

Unified framework, statistical properties and estimation of cubic transmuted Rayleigh distributions

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ABSTRACT. Transmutation is a widely used technique to enhance the flexibility of baseline probability distributions in statistical modelling. While the quadratic transmutation is unique, the cubic transmutation admits multiple formulations. This paper presents a unified investigation of cubic transmuted Rayleigh distributions from both theoretical and empirical perspectives. On the theoretical side, we revisit six existing cubic transmutation formulas and their modified versions, and we introduce a general formulation that encompasses these models while establishing their main statistical properties. To evaluate parameter estimation, simulation studies are conducted to assess the efficiency of maximum likelihood estimators, evaluating performance across different sample sizes and

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parameter settings, showing satisfactory performance across different scenarios. On the empirical side, real data analyses highlight the comparative performance of the proposed models, with some formulations providing improved fit and flexibility. Overall, this study offers a comprehensive framework that consolidates existing approaches, extends the family of cubic transmuted Rayleigh distributions, and provides practical guidance for their application in data analysis.

Keywords: Cubic transmutation, Rayleigh distribution, parameter estimation, maximum likelihood, numerical optimization.

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1. INTRODUCTION

Let $x \in \mathbb{R}$ and $G(x)$ be a cumulative distribution function (CDF) and $g(x)$ be the probability density function (PDF) linked to $G(x)$. In order to improve data fitting, some of the recent scientific research has focused on building functions R defined from $[0, 1]$ to itself such that the compound function $F(x) = R[G(x)]$ is a CDF and then fitting the data with $F(x) = R[G(x)]$ instead of $G(x)$. Such processes (known as extension or generalization of probability distributions) usually need adding one or more new parameters and have led to the emergence of new families of probability distributions (see, for example, [1, 2] for detailed reviews).

One of the most used compound functions is

$$F(x) = R[G(x)] = \sum_{i=1}^{k-1} \delta_i [G(x)]^i + \left(1 - \sum_{i=1}^{k-1} \delta_i\right) [G(x)]^k, \quad (1.1)$$

where k is an integer such that $k \geq 2$ and $\delta_1, \dots, \delta_{k-1}$ are real parameters defined to ensure that $R[G(x)]$ is also a CDF; that is (see, for example, [17]),

$$\inf_{t \in [0, 1]} \left[\sum_{i=1}^{k-1} i \delta_i t^{i-1} + k \left(1 - \sum_{i=1}^{k-1} \delta_i\right) t^{k-1} \right] \geq 0.$$

This technique, called transmutation of order $k - 1$ by some authors [15, 27, 28, 30] and transmutation of order k by others [4, 8], will be referred to in this paper as transmutation of order $k - 1$. It is worth noting that some authors (see, for example, [6, 22, 24, 32]) have used the formula (1.1) by replacing $G(x)$ by $G(x)^\alpha$ or $1 - (1 - G(x))^\alpha$, where $\alpha > 0$, and obtained new families which they also called transmuted family of distributions. However, this can be considered as a combination of transmutation and exponentiation [5], so it will not be considered in this study. In a related line of research, Balakrishnan and He [8] combined

transmutation with the theory of record values and developed record-based transmuted families of distributions. This new development has incited greater and greater interest in researchers. For example, Tanış [38] introduced the transmuted lower record type inverse Rayleigh distribution, which was shown to be suitable for modelling recovery times of Covid-19 patients. Other examples include the record-based transmuted power Lomax distribution [33] and the record-based transmuted Rayleigh distribution [21].

In the case $k = 2$ (quadratic transmutation), Shaw and Buckley [36] proved that the unique form possible is

$$R[G(x)] = (1 + \lambda)G(x) - \lambda G^2(x), \quad (1.2)$$

where λ is an additional parameter such that $|\lambda| \leq 1$ and setting $\lambda = 0$ enables to get the baseline CDF $G(x)$. Formula (1.2) has been applied to several baseline distributions, thus leading to a considerable number of quadratic transmuted distributions, all of which were proved to fit data better than the underlying baseline distributions [11, 16].

Unlike quadratic transmutation (transmutation of order 1), transmutation of order 2, called cubic transmutation (see, for example, [15]), is not unique. Six different cubic transmutation (CT) formulas have been developed respectively by Granzotto et al. [15], AL-Kadim and Mohammed [3], and Rahman et al. [27, 28, 29, 31]. They will be respectively denoted CT_G , CT_A , CT_{R18a} , CT_{R18b} , CT_{R19} and CT_{R23} . Real data applications suggest that these CTs generally fit the data better than the quadratic transmutation and the baseline distribution. Gerardo et al. [14] have made a comparative analysis of these CTs from both theoretical and empirical viewpoints and suggested improvements by modifying the parameters ranges. Their case study using Pareto distribution as baseline distribution suggests that modified CTs presented a better fit to the studied data than the unmodified CTs.

In this paper, we investigate various cubic transmutations of the Rayleigh distribution, which is an important statistical distribution with applications in survival analysis, communication engineering, acoustics and reliability theory [10]. Drawing on ideas from [14] and [17], we derive general formulas for the CDF, PDF, and key statistical properties such as moments for different cubic transmuted Rayleigh (CTR) distributions. These formulas depend on two parameters, which, once specified, allow for the derivation of the desired properties for each cubic transmutation form. We also compare the unmodified and modified versions of different CTR models using real data. Sakthivel and Vidhya [34] explored four CTR forms, using CT_G , CT_A , CT_{R18a} , and CT_{R19} formulas. In their work, they analysed each model separately, estimated

parameters via maximum likelihood, and assessed their applicability using real data. However, they did not perform a simultaneous comparison of the four models on the same datasets. Our work extends theirs in several key aspects: (a) we examine six CTR forms, including CT_{R18b} and CT_{R23} ; (b) instead of treating statistical properties separately for each model, we propose a unified general formula encompassing all cases; (c) we simultaneously compare all six models across multiple datasets; (d) we incorporate parameter range modifications suggested by Geraldo et al. [14] to enhance model performance.

The remainder of the paper is structured as follows. Section 2 provides a brief review of six CT formulas proposed in the literature, along with their modifications by Geraldo et al. [14] for improved data fitting. Section 3 introduces a general formula for CTR distributions. In Section 4, we derive formulas for the statistical properties of the general CTR distribution, which depend on certain parameters that, once specified, yield the properties of a given CTR form. Section 5 discusses the maximum likelihood estimation of model parameters, while Section 6 provides the results of a simulation study conducted to assess the performance of the maximum likelihood estimators (MLEs) for the parameters of all six CTR distributions. Section 7 presents a comparative analysis of the modified and unmodified CTR models using real data in R software [25]. Finally, Section 8 concludes with some remarks.

2. A BRIEF REVIEW OF CT FORMULAS AND THEIR MODIFICATIONS

2.1. Cubic transmutation formulas. Let $x \in \mathbb{R}$ and $G(x)$ be a baseline CDF. To the best of our knowledge, there exist six formulas for CT. In this section, we just briefly present these formulas. For a more detailed review on the construction of each formula, we refer the reader to [14].

The first CT formula (that we denote CT_G) is due to Granzotto et al. [15] who proposed the new CDF

$$F_G(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1) G^2(x) + (1 - \lambda_2) G^3(x),$$

where $(\lambda_1, \lambda_2) \in \mathcal{S}_G = [0, 1] \times [-1, 1]$. The second formula (that we denote CT_A) was proposed by AL-Kadim and Mohammed [3]. It corresponds to the CDF

$$F_A(x) = (1 + \lambda) G(x) - 2\lambda G^2(x) + \lambda G^3(x),$$

where $\lambda \in [-1, 1]$. The third formula (that we denote CT_{R18a}) was proposed by Rahman et al. [27] and corresponds to the CDF

$$F_{R18a}(x) = (1 + \lambda_1) G(x) + (\lambda_2 - \lambda_1) G^2(x) - \lambda_2 G^3(x),$$

where

$$(\lambda_1, \lambda_2) \in \mathcal{S}_{R18a} = \{(\lambda_1, \lambda_2) \in [-1, 1]^2 : -2 \leq \lambda_1 + \lambda_2 \leq 1\}.$$

The fourth formula (that we denote CT_{R18b}) was proposed by Rahman et al. [28] who obtained a new CDF in the form

$$F_{R18b}(x) = (1 + \lambda_1 + \lambda_2)G(x) - (\lambda_1 + 2\lambda_2)G^2(x) + \lambda_2 G^3(x),$$

where $(\lambda_1, \lambda_2) \in \mathcal{S}_{R18b} = [-1, 1] \times [0, 1]$. The fifth formula (that we denote CT_{R19}) was proposed by Rahman et al. [29] who defined another form of CT through the new CDF

$$F_{R19}(x) = (1 - \lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x),$$

where $\lambda \in [-1, 1]$. The sixth and final formula (that we denote CT_{R23}) was developed by Rahman et al. [31] through the CDF

$$F_{R23}(x) = [1 - \lambda(\theta - 1)]G(x) + \lambda(2\theta - 1)G^2(x) - \lambda\theta G^3(x),$$

where $(\lambda, \theta) \in \mathcal{S}_{R23} = [-1, 1] \times [0, 2]$.

