi_1 - r^2\pi_0 + 3r\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}}{12n^2b_2}$$

$$b_1 = \frac{\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}(n+1)}{2b_2n^2}$$

$$b_4 = \frac{3\,(2n+1)\,n^2b_2^2}{8n^4\pi_1 - n^2r^2\pi_0 + 16n^3\pi_1 - 2nr^2\pi_0 + 8n^2\pi_1 - r^2\pi_0}$$

$$c_2 = -\frac{1}{72b_2n^4}(32n^6\pi_1^2 + 4n^4r^2\pi_0\pi_1 - n^2r^4\pi_0^2 - 32n^4\pi_1^2 - 4n^2r^2\pi_0\pi_1 + r^4\pi_0^2),$$

$$c_4 = -\frac{1}{1728n^6b_2^2}(256n^9\pi_1^3 + 96n^7r^2\pi_0\pi_1^2 - 2n^3r^6\pi_0^3 + 384n^8\pi_1^3 + 144n^6r^2\pi_0\pi_1^2 - 3n^2r^6\pi_0^3 - 128n^6\pi_1^3 - 48n^4r^2\pi_0\pi_1^2 + r^6\pi_0^3). \tag{4.43}$$

Case 3.2

$$b_0 = -\frac{8n^3\pi_1 - nr^2\pi_0 - 3r\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}\, n}{12b_2n^2}$$
$$\frac{8n^2\pi_1 - r^2\pi_0 - 3r\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}}{12b_2n^2},$$
$$b_1 = -\frac{\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}\,(n+1)}{2b_2n^2},$$
$$b_4 = \frac{3\,(2n+1)\,n^2b_2^2}{8n^4\pi_1 - n^2r^2\pi_0 + 16n^3\pi_1 - 2nr^2\pi_0 + 8n^2\pi_1 - r^2\pi_0}$$
$$c_2 = -\frac{1}{72b_2n^4}(32n^6\pi_1^2 + 4n^4r^2\pi_0\pi_1 - n^2r^4\pi_0^2 - 32n^4\pi_1^2 - 4n^2 + r^2\pi_0\pi_1 + r^4\pi_0^2))$$
$$c_4 = -\frac{1}{1728n^6b_2^2}(256n^9\pi_1^3 + 96n^7r^2\pi_0\pi_1^2 - 2n^3r^6\pi_0^3 + 384n^8\pi_1^3$$
$$+ 144n^6r^2\pi_0\pi_1^2 - 3n^2r^6\pi_0^3 - 128n^6\pi_1^3 - 48n^4r^2\pi_0\pi_1^2 + r^6\pi_0^3). \tag{4.44}$$

Utilizing (2.1), (4.6), (4.36), and (4.44), the following hyperbolic function solutions are achieved:

$$Q_{26}(x,t) = \left[\left(-\frac{8n^3\pi_1 - nr^2\pi_0 + 3r\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}\, n}{12b_2n^2} + \frac{8n^2\pi_1 - r^2\pi_0 + 3r\sqrt{-\frac{8}{3}n^2\pi_0\pi_1 + \frac{1}{3}r^2\pi_0^2}}{12b_2n^2}\right) + \frac{\sqrt{-\frac{8}{3}n^2\pi_0\pi_1+\frac{1}{3}r^2\pi_0^2}\,(n+1)}{2b_2n^2}\left(\frac{r}{2} + \frac{r}{2}tanh\left(\frac{r}{2}(x-ct)\right)\right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)} \tag{4.45}$$

Result 4:

$$b_1 = \pm\frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{n},$$
$$c_4 = \frac{1}{2n+1}\left(b_0^2\left(\frac{16\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\, b_0b_4\right.}{\pi_0\,(2n+1)}\right.\right.$$
$$+\frac{\sqrt{-6\pi_0b_4b_0^2\, n - 4n^2\pi_0\pi_1 - 3\pi_0b_4b_0^2 - 4n\pi_0\pi_1 - \pi_0\pi_1}\,\Big)\, nb_4b_0}{\pi_0\,(2n+1)}$$
$$\times\frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{\pi_0\,(2n+1)} - \frac{8\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\, b_0b_4\right.}{\pi_0\,(2n+1)}$$
$$+\frac{\sqrt{-6\pi_0b_4b_0^2\, n - 4n^2\pi_0\pi_1 - 3\pi_0b_4b_0^2 - 4n\pi_0\pi_1 - \pi_0\pi_1}\,\Big)}{\pi_0\,(2n+1)}$$

$$\times \frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}b_4b_0}{\pi_0\left(2n+1\right)} + 10nb_4b_0^2 - 4n^2\pi_1 - 5b_4b_0^2 + \pi_1)\Bigg)\Bigg),$$

$$c_2 = -\frac{1}{2n+1}\Bigg(2b_0\Bigg(\frac{6\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}$$

$$+\frac{\left.\sqrt{-6\pi_0b_4b_0^2\ n - 4n^2\pi_0\pi_1 - 3\pi_0b_4b_0^2 - 4n\pi_0\pi_1 - \pi_0\pi_1}\right) nb_4b_0}{\pi_0\left(2n+1\right)}$$

$$\times \frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{\pi_0\left(2n+1\right)} - \frac{6\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}$$

$$+\frac{\left.\sqrt{-6\pi_0b_4b_0^2\ n - 4n^2\pi_0\pi_1 - 3\pi_0b_4b_0^2 - 4n\pi_0\pi_1 - \pi_0\pi_1}\right)}{\pi_0\left(2n+1\right)}$$

$$\times \frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}b_4b_0}{\pi_0\left(2n+1\right)} + 4nb_4b_0^2 - 2n^2\pi_1 - 4b_4b_0^2 + n\pi_1 + \pi_1\Bigg)\Bigg),$$

$$b_2 = -\frac{\frac{4b_4\left(n+1\right)\left(\frac{\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0(2n+1)} + \frac{\left.\sqrt{-6\pi_0b_4b_0^2\ n-4n^2\pi_0\pi_1-3\pi_0b_4b_0^2 -4n\pi_0\pi_1-\pi_0\pi_1}\right)}{\pi_0(2n+1)}\right)}{2n+1}}{2n+1} \times \frac{\frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{\pi_0(2n+1)} + b_0}{2n+1},$$

$$r = \pm\frac{2\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)} + \frac{\left.\sqrt{-6\pi_0b_4b_0^2\ n - 4n^2\pi_0\pi_1 - 3\pi_0b_4b_0^2 - 4n\pi_0\pi_1 - \pi_0\pi_1}\right)\ n}{\pi_0\left(2n+1\right)}. \tag{4.46}$$

Case 4.2

$$b_1 = \pm\frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{n},$$

$$c_4 = \frac{1}{2n+1}\Bigg(b_0^2\Bigg(\frac{16\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}$$

$$-\frac{\left.\sqrt{-6\pi_0b_4b_0^2\ n - 4n^2\pi_0\pi_1 - 3\pi_0b_4b_0^2 - 4n\pi_0\pi_1 - \pi_0\pi_1}\ \right) nb_4b_0}{\pi_0\left(2n+1\right)}$$

$$\times\frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{\pi_0\left(2n+1\right)}-\frac{8\ \left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}$$

$$\left.-\frac{\sqrt{-6\pi_0b_4b_0^2\ n-4n^2\pi_0\pi_1-3\pi_0b_4b_0^2\ -4n\pi_0\pi_1-\pi_0\pi_1}\ \right)}{\pi_0\left(2n+1\right)}$$

$$\left.\left.\times\ \frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}b_4b_0}{\pi_0\left(2n+1\right)}+10nb_4b_0^2\quad-4n^2\pi_1-5b_4b_0^2\quad+\pi_1)\right)\right),$$

$$c_2=-\frac{1}{2n+1}\left(2b_0\left(\frac{6\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}\right.\right.$$

$$-\frac{\sqrt{-6\pi_0b_4b_0^2\ n-4n^2\pi_0\pi_1-3\pi_0b_4b_0^2\ -4n\pi_0\pi_1-\pi_0\pi_1})nb_4b_0}{\pi_0\left(2n+1\right)}$$

$$\times\ \frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{\pi_0\left(2n+1\right)}-\frac{6\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}$$

$$-\frac{\sqrt{-6\pi_0b_4b_0^2\ n-4n^2\pi_0\pi_1-3\pi_0b_4b_0^2\quad-4n\pi_0\pi_1-\pi_0\pi_1})}{\pi_0\left(2n+1\right)}$$

$$\left.\left.\times\ \frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}b_4b_0}{\pi_0\left(2n+1\right)}+4nb_4b_0^2\quad-2n^2\pi_1-4b_4b_0^2\quad+n\pi_1+\pi_1\right)\right),$$

$$b_2=-\frac{4b_4\left(n+1\right)\ \left(\frac{\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0(2n+1)}-\right.}{2n+1}$$

$$\frac{\left.\frac{\sqrt{-6\pi_0b_4b_0^2\ n-4n^2\pi_0\pi_1-3\pi_0b_4b_0^2\ \ -4n\pi_0\pi_1-\pi_0\pi_1})}{\pi_0(2n+1)}\right)}{2n+1}$$

$$\times\frac{\frac{\sqrt{-\frac{2n\pi_0+\pi_0}{4b_4}}}{\pi_0(2n+1)}+b_0}{2n+1},$$