2.2. Comparison and modifications of CT formulas. Let $G(x)$ be a CDF. Geraldo et al. [14] studied the six different CT formulas. Their first very important result is that there exist values $(\lambda_1, \lambda_2) \in [0, 1] \times [-1, 1]$ such that the CDF $F_G(x)$ of the CT_G distribution with baseline CDF $G(x)$ does not satisfy the properties of a CDF. They then proposed a modified CT_G distribution (denoted CT_{MG}) with the same CDF $F_G(x)$ but with the modified parameters range

$$(\lambda_1, \lambda_2) \in \mathcal{S}_{MG} = \{(\lambda_1, \lambda_2) \in [0, 3]^2 : 0 \leq \lambda_1 + \lambda_2 \leq 3\}.$$

A second important result from Geraldo et al. [14] is that although the other five CT formulas yield well-defined CDFs for all parameters values, the fit to data of four of them can be improved by extending their respective parameter ranges. They proved that the CT_A distribution is also well defined under the extended condition $\lambda \in [-1, 3]$ and named it the modified AL-Kadim Cubic Transmutation (denoted CT_{MA}). They also proved that the CT_{R19} distribution is well defined under the extended condition $\lambda \in [-2, 1]$ and named it the modified cubic transmutation of Rahman et al. [29] (denoted CT_{MR19}). They also extended the parameter ranges of the CT_{R18a} and CT_{R18b} families respectively as

$$\mathcal{S}_{MR18a} = \{(\lambda_1, \lambda_2) \in [-1, 2]^2 : -2 \leq \lambda_1 + \lambda_2 \leq 1\}$$

and

$$\mathcal{S}_{MR18b} = \{(\lambda_1, \lambda_2) \in [-2, 1]^2 : -1 \leq \lambda_1 + \lambda_2 \leq 2\},$$

and respectively named them modified CT_{R18a} (denoted CT_{MR18a}) and modified CT_{R18b} (denoted CT_{MR18b}). The third important result from the paper [14], is that for all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$, the modified distributions

$CT_{MG}(\lambda_1, \lambda_2)$, $CT_{MR18a}(\lambda_1 - 1, \lambda_2 - 1)$, and $CT_{MR18b}(\lambda_1 + \lambda_2 - 2, 1 - \lambda_2)$ linked to $G(x)$ are equal. No modification has been proposed by the authors for the CT_{R23} formula.

3. UNIFIED FORMULA FOR CUBIC TRANSMUTED RAYLEIGH DISTRIBUTIONS

The Rayleigh distribution (also called the one-parameter Rayleigh distribution) is defined by its CDF

$$G(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0, \quad (3.1)$$

and its PDF

$$g(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0,$$

where $\sigma > 0$ is the scale parameter. For all $\mu > 0$, the random variable $Y = X + \mu$ follows the two-parameter Rayleigh distribution and μ is then called the location parameter [9].

Using the quadratic transmutation proposed by Shaw and Buckley [36], Merovci [20] developed the transmuted Rayleigh (TR) distribution. Dey et al. [10] studied different methods for estimating the parameters of the TR distribution. More recently, Malik and Ahmad [19] developed a new version of the TR distribution using a combination of quadratic transmutation and exponentiation. Khan et al. [18] developed the transmuted version of the two-parameter Rayleigh distribution.

Rahman [26] proposed a CTR distribution using the CT_{R19} formula. Rahman et al. [31] applied their own CT_{R23} formula to develop another CTR distribution. Sakthivel and Vidhya [34] studied simultaneously four CTR distributions by applying the CT_G , CT_A , CT_{R18a} , and CT_{R19} formulas.

Beyond the Rayleigh baseline, the development of cubic transmuted distributions has attracted considerable attention. For instance, Saraçoğlu and Tanış [35] proposed a CT of the Kumaraswamy distribution and derived several statistical properties together with maximum likelihood estimation supported by simulation studies. More recently, Tanış and Saraçoğlu [40] introduced a CT of the generalized Gompertz distribution, further demonstrating the flexibility of the CT approach through three real data applications. Other works, such as the CT of the inverse Rayleigh distribution by Tanış and Saraçoğlu [39], illustrate the broader applicability of CT techniques across different baseline distributions.

In the remainder of this paper, the CTR distributions corresponding to the CT_G , CT_A , CT_{R18a} , CT_{R18b} and CT_{R19} formulas will be respectively denoted CTR_G , CTR_A , CTR_{R18a} , CTR_{R18b} and CTR_{R19}

and the corresponding modifications will be denoted CTR_{MG} , CTR_{MA} , CTR_{MR18a} , CTR_{MR18b} and CTR_{MR19} . The CTR of Rahman et al. [31] will be denoted CTR_{R23} .

The following proposition gives the respective general forms of the CDF and PDF of any CTR distribution.

Proposition 3.1. *For any CTR distribution, there exist two reals δ_1 and δ_2 such that*

$$\inf_{t \in [0,1]} \left[\delta_1 + 2\delta_2 t + 3(1 - \delta_1 - \delta_2)t^2 \right] \geq 0, \quad (3.2)$$

and the CDF and the PDF are respectively given for all $x > 0$ by

$$\begin{aligned} F(x) = 1 + (2\delta_1 + \delta_2 - 3) \exp\left(-\frac{x^2}{2\sigma^2}\right) + (3 - 3\delta_1 - 2\delta_2) \exp\left(-\frac{x^2}{\sigma^2}\right) \\ + (\delta_1 + \delta_2 - 1) \exp\left(-\frac{3x^2}{2\sigma^2}\right) \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[(3 - 2\delta_1 - \delta_2) + (6\delta_1 + 4\delta_2 - 6) \exp\left(-\frac{x^2}{\sigma^2}\right) \right. \\ \left. + (3 - 3\delta_1 - 3\delta_2) \exp\left(-\frac{3x^2}{\sigma^2}\right) \right], \end{aligned} \quad (3.4)$$

where $\sigma > 0$.

Proof. Katchekpele et al. [17] proved in Section 2.3 of their paper, that a CDF $F(x)$ is a cubic transmutation of a baseline CDF $G(x)$ if and only if it has the form

$$F(x) = \delta_1 G(x) + \delta_2 G^2(x) + (1 - \delta_1 - \delta_2) G^3(x), \quad (3.5)$$

where δ_1 and δ_2 satisfy Equation (3.2). Now, let F and f be the respective CDF and PDF of a CTR distribution. By replacing Equation (3.1)

in (3.5), we have

$$\begin{aligned}
 F(x) &= \delta_1 \left[1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \right] + \delta_2 \left[1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \right]^2 \\
 &\quad + (1 - \delta_1 - \delta_2) \left[1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \right]^3 \\
 &= \delta_1 \left[1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \right] + \delta_2 \left[1 - 2\exp\left(-\frac{x^2}{2\sigma^2}\right) + \exp\left(-\frac{2x^2}{2\sigma^2}\right) \right] \\
 &\quad + (1 - \delta_1 - \delta_2) \left[1 - 3\exp\left(-\frac{x^2}{2\sigma^2}\right) \right. \\
 &\quad \quad \left. + 3\exp\left(-\frac{2x^2}{2\sigma^2}\right) - \exp\left(-\frac{3x^2}{2\sigma^2}\right) \right] \\
 &= 1 + (2\delta_1 + \delta_2 - 3) \exp\left(-\frac{x^2}{2\sigma^2}\right) + (3 - 3\delta_1 - 2\delta_2) \exp\left(-\frac{2x^2}{2\sigma^2}\right) \\
 &\quad + (\delta_1 + \delta_2 - 1) \exp\left(-\frac{3x^2}{2\sigma^2}\right),
 \end{aligned}$$

which completes the proof of Equation (3.3). Equation (3.4) is then deduced using the relation $f(x) = F'(x)$. $\square$

Table 1 gives a summary of the values of (δ_1, δ_2) corresponding to the six CT formulas considered in this paper.

Geraldo et al. [14] proved that all six CTs satisfy Equation (3.2) (whether the parameter range is modified or not) except the CT_G under the initial parameter range. This further highlights the importance of the modified parameter range for the CT_G .

Remark 3.2. By applying the formulas (3.3) and (3.4) for the values of (δ_1, δ_2) from Table 1, one gets the respective CDFs and PDFs of the different CTR distributions (see [34] for CTR_G , CTR_A , CTR_{R18a} , CTR_{R19} , and [31] for CTR_{R23}).

- The CDF and PDF of both the CTR_G and CTR_{MG} distributions are

$$\begin{aligned}
 F_G(x) &= 1 + (\lambda_1 + \lambda_2 - 3) \exp\left(-\frac{x^2}{2\sigma^2}\right) \\
 &\quad + (3 - \lambda_1 - 2\lambda_2) \exp\left(-\frac{x^2}{\sigma^2}\right) + (\lambda_2 - 1) \exp\left(-\frac{3x^2}{2\sigma^2}\right)
 \end{aligned}$$

TABLE 1. Summary of the values of (δ_1, δ_2) for the six CT formulas and their corresponding initial and modified parameter ranges (if any)

Formula	(δ_1, δ_2)	Initial and modified ranges (if any)
CT_G	$(\lambda_1, \lambda_2 - \lambda_1)$	Initial: $(\lambda_1, \lambda_2) \in [0, 1] \times [-1, 1]$ Modified: $(\lambda_1, \lambda_2) \in [0, 3]^2$ and $0 \leq \lambda_1 + \lambda_2 \leq 3$
CT_A	$(1 + \lambda, -2\lambda)$	Initial: $\lambda \in [-1, 1]$ Modified: $\lambda \in [-1, 3]$
CT_{R18a}	$(1 + \lambda_1, \lambda_2 - \lambda_1)$	Initial: $(\lambda_1, \lambda_2) \in [-1, 1]^2$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$ Modified: $(\lambda_1, \lambda_2) \in [-1, 2]^2$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$
CT_{R18b}	$(1 + \lambda_1 + \lambda_2, -\lambda_1 - 2\lambda_2)$	Initial: $(\lambda_1, \lambda_2) \in [-1, 1] \times [0, 1]$ Modified: $(\lambda_1, \lambda_2) \in [-2, 1]^2$ and $-1 \leq \lambda_1 + \lambda_2 \leq 2$
CT_{R19}	$(1 - \lambda, 3\lambda)$	Initial: $\lambda \in [-1, 1]$ Modified: $\lambda \in [-2, 1]$
CT_{R23}	$(1 + \lambda - \lambda\theta, 2\lambda\theta - \lambda)$	Initial: $(\lambda, \theta) \in [-1, 1] \times [0, 2]$

and

$$f_G(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[(3 - \lambda_1 - \lambda_2) + 2(\lambda_1 + 2\lambda_2 - 3) \exp\left(-\frac{x^2}{2\sigma^2}\right) + 3(1 - \lambda_2) \exp\left(-\frac{x^2}{\sigma^2}\right) \right].$$

- The CDF and PDF of both the CTR_A and CTR_{MA} distributions are

$$F_A(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) + \lambda \exp\left(-\frac{x^2}{\sigma^2}\right) - \lambda \exp\left(-\frac{3x^2}{2\sigma^2}\right)$$

and

$$f_A(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[1 - 2\lambda \exp\left(-\frac{x^2}{2\sigma^2}\right) + 3\lambda \exp\left(-\frac{x^2}{\sigma^2}\right) \right].$$

- The CDF and PDF of both the CTR_{R18a} and CTR_{MR18a} distributions are

$$F_{R18a}(x) = 1 + (\lambda_1 + \lambda_2 - 1) \exp\left(-\frac{x^2}{2\sigma^2}\right) - (\lambda_1 + 2\lambda_2) \exp\left(-\frac{x^2}{\sigma^2}\right) + \lambda_2 \exp\left(-\frac{3x^2}{2\sigma^2}\right)$$

and

$$f_{R18a}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[(1 - \lambda_1 - \lambda_2) + 2(\lambda_1 + 2\lambda_2) \exp\left(-\frac{x^2}{\sigma^2}\right) - 3\lambda_2 \exp\left(-\frac{x^2}{\sigma^2}\right) \right].$$

- The CDF and PDF of both the CTR_{R18b} and CTR_{MR18b} distributions are

$$F_{R18b}(x) = 1 + (\lambda_1 - 1) \exp\left(-\frac{x^2}{2\sigma^2}\right) + (\lambda_2 - \lambda_1) \exp\left(-\frac{x^2}{\sigma^2}\right) - \lambda_2 \exp\left(-\frac{3x^2}{2\sigma^2}\right)$$

and

$$f_{R18b}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[(1 - \lambda_1) + 2(\lambda_1 - \lambda_2) \exp\left(-\frac{x^2}{\sigma^2}\right) + 3\lambda_2 \exp\left(-\frac{x^2}{\sigma^2}\right) \right].$$

- The CDF and PDF of both the CTR_{R19} and CTR_{MR19} distributions are

$$F_{R19}(x) = 1 + (\lambda - 1) \exp\left(-\frac{x^2}{2\sigma^2}\right) - 3\lambda \exp\left(-\frac{x^2}{\sigma^2}\right) + 2\lambda \exp\left(-\frac{3x^2}{2\sigma^2}\right)$$

and

$$f_{R19}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[(1 - \lambda) + 6\lambda \exp\left(-\frac{x^2}{\sigma^2}\right) - 6\lambda \exp\left(-\frac{x^2}{\sigma^2}\right) \right].$$

- The CDF and PDF of the CTR_{R23} distribution are

$$F_{R23}(x) = 1 + (\lambda - 1) \exp\left(-\frac{x^2}{2\sigma^2}\right) - \lambda(1 + \theta) \exp\left(-\frac{x^2}{\sigma^2}\right) + \lambda\theta \exp\left(-\frac{3x^2}{2\sigma^2}\right)$$

and

$$f_{R23}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left[(1-\lambda) + 2\lambda(1+\theta) \exp\left(-\frac{x^2}{2\sigma^2}\right) - 3\lambda\theta \exp\left(-\frac{x^2}{\sigma^2}\right) \right].$$

4. MOMENTS AND OTHER RESULTS

4.1. Moments.

Theorem 4.1. *Let X be a random variable following a CTR distribution as defined by Proposition 3.1. Then, for all $k \in \mathbb{N}^*$, the k^{th} moment is given by*

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(3-2\delta_1-\delta_2) + \frac{(3\delta_1+2\delta_2-3)}{2^{k/2}} + \frac{(1-\delta_1-\delta_2)}{3^{k/2}} \right], \quad (4.1)$$

where Γ is the Gamma function. The mean and variance are respectively given by

$$E(X) = \sigma\sqrt{2} \Gamma\left(\frac{3}{2}\right) \left[(3-2\delta_1-\delta_2) + \frac{(3\delta_1+2\delta_2-3)}{2^{1/2}} + \frac{(1-\delta_1-\delta_2)}{3^{1/2}} \right]$$

and

$$V(X) = 2\sigma^2 \left\{ \left[(3-2\delta_1-\delta_2) + \frac{(3\delta_1+2\delta_2-3)}{2} + \frac{(1-\delta_1-\delta_2)}{3} \right] - \left(\Gamma\left(\frac{3}{2}\right) \left[(3-2\delta_1-\delta_2) + \frac{(3\delta_1+2\delta_2-3)}{2^{1/2}} + \frac{(1-\delta_1-\delta_2)}{3^{1/2}} \right] \right)^2 \right\}.$$

Proof. Let $k \in \mathbb{N}^*$. We have

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx.$$

From Equation (3.4), we have

$$\begin{aligned}
 E(X^k) &= \int_0^\infty \frac{x^{k+1}}{\sigma^2} \left[(3 - 2\delta_1 - \delta_2) \exp\left(-\frac{x^2}{2\sigma^2}\right) \right. \\
 &\quad \left. + (6\delta_1 + 4\delta_2 - 6) \exp\left(-\frac{x^2}{\sigma^2}\right) \right. \\
 &\quad \left. + (3 - 3\delta_1 - 3\delta_2) \exp\left(-\frac{3x^2}{2\sigma^2}\right) \right] dx \\
 &= (3 - 2\delta_1 - \delta_2) \int_0^\infty \frac{x^{k+1}}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
 &\quad + (6\delta_1 + 4\delta_2 - 6) \int_0^\infty \frac{x^{k+1}}{\sigma^2} \exp\left(-\frac{x^2}{\sigma^2}\right) dx \\
 &\quad + (3 - 3\delta_1 - 3\delta_2) \int_0^\infty \frac{x^{k+1}}{\sigma^2} \exp\left(-\frac{3x^2}{2\sigma^2}\right) dx.
 \end{aligned}$$

By making respectively the changes of variables

$$s = \frac{x^2}{2\sigma^2}, \quad t = \frac{x^2}{\sigma^2} \quad \text{and} \quad u = \frac{3x^2}{2\sigma^2},$$

we have

$$\begin{aligned}
 E(X^k) &= (3 - 2\delta_1 - \delta_2) \int_0^\infty \left(\sigma 2^{1/2} s^{1/2}\right)^k e^{-s} ds \\
 &\quad + (6\delta_1 + 4\delta_2 - 6) \int_0^\infty \frac{1}{2} \left(\sigma t^{1/2}\right)^k e^{-t} dt \\
 &\quad + (3 - 3\delta_1 - 3\delta_2) \int_0^\infty \frac{1}{3} \left(\sigma 2^{1/2} \left(\frac{u}{3}\right)^{1/2}\right)^k e^{-u} du \\
 &= (3 - 2\delta_1 - \delta_2) \int_0^\infty \sigma^k 2^{k/2} s^{k/2} e^{-s} ds \\
 &\quad + (3\delta_1 + 2\delta_2 - 3) \int_0^\infty \sigma^k t^{k/2} e^{-t} dt \\
 &\quad + (1 - \delta_1 - \delta_2) \int_0^\infty \frac{\sigma^k 2^{k/2}}{3^{k/2}} u^{k/2} e^{-u} du \\
 &= (3 - 2\delta_1 - \delta_2) \sigma^k 2^{k/2} \Gamma\left(\frac{k}{2} + 1\right) + (3\delta_1 + 2\delta_2 - 3) \sigma^k \Gamma\left(\frac{k}{2} + 1\right) \\
 &\quad + (1 - \delta_1 - \delta_2) \frac{\sigma^k 2^{k/2}}{3^{k/2}} \Gamma\left(\frac{k}{2} + 1\right),
 \end{aligned}$$

and, finally,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(3 - 2\delta_1 - \delta_2) + \frac{(3\delta_1 + 2\delta_2 - 3)}{2^{k/2}} + \frac{(1 - \delta_1 - \delta_2)}{3^{k/2}} \right].$$

The expectation $E(X)$ corresponds to the case $k = 1$ and the variance $V(X)$ is deduced by considering $k = 2$ and using the relation $V(X) = E(X^2) - [E(X)]^2$. $\square$

Remark 4.2. By setting the values of δ_1 and δ_2 as in Table 1, one can obtain the k^{th} moment for each of the six CT formulas considered in this paper (see [34] for CTR_G , CTR_A , CTR_{R18a} , CTR_{R19} and [31] for CTR_{R23}).