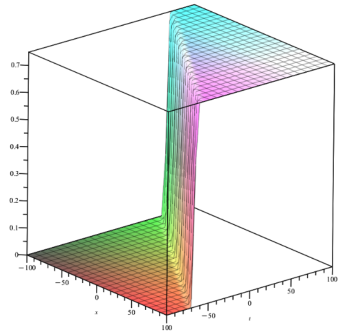
$$r=\pm\frac{2\left(3\sqrt{-\frac{\pi_0(2n+1)}{b_4}}\ b_0b_4\right.}{\pi_0\left(2n+1\right)}-$$

$$\frac{\sqrt{-6\pi_0b_4b_0^2\ n-4n^2\pi_0\pi_1-3\pi_0b_4b_0^2\quad-4n\pi_0\pi_1-\pi_0\pi_1}\Big)\ n}{\pi_0\left(2n+1\right)}. \tag{4.47}$$

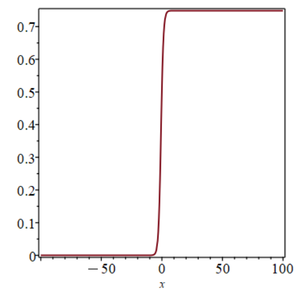
Utilizing (2.1), (4.6), (4.36), and (4.49), the following hyperbolic function solutions are achieved:

$$Q_{27}\left(x,t\right)\ \ =\left[b_0\pm\frac{\sqrt{-\frac{2\pi_0+\pi_0}{4b_4}}}{n}\right.$$

$$\times \left(\frac{r}{2} + tanh\left(\frac{r}{2}(x-ct)\right)\right)\Bigg]^{\frac{1}{2n}}$$
$$\times\, e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}. \tag{4.48}$$



(A)



(B)

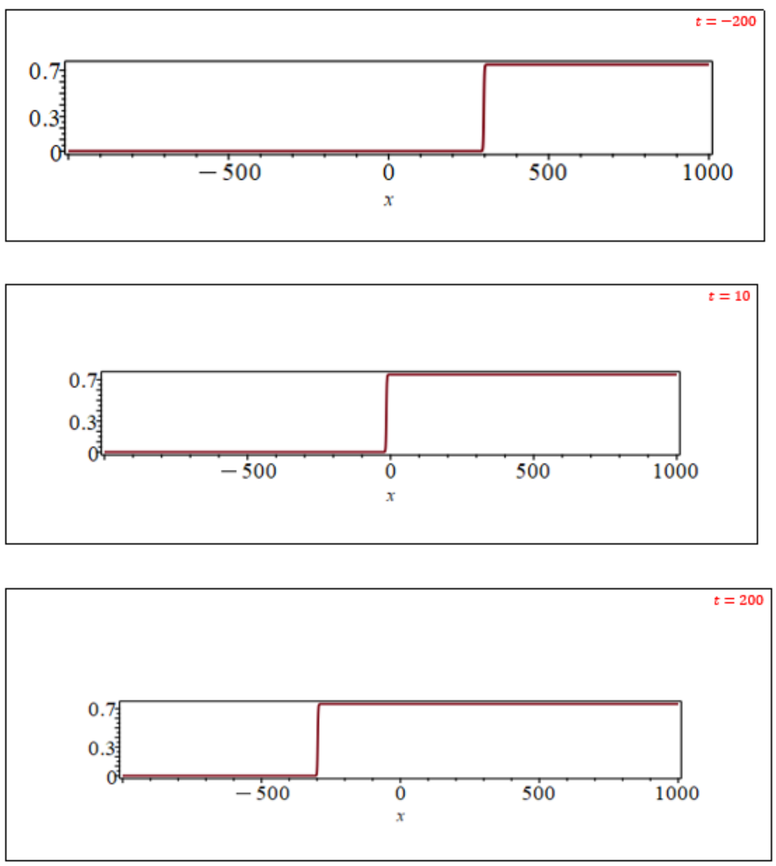


FIGURE 11. The kink type plots of $Q_{27}(x,t)$ where $r=1$, $H=-0.2$, $n=\frac{1}{2}$, $\alpha=0.3$, $\beta=-0.6$, $\omega=0.35$, $\delta=1$, $\gamma=0.5$, $W(t)=3t$, $\chi=1$, $\sigma=-1$, $\kappa=-3$, $b_0=0$, $b_4=0.33$.