- For both the CTR_G and CTR_{MG} distributions,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(3 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 + 2\lambda_2 - 3)}{2^{k/2}} + \frac{(1 - \lambda_2)}{3^{k/2}} \right].$$

- For both the CTR_A and CTR_{MA} distributions,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[1 - \frac{\lambda}{2^{k/2}} + \frac{\lambda}{3^{k/2}} \right].$$

- For both the CTR_{R18a} and CTR_{MR18a} distributions,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(1 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 + 2\lambda_2)}{2^{k/2}} - \frac{\lambda_2}{3^{k/2}} \right].$$

- For both the CTR_{R18b} and CTR_{MR18b} distributions,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(1 - \lambda_1) + \frac{(\lambda_1 - \lambda_2)}{2^{k/2}} + \frac{\lambda_2}{3^{k/2}} \right].$$

- For both the CTR_{R19} and CTR_{MR19} distributions,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(1 - \lambda) + \frac{3\lambda}{2^{k/2}} - \frac{2\lambda}{3^{k/2}} \right].$$

- For the CTR_{R23} distribution,

$$E(X^k) = \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(1 - \lambda) + \frac{\lambda(1 + \theta)}{2^{k/2}} - \frac{\lambda\theta}{3^{k/2}} \right].$$

4.2. Moment generating function.

Proposition 4.3. *Let X be a random variable following a CTR distribution as defined by Proposition 3.1. Then, the moment generating function is defined for all $t \in \mathbb{R}$ by:*

$$M(t) = \sum_{k=0}^{\infty} \left\{ \frac{t^k}{k!} \sigma^k 2^{k/2} \Gamma\left(\frac{k+2}{2}\right) \left[(3 - 2\delta_1 - \delta_2) + \frac{(3\delta_1 + 2\delta_2 - 3)}{2^{k/2}} + \frac{(1 - \delta_1 - \delta_2)}{3^{k/2}} \right] \right\}.$$

Proof. The proof, largely inspired from the papers on CTR distributions (see, for example, [26, 31, 34]), uses the power series expansion of the exponential function. Indeed, we have

$$\begin{aligned} M(t) &= E(e^{tX}) = \int_0^{\infty} e^{tx} f(x) dx \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \frac{t^k x^k}{k!} f(x) dx \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k), \end{aligned}$$

where $E(X^k)$ is given by Equation (4.1). □

5. PARAMETER ESTIMATION

Let $x_1, \dots, x_n$ be a random sample of size n from the CTR with PDF $f(x)$ defined by Equation (3.4). The likelihood and its logarithm (log-likelihood) are respectively defined by:

$$L = \prod_{i=1}^n f(x_i) \quad \text{and} \quad \ell = \log L = \sum_{i=1}^n \log [f(x_i)].$$

Since the parameters of the six CTR distributions are not the same, it is necessary to detail the log-likelihood for each.

- The log-likelihood corresponding to both the CTR_G and the CTR_{MG} distributions is

$$\begin{aligned} \ell_G(\sigma, \lambda_1, \lambda_2) = & -2n \log(\sigma) + \sum_{i=1}^n \log(x_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \log \left[(3 - \lambda_1 - \lambda_2) + 2(\lambda_1 + 2\lambda_2 - 3) \exp \left(-\frac{x_i^2}{2\sigma^2} \right) \right. \\ & \left. + 3(1 - \lambda_2) \exp \left(-\frac{x_i^2}{\sigma^2} \right) \right]. \end{aligned}$$

The MLE $(\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2)$ of $(\sigma, \lambda_1, \lambda_2)$ is

$$(\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2) = \underset{(\sigma, \lambda_1, \lambda_2) \in \mathbb{R}_+^* \times \mathcal{S}}{\operatorname{argmax}} \ell_G(\sigma, \lambda_1, \lambda_2),$$

where $\mathcal{S} = \mathcal{S}_G$ for the CTR_G distribution and $\mathcal{S} = \mathcal{S}_{MG}$ for the CTR_{MG} distribution.

- The log-likelihood corresponding to both the CTR_A and the CTR_{MA} distributions is

$$\begin{aligned} \ell_A(\sigma, \lambda) = & -2n \log(\sigma) + \sum_{i=1}^n \log(x_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \log \left[1 - 2\lambda \exp \left(-\frac{x_i^2}{2\sigma^2} \right) + 3\lambda \exp \left(-\frac{x_i^2}{\sigma^2} \right) \right]. \end{aligned}$$

The MLE $(\hat{\sigma}, \hat{\lambda})$ of (σ, λ) is

$$(\hat{\sigma}, \hat{\lambda}) = \underset{(\sigma, \lambda) \in \mathbb{R}_+^* \times \mathcal{S}}{\operatorname{argmax}} \ell_A(\sigma, \lambda),$$

where $\mathcal{S} = [-1, 1]$ for the CTR_A distribution and $\mathcal{S} = [-1, 3]$ for the CTR_{MA} distribution.

- The log-likelihood corresponding to both the CTR_{R18a} and the CTR_{MR18a} distributions is

$$\begin{aligned} \ell_{R18a}(\sigma, \lambda_1, \lambda_2) = & -2n \log(\sigma) + \sum_{i=1}^n \log(x_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \log \left[(1 - \lambda_1 - \lambda_2) + 2(\lambda_1 + 2\lambda_2) \exp \left(-\frac{x_i^2}{2\sigma^2} \right) \right. \\ & \left. - 3\lambda_2 \exp \left(-\frac{x_i^2}{\sigma^2} \right) \right]. \end{aligned}$$

The MLE $(\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2)$ of $(\sigma, \lambda_1, \lambda_2)$ is

$$(\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2) = \underset{(\sigma, \lambda_1, \lambda_2) \in \mathbb{R}_+^* \times \mathcal{S}}{\operatorname{argmax}} \ell_{R18a}(\sigma, \lambda_1, \lambda_2),$$

where $\mathcal{S} = \mathcal{S}_{R18a}$ for the CTR_{R18a} distribution and $\mathcal{S} = \mathcal{S}_{MR18a}$ for the CT_{MR18a} distribution.

- The log-likelihood corresponding to both the CTR_{R18b} and the CTR_{MR18b} distributions is

$$\begin{aligned} \ell_{R18b}(\sigma, \lambda_1, \lambda_2) = & -2n \log(\sigma) + \sum_{i=1}^n \log(x_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \log \left[(1 - \lambda_1) + 2(\lambda_1 - \lambda_2) \exp \left(-\frac{x_i^2}{2\sigma^2} \right) + 3\lambda_2 \exp \left(-\frac{x_i^2}{\sigma^2} \right) \right]. \end{aligned}$$

The MLE $(\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2)$ of $(\sigma, \lambda_1, \lambda_2)$ is

$$(\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2) = \underset{(\sigma, \lambda_1, \lambda_2) \in \mathbb{R}_+^* \times \mathcal{S}}{\operatorname{argmax}} \ell_{R18b}(\sigma, \lambda_1, \lambda_2),$$

where $\mathcal{S} = \mathcal{S}_{R18b}$ for the CTR_{R18b} distribution and $\mathcal{S} = \mathcal{S}_{MR18b}$ for the CT_{MR18b} distribution.

- The log-likelihood corresponding to both the CTR_{R19} and the CTR_{MR19} distributions is

$$\begin{aligned} \ell_{R19}(\sigma, \lambda) = & -2n \log(\sigma) + \sum_{i=1}^n \log(x_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \log \left[(1 - \lambda) + 6\lambda \exp \left(-\frac{x_i^2}{2\sigma^2} \right) - 6\lambda \exp \left(-\frac{x_i^2}{\sigma^2} \right) \right]. \end{aligned}$$

The MLE $(\hat{\sigma}, \hat{\lambda})$ of (σ, λ) is

$$(\hat{\sigma}, \hat{\lambda}) = \underset{(\sigma, \lambda) \in \mathbb{R}_+^* \times \mathcal{S}}{\operatorname{argmax}} \ell_{R19}(\sigma, \lambda),$$

where $\mathcal{S} = [-1, 1]$ for the CTR_{R19} distribution and $\mathcal{S} = [-2, 1]$ for the CTR_{MR19} distribution.

- The log-likelihood corresponding to CTR_{R23} distribution is

$$\begin{aligned} \ell_{R23}(\sigma, \lambda, \theta) = & -2n \log(\sigma) + \sum_{i=1}^n \log(x_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \\ & + \sum_{i=1}^n \log \left[(1 - \lambda) + 2\lambda(1 + \theta) \exp \left(-\frac{x_i^2}{2\sigma^2} \right) - 3\lambda\theta \exp \left(-\frac{x_i^2}{\sigma^2} \right) \right]. \end{aligned}$$

The MLE $(\hat{\sigma}, \hat{\lambda}, \hat{\theta})$ of $(\sigma, \lambda, \theta)$ is

$$(\hat{\sigma}, \hat{\lambda}, \hat{\theta}) = \underset{(\sigma, \lambda, \theta) \in \mathbb{R}_+^* \times [-1, 1] \times [0, 2]}{\operatorname{argmax}} \ell_{R23}(\sigma, \lambda, \theta).$$

The resolution of these optimization problems requires the use of a numerical optimization algorithm which can handle inequality constraints. As in [14], we will use the R function "constrOptim" because it not only provides algorithms performing numerical optimization under inequality constraints, but also returns the Hessian matrix of the log-likelihood (the matrix of partial derivatives of order 2) thus allowing the easy estimation of standard errors as the diagonal elements of the inverse of the observed information matrix (the opposite of the Hessian matrix).

6. SIMULATION STUDY

This section presents the results of a simulation study to evaluate the performance of the MLEs of the parameters for the six CTR distributions. The true parameter values were set according to the following scenarios:

- CTR_{MG} distribution:
 - Scenario 1: $(\alpha, \lambda_1, \lambda_2) = (1.7, 1.3, 0.5)$;
 - Scenario 2: $(\alpha, \lambda_1, \lambda_2) = (3.0, 2.0, 0.0)$;
- CTR_{MA} distribution:
 - Scenario 3: $(\alpha, \lambda) = (1.5, -0.5)$;
 - Scenario 4: $(\alpha, \lambda) = (3.0, 1.8)$;
- CTR_{MR18a} distribution:
 - Scenario 5: $(\alpha, \lambda_1, \lambda_2) = (1.4, -0.5, 0.7)$;
 - Scenario 6: $(\alpha, \lambda_1, \lambda_2) = (3.0, 0.8, -0.8)$;
- CTR_{MR18b} distribution:
 - Scenario 7: $(\alpha, \lambda_1, \lambda_2) = (1.0, -0.5, 0.9)$;
 - Scenario 8: $(\alpha, \lambda_1, \lambda_2) = (2.0, 0.8, 0.5)$;
- CTR_{MR19} distribution:
 - Scenario 9: $(\alpha, \lambda) = (1.2, -1.8)$;
 - Scenario 10: $(\alpha, \lambda) = (2.8, 0.9)$;
- CTR_{R23} distribution:
 - Scenario 11: $(\alpha, \lambda, \theta) = (1.7, -0.5, 1.0)$;
 - Scenario 12: $(\alpha, \lambda, \theta) = (3.0, -1.0, 2.0)$.

For each CTR distribution, the random sample $x_1, \dots, x_n$ was simulated using the inversion method for random number generation, i.e., by generating n random samples $u_1, \dots, u_n$ from the uniform distribution on $[0, 1]$ (using the "runif" function in R software [25]) and then, for

each $i = 1, \dots, n$, computing the unique real x_i such that $F(x_i) = u_i$ (using the "uniroot" function).

To observe the convergence properties of the MLEs, we fixed the sample sizes as follows: $n \in \{50, 100, 250, 500, 1000\}$. For each CTR distribution and each combination of true parameter values and sample size, we replicated $R = 1000$ times, the process consisting of generating a sample of size n , and then computing the MLEs of the parameters using the BFGS algorithm. In past studies (see, for example, [12, 13]), it was observed that the log-likelihoods of transmuted distributions could have local maxima (which are not the true MLEs), to which optimization algorithms can sometimes converge, depending on the starting values. To address this issue, we used a multi-start optimization approach, testing $m = 10$ different starting values for each replication and retaining the parameter estimates that produced the highest log-likelihood value. The mean and the mean squared errors (MSEs) of the MLEs are respectively estimated as

$$\text{Mean}(\hat{a}) = \frac{1}{R} \sum_{i=1}^R \hat{a}^{(i)} \quad \text{and} \quad \text{MSE}(\hat{a}) = \frac{1}{R} \sum_{i=1}^R \left(\hat{a}^{(i)} - a \right)^2,$$

where a stands for the true value of any of the parameters σ , λ , λ_1 , λ_2 or θ , and $\hat{a}^{(i)}$ represents the estimate of that parameter from replication $i \in \{1, \dots, R\}$.

Tables 2 to 7 present the results of our simulation study.

TABLE 2. Means and MSEs for the MLEs of the parameters of the CTR_{MG} distribution

Scenario	n	$\hat{\alpha}$		$\hat{\lambda}_1$		$\hat{\lambda}_2$	
		Mean	MSE	Mean	MSE	Mean	MSE
1	50	1.766	0.087	1.395	0.269	0.451	0.315
	100	1.776	0.071	1.426	0.181	0.444	0.195
	250	1.738	0.040	1.366	0.086	0.465	0.106
	500	1.713	0.014	1.332	0.031	0.472	0.056
	1000	1.704	0.006	1.314	0.015	0.475	0.032
2	50	3.021	0.165	1.896	0.183	0.146	0.113
	100	3.029	0.096	1.931	0.098	0.120	0.064
	250	3.019	0.036	1.953	0.038	0.096	0.032
	500	3.006	0.016	1.961	0.020	0.077	0.020
	1000	3.009	0.007	1.972	0.010	0.060	0.011

TABLE 3. Means and MSEs for the MLEs of the parameters of the CTR_{MA} distribution

Scenario	n	$\hat{\alpha}$		$\hat{\lambda}$	
		Mean	MSE	Mean	MSE
3	50	1.508	0.026	-0.499	0.197
	100	1.499	0.006	-0.517	0.061
	250	1.498	0.002	-0.507	0.024
	500	1.502	0.005	-0.493	0.026
	1000	1.498	0.001	-0.505	0.006
4	50	2.984	0.103	1.783	0.257
	100	2.997	0.042	1.799	0.115
	250	2.998	0.015	1.801	0.039
	500	3.001	0.008	1.807	0.020
	1000	3.001	0.004	1.806	0.010

 TABLE 4. Means and MSEs for the MLEs of the parameters of the CTR_{MR18a} distribution

Scenario	n	$\hat{\alpha}$		$\hat{\lambda}_1$		$\hat{\lambda}_2$	
		Mean	MSE	Mean	MSE	Mean	MSE
5	50	1.401	0.052	-0.433	0.247	0.447	0.746
	100	1.408	0.041	-0.419	0.152	0.480	0.527
	250	1.423	0.023	-0.440	0.062	0.644	0.207
	500	1.427	0.017	-0.451	0.033	0.688	0.128
	1000	1.423	0.010	-0.463	0.018	0.706	0.078
6	50	3.056	0.204	0.767	0.206	-0.765	0.143
	100	3.035	0.102	0.801	0.121	-0.781	0.094
	250	3.008	0.042	0.802	0.045	-0.793	0.048
	500	3.019	0.022	0.806	0.027	-0.787	0.031
	1000	3.007	0.009	0.802	0.012	-0.799	0.017

It is noticed that, when the sample size increases, the means of the MLEs get closer to the true values of the parameters and the MSEs decrease.

7. COMPARISON OF MODELS ON REAL DATASETS

In this section, we compare the Rayleigh, TR, CTR_G , CTR_A , CTR_{R18a} , CTR_{R18b} , CTR_{R19} and CTR_{R23} distributions and their modifications on real datasets using R software [25]. As stated earlier, we will use

TABLE 5. Means and MSEs for the MLEs of the parameters of the CTR_{MR18b} distribution

Scenario	n	$\hat{\alpha}$		$\hat{\lambda}_1$		$\hat{\lambda}_2$	
		Mean	MSE	Mean	MSE	Mean	MSE
7	50	1.043	0.020	-0.331	0.269	0.728	0.227
	100	1.027	0.014	-0.378	0.186	0.773	0.138
	250	1.015	0.005	-0.428	0.079	0.822	0.065
	500	1.006	0.001	-0.464	0.033	0.848	0.034
	1000	1.003	0.001	-0.484	0.014	0.875	0.017
8	50	1.690	0.224	0.253	0.593	0.503	0.361
	100	1.775	0.153	0.428	0.311	0.472	0.280
	250	1.888	0.090	0.605	0.111	0.503	0.166
	500	1.942	0.058	0.685	0.052	0.524	0.117
	1000	1.980	0.034	0.741	0.022	0.533	0.071

TABLE 6. Means and MSEs for the MLEs of the parameters of the CTR_{MR19} distribution

Scenario	n	$\hat{\alpha}$		$\hat{\lambda}$	
		Mean	MSE	Mean	MSE
9	50	1.201	0.003	-1.814	0.035
	100	1.199	0.002	-1.807	0.021
	250	1.199	0.001	-1.803	0.009
	500	1.199	0.000	-1.803	0.004
	1000	1.200	0.000	-1.800	0.002
10	50	2.803	0.022	0.875	0.026
	100	2.797	0.013	0.891	0.016
	250	2.798	0.005	0.897	0.008
	500	2.804	0.002	0.903	0.004
	1000	2.802	0.001	0.901	0.002

the R function "constrOptim" and the BFGS algorithm to compute the different MLEs. To avoid problems of convergence of the algorithm towards wrong values that are not the MLEs, we tested for each model and each dataset, $m = 100$ initial values randomly chosen in the parameter space and retained the best initial parameters to start the BFGS algorithm.