5. Stability Analysis

From Eq(2.7),by using the Galilean transformation [27],we obtain:

$$\begin{cases} \frac{d\Phi}{dy} = U, \\ \frac{dU}{dy} = \frac{1}{2n\pi_0\Phi}(-(1-2n)\pi_0(\Phi')^2 - 4n^2c_4 - 4n^2c_2\Phi - 4n^2\pi_1\Phi^2 - 4n^2b_2\Phi^3 - 4n^2b_4\Phi^4), \end{cases} \tag{5.1}$$

Which is not the Hamiltonian system as it does not satisfies the Hamiltonian's condition [28]. By putting $n = 1/4$, the Eq.(5.1) becomes,

$$\begin{cases} \frac{d\Phi}{dy} = U, \\ \frac{dU}{dy} = \frac{(\Phi')^2+B+C\Phi+D\Phi^2+E\Phi^3+F\Phi^4}{\Phi}, \end{cases} \tag{5.2}$$

Where,$B = -\frac{c_4}{2\pi_0}$, $C = -\frac{c_2}{2\pi_0}$, $D = -\frac{\pi_1}{2\pi_0}$, $E = -\frac{b_2}{2\pi_0}$,$F = -\frac{b_4}{2\pi_0}$.
By using the Eq.(5.2), we obtain the following equation,

$$\frac{dU^2}{d\Phi} = \frac{2U^2}{\Phi} + 2B\Phi^{-1} + 2C + 2D\Phi + 2E\Phi^2 + 2F\Phi^3. \tag{5.3}$$

By solving the above Eq.(5.3), the following solution is obtained:

$$\frac{U^2}{2} - \left(\frac{B}{2} + 2C\Phi + D\Phi^2 + \frac{2}{3}E\Phi^3 + F\frac{\Phi^4}{2}\right) = 0. \tag{5.4}$$

We can see that Eq.(5.4), satisfies the Hamiltonian condition, we can write Eq.(5.4), as:

$$H(U,\Phi) = \frac{U^2}{2} - \left(\frac{B}{2} + 2C\Phi + D\Phi^2 + \frac{2}{3}E\Phi^3 + F\frac{\Phi^4}{2}\right) = h, \tag{5.5}$$

From Eq.(5.5), we have:

$$U = \pm\sqrt{2}\sqrt{h - \frac{B}{2} - 2C\Phi - D\Phi^2 - \frac{2}{3}E\Phi^3 - F\frac{\Phi^4}{2}}, \tag{5.6}$$

Now,

$$\frac{dU}{\pm\sqrt{2}\sqrt{h - \frac{B}{2} - 2C\Phi - D\Phi^2 - \frac{2}{3}E\Phi^3 - F\frac{\Phi^4}{2}}} = dy$$

$$\Longrightarrow \int \frac{dU}{\sqrt{h - \frac{B}{2} - 2C\Phi - D\Phi^2 - \frac{2}{3}E\Phi^3 - F\frac{\Phi^4}{2}}} = \pm\sqrt{2}\ (y + y_0). \tag{5.7}$$

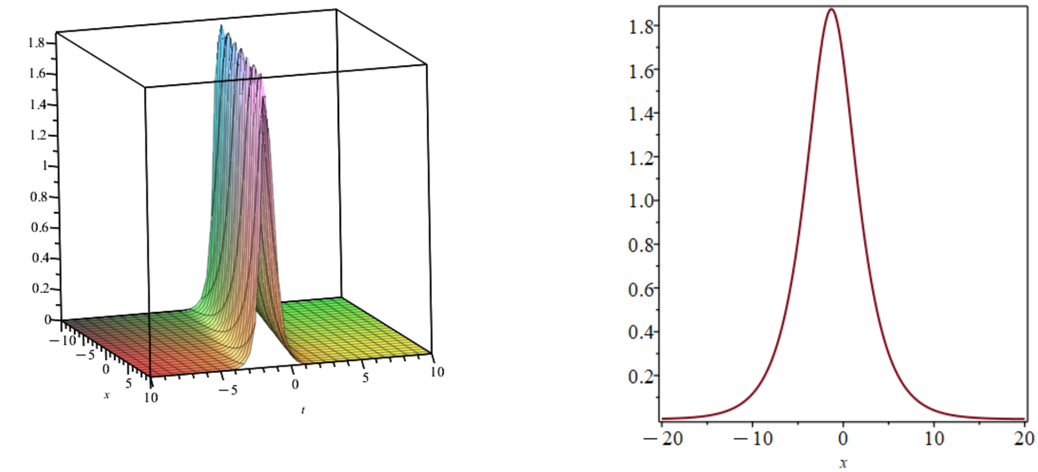
Eq.(5.6), let $f_0 = h - \frac{B}{2}$, $f_1 = -2C$, $f_2 = -D$, $f_3 = -\frac{2}{3}E$, $f_4 = -\frac{F}{2}$, then by follows [29], we have the following solutions,
When $f_0 = f_1 = f_3 = 0$, we have the following solution:

$$Q_{28}(x,t) = (\sqrt{\frac{-f_2}{f_4}}\ sech\ \sqrt{f_2}\ (y)) \times e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}. \tag{5.8}$$