TABLE 7. Means and MSEs for the MLEs of the parameters of the CTR_{R23} distribution

Scenario	n	$\hat{\alpha}$		$\hat{\lambda}$		$\hat{\theta}$	
		Mean	MSE	Mean	MSE	Mean	MSE
11	50	1.793	0.078	-0.330	0.543	1.094	0.359
	100	1.780	0.066	-0.361	0.421	1.066	0.307
	250	1.773	0.056	-0.373	0.311	1.018	0.209
	500	1.742	0.035	-0.434	0.180	1.007	0.133
	1000	1.725	0.020	-0.455	0.097	0.999	0.073
12	50	3.016	0.044	-0.908	0.053	1.851	0.104
	100	3.007	0.019	-0.946	0.019	1.915	0.036
	250	3.005	0.007	-0.963	0.006	1.947	0.012
	500	3.003	0.004	-0.973	0.003	1.961	0.006
	1000	3.002	0.002	-0.981	0.002	1.973	0.003

We use the following likelihood-based model selection criteria to select the distributions that best fit each dataset: the negative maximum log-likelihood ($-\ell^*$), the Akaike Information Criterion (AIC), the Akaike Information Criterion Corrected (AICC) and the Bayesian Information Criterion (BIC), which are defined by:

$$\text{AIC} = -2\ell^* + 2k; \quad \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1}; \quad \text{BIC} = -2\ell^* + k \log n,$$

where k is the number of parameters and n is the size (number of observations) of the dataset. The AIC, AICC, and BIC criteria incorporate penalty terms for model complexity and are among the most used model selection criteria. For each dataset, the best model is the one with the smallest values for the criteria.

7.1. Life of fatigue fracture of Kevlar 373/epoxy. These data taken from [7], consist of 76 observations of the life of fatigue fracture of Kevlar 373/epoxy. Descriptive statistics for the Kevlar dataset are presented in Table 8.

TABLE 8. Descriptive statistics for the Kevlar dataset

Min	Q_1	Median	Mean	Q_3	Max
0.0251	0.9048	1.7362	1.9592	2.2959	9.0960

Tables 9 and 10 respectively present the different model selection criteria for unmodified and modified models for the Kevlar dataset. In these tables, the ranks of the models, from best to worse, are given in

parentheses, i.e. a lower rank indicates a better fit. Among the unmodified models (Table 9), the CTR_{R18b} distribution provides the best fit to the Kevlar dataset, as it ranks first for all four model selection criteria. However, when considering the modified models (Table 10), they fit the data better than their unmodified counterparts, as evidenced by a decrease in their corresponding model selection criteria values. For each distribution, the corresponding criteria values of the modified version (Table 10) are lower than or equal to those of the unmodified version (Table 9). The overall model selection should therefore be done mainly using Table 10. Consequently, the modified CTR_{MA} distribution is the best-fitting distribution for the Kevlar dataset, as it has the lowest values for all the criteria among all models considered.

TABLE 9. Model selection criteria and their corresponding ranks (in parentheses) for unmodified models on the Kevlar dataset

Distributions	$-\ell^*$	AIC	AICC	BIC
CTR_G	137.320 ⁽⁷⁾	280.640 ⁽⁸⁾	280.973 ⁽⁸⁾	287.632 ⁽⁸⁾
CTR_A	128.706 ⁽³⁾	261.412 ⁽³⁾	261.576 ⁽²⁾	266.073 ⁽²⁾
CTR_{R18a}	127.701 ⁽²⁾	261.402 ⁽²⁾	261.735 ⁽³⁾	268.394 ⁽³⁾
CTR_{R18b}	124.844 ⁽¹⁾	255.688 ⁽¹⁾	256.021 ⁽¹⁾	262.680 ⁽¹⁾
CTR_{R19}	135.502 ⁽⁶⁾	275.004 ⁽⁶⁾	275.168 ⁽⁶⁾	279.665 ⁽⁷⁾
CTR_{R23}	130.873 ⁽⁴⁾	267.746 ⁽⁵⁾	268.079 ⁽⁵⁾	274.738 ⁽⁵⁾
TR	130.873 ⁽⁴⁾	265.746 ⁽⁴⁾	265.910 ⁽⁴⁾	270.407 ⁽⁴⁾
Rayleigh	137.320 ⁽⁷⁾	276.640 ⁽⁷⁾	276.694 ⁽⁷⁾	278.971 ⁽⁶⁾

TABLE 10. Model selection criteria and their corresponding ranks (in parentheses) for modified models on the Kevlar dataset

Distributions	$-\ell^*$	AIC	AICC	BIC
CTR_{MG}	124.844 ⁽²⁾	255.688 ⁽²⁾	256.021 ⁽²⁾	262.680 ⁽²⁾
CTR_{MA}	123.532 ⁽¹⁾	251.064 ⁽¹⁾	251.228 ⁽¹⁾	255.725 ⁽¹⁾
CTR_{MR18a}	124.844 ⁽²⁾	255.688 ⁽²⁾	256.021 ⁽²⁾	262.680 ⁽²⁾
CTR_{MR18b}	124.844 ⁽²⁾	255.688 ⁽²⁾	256.021 ⁽²⁾	262.680 ⁽²⁾
CTR_{MR19}	135.502 ⁽⁷⁾	275.004 ⁽⁷⁾	275.168 ⁽⁷⁾	279.665 ⁽⁸⁾
CTR_{R23}	130.873 ⁽⁵⁾	267.746 ⁽⁶⁾	268.079 ⁽⁶⁾	274.738 ⁽⁶⁾
TR	130.873 ⁽⁵⁾	265.746 ⁽⁵⁾	265.910 ⁽⁵⁾	270.407 ⁽⁵⁾
Rayleigh	137.320 ⁽⁸⁾	276.640 ⁽⁸⁾	276.694 ⁽⁸⁾	278.971 ⁽⁷⁾

Since the modified models fit the data better than the unmodified ones, we present in Table 11, the MLEs and standard errors (SEs) of the parameters for the modified CTR models.

TABLE 11. MLEs of the parameters for the modified models on the Kevlar dataset (SEs are in parentheses)

Distributions	Estimations		
CTR_{MG}	$\hat{\sigma} = 2.289$ (0.207)	$\hat{\lambda}_1 = 2.592$ (0.470)	$\hat{\lambda}_2 = 0.000$ (0.523)
CTR_{MA}	$\hat{\sigma} = 2.430$ (0.220)	$\hat{\lambda} = 2.309$ (0.309)	
CTR_{MR18a}	$\hat{\sigma} = 2.289$ (0.207)	$\hat{\lambda}_1 = 1.592$ (0.470)	$\hat{\lambda}_2 = -1.000$ (0.523)
CTR_{MR18b}	$\hat{\sigma} = 2.289$ (0.207)	$\hat{\lambda}_1 = 0.592$ (0.190)	$\hat{\lambda}_2 = 1.000$ (0.523)
CTR_{MR19}	$\hat{\sigma} = 1.819$ (0.122)	$\hat{\lambda} = -0.455$ (0.247)	
CTR_{R23}	$\hat{\sigma} = 2.076$ (0.248)	$\hat{\lambda} = 0.704$ (0.175)	$\hat{\theta} = 0.000$ (0.759)
TR	$\hat{\sigma} = 2.076$ (0.156)	$\hat{\lambda} = 0.704$ (0.150)	
Rayleigh	$\hat{\sigma} = 1.772$ (0.102)		

Figure 1 represents the fitting of the different modified densities to the Kevlar dataset. In this figure, the graphs for the CTR_{MG} , CTR_{MR18a} , and CTR_{MR18b} distributions are identical. This is a consequence of a theoretical result obtained by Geraldo et al. [14] who proved that, for all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$, the modified distributions $CT_{MG}(\lambda_1, \lambda_2)$, $CT_{MR18a}(\lambda_1 - 1, \lambda_2 - 1)$, and $CT_{MR18b}(\lambda_1 + \lambda_2 - 2, 1 - \lambda_2)$ linked to any baseline CDF $G(x)$ are equal. Since $\hat{\theta} = 0$ for the CTR_{R23} distribution, its graph is identical to that of the TR distribution. The PDF of the CTR_{MA} distribution, represented by the black dashed line, provides the best fit to the histogram of the Kevlar dataset.

7.2. Carbon fibres dataset. This dataset comes from a study on the breaking stress of carbon fibres [23]. It contains $n = 100$ observations. Descriptive statistics for this dataset are presented by Table 12.

Tables 13 and 14 respectively present the different model selection criteria for unmodified and modified models for the carbon fibres dataset. In these tables also, the ranks of the models, from best to worse, are given in parentheses. From the results for the unmodified models (Table

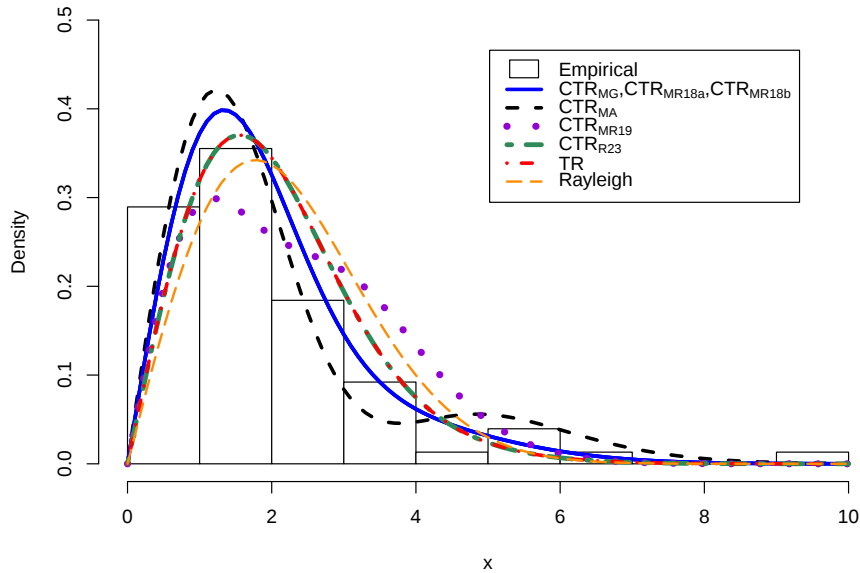


FIGURE 1. Graphical comparison of the modified CTR PDFs fitted to the Kevlar dataset

TABLE 12. Descriptive statistics for the carbon fibres dataset

Min	Q_1	Median	Mean	Q_3	Max
0.390	1.840	2.700	2.621	3.220	5.560

13) and modified models (Table 14), none of the models simultaneously has the lowest values for all the model selection criteria. According to the $-\ell^*$ criterion, the CTR_{MG} , CTR_{MA} , CTR_{MR18a} and CTR_{R23} distributions provide the best fit to the carbon fibres dataset while, according to the AIC, AICC and BIC criteria, the best-fitting model is the TR distribution. It is well known (see, for example, [15]) that, for the same dataset, the penalty terms incorporated in AIC, AICC, and BIC increase with the number k of parameters and, therefore, tend to favour distributions with fewer parameters. For the carbon fibres dataset, these three criteria tend to present the TR distribution as a good compromise between model complexity (a greater number of parameters) and a reasonable log-likelihood value.

Table 15 presents the MLEs and SEs (in parentheses) of the parameters for the modified models.

TABLE 13. Model selection criteria and their corresponding ranks (in parentheses) for unmodified models on the carbon fibres dataset

Distributions	$-\ell^*$	AIC	AICC	BIC
CTR_G	141.362 ⁽²⁾	288.724 ⁽⁵⁾	288.974 ⁽⁵⁾	296.540 ⁽⁵⁾
CTR_A	142.176 ⁽⁷⁾	288.352 ⁽³⁾	288.476 ⁽³⁾	293.562 ⁽³⁾
CTR_{R18a}	141.362 ⁽²⁾	288.724 ⁽⁵⁾	288.974 ⁽⁵⁾	296.540 ⁽⁵⁾
CTR_{R18b}	141.395 ⁽⁴⁾	288.790 ⁽⁷⁾	289.040 ⁽⁷⁾	296.606 ⁽⁷⁾
CTR_{R19}	141.515 ⁽⁶⁾	287.030 ⁽²⁾	287.154 ⁽²⁾	292.240 ⁽²⁾
CTR_{R23}	141.357 ⁽¹⁾	288.714 ⁽⁴⁾	288.964 ⁽⁴⁾	296.530 ⁽⁴⁾
TR	141.443 ⁽⁵⁾	286.886 ⁽¹⁾	287.010 ⁽¹⁾	292.096 ⁽¹⁾
Rayleigh	149.501 ⁽⁸⁾	301.002 ⁽⁸⁾	301.043 ⁽⁸⁾	303.607 ⁽⁸⁾

TABLE 14. Model selection criteria and their corresponding ranks (in parentheses) for modified models on the carbon fibres dataset

Distributions	$-\ell^*$	AIC	AICC	BIC
CTR_{MG}	141.357 ⁽¹⁾	288.714 ⁽⁴⁾	288.964 ⁽⁴⁾	296.530 ⁽⁴⁾
CTR_{MA}	142.176 ⁽⁷⁾	288.352 ⁽³⁾	288.476 ⁽³⁾	293.562 ⁽³⁾
CTR_{MR18a}	141.357 ⁽¹⁾	288.714 ⁽⁴⁾	288.964 ⁽⁴⁾	296.530 ⁽⁴⁾
CTR_{MR18b}	141.357 ⁽¹⁾	288.714 ⁽⁴⁾	288.964 ⁽⁴⁾	296.530 ⁽⁴⁾
CTR_{MR19}	141.515 ⁽⁶⁾	287.030 ⁽²⁾	287.154 ⁽²⁾	292.240 ⁽²⁾
CTR_{R23}	141.357 ⁽¹⁾	288.714 ⁽⁴⁾	288.964 ⁽⁴⁾	296.530 ⁽⁴⁾
TR	141.443 ⁽⁵⁾	286.886 ⁽¹⁾	287.010 ⁽¹⁾	292.096 ⁽¹⁾
Rayleigh	149.501 ⁽⁸⁾	301.002 ⁽⁸⁾	301.043 ⁽⁸⁾	303.607 ⁽⁸⁾

Figure 2 represents the fitting of the modified PDFs to the carbon fibres dataset. In this figure, the graphs for the CTR_{MG} , CTR_{MR18a} , and CTR_{MR18b} distributions are identical for the same reasons as those mentioned for the Kevlar dataset. Except the PDFs of the CTR_{MA} and Rayleigh distributions, all the others provide almost the same fit to the histogram of the carbon fibres dataset. This clearly illustrates the results of the model selection for the carbon fibres dataset.

7.3. Guinea pig dataset. This dataset consists of the survival times of guinea pigs injected with different doses of tubercle bacilli. In this paper, we consider the dataset in thousands of days containing $n = 72$ observations, as reported by Singh et al. [37]. Table 16 presents descriptive statistics for the guinea pig dataset.

TABLE 15. MLEs of the parameters for modified models on the carbon fibres dataset (SEs are in parentheses)

Distributions	Estimations		
CTR_{MG}	$\hat{\sigma} = 2.216$ (0.522)	$\hat{\lambda}_1 = 0.179$ (0.412)	$\hat{\lambda}_2 = 2.821$ (0.766)
CTR_{MA}	$\hat{\sigma} = 1.863$ (0.079)	$\hat{\lambda} = -0.938$ (0.141)	
CTR_{MR18a}	$\hat{\sigma} = 2.216$ (0.522)	$\hat{\lambda}_1 = -0.821$ (0.412)	$\hat{\lambda}_2 = 1.821$ (0.766)
CTR_{MR18b}	$\hat{\sigma} = 2.216$ (0.522)	$\hat{\lambda}_1 = 1.000$ (1.073)	$\hat{\lambda}_2 = -1.821$ (0.765)
CTR_{MR19}	$\hat{\sigma} = 2.158$ (0.086)	$\hat{\lambda} = 0.892$ (0.153)	
CTR_{R23}	$\hat{\sigma} = 2.216$ (0.525)	$\hat{\lambda} = 1.000$ (1.080)	$\hat{\theta} = 1.821$ (1.253)
TR	$\hat{\sigma} = 1.646$ (0.071)	$\hat{\lambda} = -0.919$ (0.121)	
Rayleigh	$\hat{\sigma} = 1.986$ (0.099)		

TABLE 16. Descriptive statistics for the guinea pig dataset

Min	Q_1	Median	Mean	Q_3	Max
0.01200	0.05475	0.07000	0.09982	0.11275	0.37600

Tables 17 and 18 respectively present the different model selection criteria for unmodified and modified models for the guinea pig dataset. In these tables, the ranks of the models, from best to worse, are given in parentheses. From the results for the unmodified models (Table 17) and the modified models (Table 18), the modified CTR_{MA} distribution is the best-fitting distribution for the guinea pig dataset, as it has the lowest values for all the criteria among all models considered.