With constraint condition $f_2 > 0$ and $f_4 < 0$.
When $f_1 = f_3 = 0$, we have the following solution,

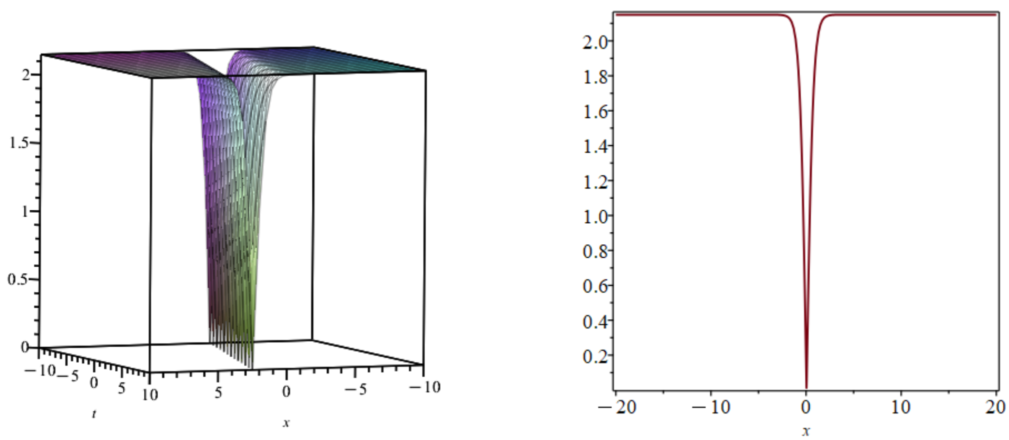
$$Q_{29}(x,t) = (\sqrt{\frac{-f_2}{2f_4}}\ tanh\sqrt{\frac{-f_2}{2}}\ (y))\, e^{i\left(-\kappa x+\omega t+\chi W(t)-\chi^2 t\right)}. \tag{5.9}$$

With constraint condition $f_2 < 0$ and $f_4 > 0$.



(A) (B)

FIGURE 12. bright soliton for $Q_{28}(x,t)$ where $\alpha = 0.3$, $\beta = -0.4$, $\omega = 0.35$, $\delta = 0.75$, $\gamma = 0.5$, $W(t) = 3t$, $\chi = 1$, $\kappa = -3$, $b_4 = 0.33$.



(A) (B)

FIGURE 13. dark soliton for $Q_{29}(x,t)$, where $\alpha = -0.2$, $\beta = 0.5$, $\omega = 0.35$, $\delta = 1$, $\gamma = 0.5$, $W(t) = 3t$, $\chi = 1$, $\kappa = 3$, $b_4 = 0.33$.

6. Conclusion

This article reported on the optical soliton solutions of the stochastic resonant nonlinear Schrödinger equation, integrating spatio-temporal dispersion, inter-modal dispersion, nonlinearity, and multiplicative white noise under generalized Kudryashov's law. This paper used the following methods: $(\frac{G\prime}{G^2})$-expansion method and the new kudryashov method, we got the extraction of numerous soliton solutions, from which bright and dark, periodic, and even singular solitons are attained, providing interesting details on how stochastic effects drive, complicated dynamics within the soliton's nonlinear optics systems.The presence of multiplicative noise significantly affects the propagation and stability of solitons in the interplay between STD and IMD. Graphical representations of the soliton solutions illustrate how the noise and dispersion interactions shape the dynamics of the solitons. The exact solutions of the SRNLSE, under the influence of multiplicative noise, also reveal its effects on the profiles of the solitons. We have analyzed the stability characteristics of critical points through stability analysis near equilibrium points, which gives a deeper insight into the dynamical behavior of the system. This analysis underpins the importance of nonlinear interactions and noise in determining the robustness of solitons. This research contributes to advancing the understanding of soliton behavior in nonlinear optical systems with stochastic influences. Future work could extend these ideas by considering fractional and stochastic fractional variants of the SRNLSE, as well as practical implications in optical communication systems, fiber lasers, and other nonlinear photonic devices.

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