Table 19 presents the MLEs and SEs (in parentheses) for the parameters of the modified models.

Figure 3 represents the fitting of the different modified PDFs to the guinea pig dataset. As in previous figures, the graphs for the CTR_{MG} , CTR_{MR18a} , and CTR_{MR18b} distributions are identical. Since $\hat{\theta} = 0$ for the CTR_{R23} distribution, its graph (represented in green dashed line) is identical to that of the TR distribution (represented in red dot-dashed line). The PDF of the CTR_{MA} distribution, represented by the black

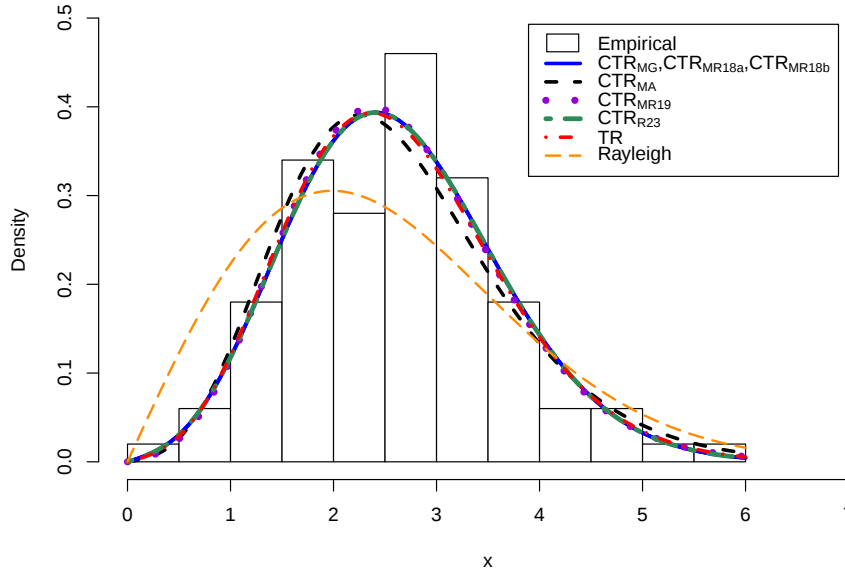


FIGURE 2. Graphical comparison of the modified CTR PDFs fitted to the carbon fibres dataset

TABLE 17. Model selection criteria and their corresponding ranks (in parentheses) for unmodified models on the guinea pig dataset

Distributions	$-\ell^*$	AIC	AICC	BIC
CTR_G	$-89.062^{(7)}$	$-172.124^{(8)}$	$-171.771^{(8)}$	$-165.294^{(8)}$
CTR_A	$-100.786^{(3)}$	$-197.572^{(2)}$	$-197.398^{(2)}$	$-193.019^{(2)}$
CTR_{R18a}	$-101.036^{(2)}$	$-196.072^{(3)}$	$-195.719^{(3)}$	$-189.242^{(3)}$
CTR_{R18b}	$-104.245^{(1)}$	$-202.490^{(1)}$	$-202.137^{(1)}$	$-195.660^{(1)}$
CTR_{R19}	$-94.309^{(6)}$	$-184.618^{(6)}$	$-184.444^{(6)}$	$-180.065^{(5)}$
CTR_{R23}	$-95.760^{(4)}$	$-185.520^{(5)}$	$-185.167^{(5)}$	$-178.690^{(6)}$
TR	$-95.760^{(4)}$	$-187.520^{(4)}$	$-187.346^{(4)}$	$-182.967^{(4)}$
Rayleigh	$-89.062^{(7)}$	$-176.124^{(7)}$	$-176.067^{(7)}$	$-173.847^{(7)}$

dashed line, provides the best fit to the histogram of the guinea pig dataset.

TABLE 18. Model selection criteria and their corresponding ranks (in parentheses) for modified models on the guinea pig dataset

Distributions	$-\ell^*$	AIC	AICC	BIC
CTR_{MG}	$-104.245^{(2)}$	$-202.490^{(2)}$	$-202.137^{(2)}$	$-195.660^{(2)}$
CTR_{MA}	$-110.079^{(1)}$	$-216.158^{(1)}$	$-215.984^{(1)}$	$-211.605^{(1)}$
CTR_{MR18a}	$-104.245^{(2)}$	$-202.490^{(2)}$	$-202.137^{(2)}$	$-195.660^{(2)}$
CTR_{MR18b}	$-104.245^{(2)}$	$-202.490^{(2)}$	$-202.137^{(2)}$	$-195.660^{(2)}$
CTR_{MR19}	$-94.374^{(7)}$	$-184.748^{(7)}$	$-184.574^{(7)}$	$-180.195^{(6)}$
CTR_{R23}	$-95.760^{(5)}$	$-185.520^{(6)}$	$-185.167^{(6)}$	$-178.690^{(7)}$
TR	$-95.760^{(5)}$	$-187.520^{(5)}$	$-187.346^{(5)}$	$-182.967^{(5)}$
Rayleigh	$-89.062^{(8)}$	$-176.124^{(8)}$	$-176.067^{(8)}$	$-173.847^{(8)}$

TABLE 19. MLEs of the parameters for modified models on the guinea pig dataset (SEs are in parentheses)

Distributions	Estimations		
CTR_{MG}	$\hat{\sigma} = 0.111$ (0.009)	$\hat{\lambda}_1 = 2.509$ (0.468)	$\hat{\lambda}_2 = 0.000$ (0.539)
CTR_{MA}	$\hat{\sigma} = 0.121$ (0.008)	$\hat{\lambda} = 2.592$ (0.253)	
CTR_{MR18a}	$\hat{\sigma} = 0.111$ (0.009)	$\hat{\lambda}_1 = 1.509$ (0.468)	$\hat{\lambda}_2 = -1.000$ (0.539)
CTR_{MR18b}	$\hat{\sigma} = 0.111$ (0.009)	$\hat{\lambda}_1 = 0.509$ (0.190)	$\hat{\lambda}_2 = 1.000$ (0.539)
CTR_{MR19}	$\hat{\sigma} = 0.105$ (0.009)	$\hat{\lambda} = -1.115$ (0.313)	
CTR_{R23}	$\hat{\sigma} = 0.103$ (0.014)	$\hat{\lambda} = 0.648$ (0.205)	$\hat{\theta} = 0.000$ (1.065)
TR	$\hat{\sigma} = 0.103$ (0.007)	$\hat{\lambda} = 0.648$ (0.141)	
Rayleigh	$\hat{\sigma} = 0.091$ (0.005)		

8. CONCLUSION

In this paper, we have considered six different CTR distributions and their modified versions based on the parameters range correction suggested by Geraldo et al. [14]. Inspired by the previous works of Geraldo et al. [14] and Katchekpele et al. [17], we have derived general formulas for the CDF, PDF and the statistical properties such as moments for

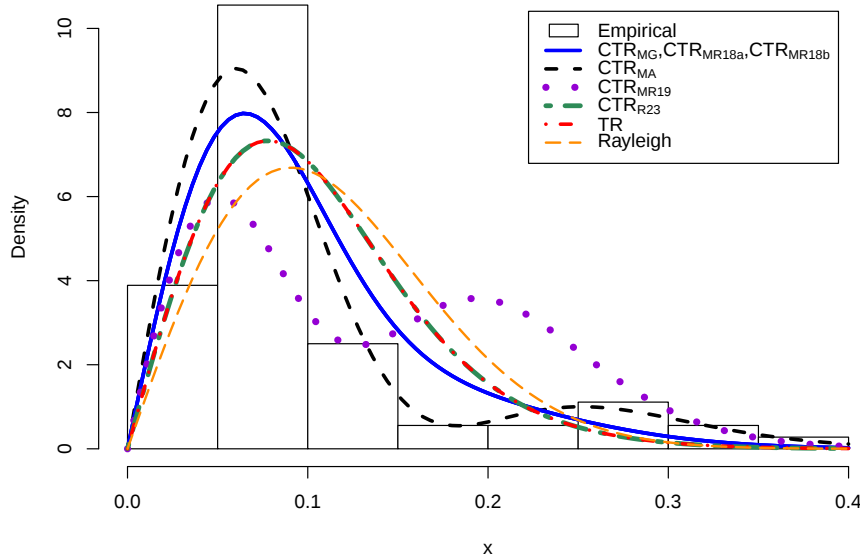


FIGURE 3. Graphical comparison of modified CTR PDFs fitted to the guinea pig dataset

the different CTR distributions. These formulas depend on two parameters which, once fixed, make it possible to find the desired property for the cubic transmutation form concerned. We have also compared the unmodified and modified versions of the different CTR distributions on real data.

Our work goes further than that of Sakthivel and Vidhya [34] since: (a) we consider six different CTR distributions instead of four; (b) instead of detailing the statistical properties model by model, we propose a general formula including all CTR distributions already available in the literature and also any other new form of CTR distribution that could be proposed in the future; (c) we simultaneously compare the six formulas on several real datasets; (d) we take into account the modifications of the parameter ranges proposed by Geraldo et al. [14] to improve the different models.

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