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Milstein scheme for the numerical solution of first-order uncertain stochastic differential equations in stock price simulation

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Abstract. Uncertain stochastic calculus is a developing branch of mathematics that aims to create models incorporating both aleatory (random) and epistemic (knowledge-based) uncertainties within dynamic systems. In essence, it recognizes two types of uncertainty related to dynamical systems: randomness and belief degree. The uncertain stochastic differential equation (USDE) models such systems by simultaneously integrating randomness and human uncertainty, expressed as belief degree. This growing field has introduced a novel class of equations called USDEs. Since finding exact or analytical solutions to these equations is often difficult, numerical methods provide a practical alternative for approximating solutions. This paper investigates the use of the Milstein method for solving USDEs. Specifically, the Milstein scheme is employed to address a stock pricing problem, and its results are compared with those obtained through the fourth-order Runge-Kutta and Euler methods. The results indicate that the Milstein method yields subtle estimates of stock prices.

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1. Introduction

Differential equations are fundamental tools for modeling dynamical systems across various fields, including biology, engineering, physics, and finance. Dynamical systems, which change over time, are influenced by two main types of uncertainty: randomness and belief degree. Systems affected solely by randomness are described by stochastic differential equations (SDEs), grounded in probability theory, whereas those governed by belief degrees are modeled using uncertain differential equations (UDEs), based on uncertainty theory.

The origins of SDEs trace back to Kolmogorov (1933), with significant advancements by Itô (1949), Black, and Scholes (1973). SDEs have proven especially valuable in financial modeling, exemplified by Black and Scholes' Nobel Prize-winning option pricing model. Conversely, UDEs stem from uncertainty theory, introduced by Liu (2007), and have found applications in areas such as stock modeling. Over time, it became evident that both randomness and belief degree could simultaneously influence a system, leading to the development of a new class of equations: uncertain stochastic differential equations (USDEs) [2]. USDEs are governed by chance theory, which integrates probability and uncertainty measures, and are employed to model processes driven by uncertain random variables [5]. For numerical solutions of USDEs, references include [1], which utilizes the fourth-order Runge-Kutta method (RK4), and [7], where Euler simulation is applied.

This paper is organized as follows: Section 2 introduces the fundamental concepts of chance theory and discusses theorems related to the existence, uniqueness, and stability of USDEs. The main contribution of this paper is the application of the Milstein method to approximate stock prices in uncertain stochastic markets, as detailed in Section 3. To the best of our knowledge, this approach has not been previously applied to this problem. Section 4 presents numerical simulations comparing the performance of the Milstein scheme with the RK4 and Euler methods.

2. Preliminaries

In this section, we present key concepts that are fundamental to this paper.

Definition 2.1. (Liu, [5]): Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ and $\mathcal{F}$ be a σ-algebra on a nonempty set Ω. If the triples $(\Gamma, \mathcal{L}, \mathcal{M})$ and $(\Omega, \mathcal{F}, P)$ are an uncertain space and a probability space respectively, then the chance space is the product measure $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, P)$.

An uncertain random variable is a function ξ defined on the chance space.

Definition 2.2. (Liu [5]): Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, P)$ be a chance space. A function ξ from a chance space to $\mathbb{R}$ such that for every Borel subset B of $\mathbb{R}$, $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{F}$, is called an uncertain random variable.

Theorem 2.3. *(Liu, [5]): If ξ is an uncertain random variable and B is a Borel set of $\mathbb{R}$, then*

$$\mathrm{Ch}\{\xi \in B\} = \int_0^1 P\{\omega \in \Omega \mid \mathcal{M}\{\xi(\omega) \in B\} \geq x\} dx$$

is called the chance measure of an uncertain random event $\{\xi \in B\}$.

Definition 2.4. (Liu [5]) Let ξ be an uncertain random variable. Then for any $x \in \mathbb{R}$, its chance distribution is defined by

$$\Phi(x) = \mathrm{Ch}\{\xi \leq x\}.$$

According to Gao and Yao [3] an uncertain stochastic process represents uncertain random variables that develop over time.

Definition 2.5. (Gao and Yao, [3]): Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, P)$ be a chance space and let T be a totally ordered set. An uncertain random process is a function $X_t(\gamma, \omega)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, P)$ to $\mathbb{R}$ that $\{X_t \in B\}$ is an event in $\mathcal{L} \times \mathcal{F}$ for any Borel set B of real numbers at each time t.

If X_t is a stochastic process (for example, Brownian motion) and Y_t is an uncertain process, then for every measurable function f, the process $Z_t = f(X_t, Y_t)$ is an uncertain random process. An example of an uncertain process is the Liu canonical process.

Definition 2.6. (Liu, [5]): An uncertain process C_t is said to be a canonical Liu process if

- $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
- C_t has stationary and independent increments,
- every increment $C_t - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

Definition 2.7. (Gao and Yao, [3]): Assume X_t is an uncertain random process on a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{F}, P)$. Then for each fixed

$\gamma^* \in \Gamma$ and $\omega^* \in \Omega$, the function $X_t(\gamma^*, \omega^*)$ is called a sample path of an uncertain random process of X_t.

Definition 2.8. (Fei, [2]): An uncertain stochastic process X_t is called continuous if the sample paths of X_t are all continuous functions of t for almost all $(\gamma, \omega) \in (\Gamma, \Omega)$.

The Itô-Liu integral, which extends both Itô's and Liu's integral as described in Liu (2009), is defined as follows. Let X_t be a stochastic process and Y_t be an uncertain process.

Definition 2.9. (Itô-Liu integral, Fei, [2]): Let $Z_t = (X_t, Y_t)$ be an uncertain stochastic process. For any partition $P = \{a = t_1, t_2, \ldots, t_{k+1} = b\}$ of $[a, b]$, with $a = t_1 < t_2 < \ldots < t_{k+1} = b$, the mesh is written as $\Delta = \max_{1<i\leq k} |t_{i+1} - t_i|$. Then the Itô-Liu integral of X_t with respect to $G_t = (W_t, C_t)$ is defined as follows,

$$\int_a^b Z_s dG_s = \lim_{\Delta\to 0} \sum_{i=1}^{N} X_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) + \lim_{\Delta\to 0} \sum_{i=1}^{N} Y_{t_i}\left(C_{t_{i+1}} - C_{t_i}\right)$$

provided that it exists in mean square and is an uncertain random variable, where C_t and W_t are one-dimensional canonical process and Brownian motion, respectively. In this case, Z_t is called Itô-Liu integrable. In particular, when $Y_t \equiv 0, Z_t$ is called Itô integrable, and when $X_t \equiv 0, Z_t$ is called Liu integrable.

The above integral plays a crucial role in formulating USDEs, as proposed by Matenda and Chikodza [6].

Uncertain Stochastic Differential Equations. This newly introduced class of differential equations is driven by a combination of Brownian motion and the canonical Liu process.

Definition 2.10. (Matenda and Chikodza, [6]): Let C_t and W_t be one-dimensional canonical process and Brownian motion respectively. For functions, f, g and h, the differential equation

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dC_t + h(t, X_t)\, dW_t, \tag{2.1}$$

is called an USDE. The solution to equation (2.1) is an uncertain random process X_t. In the absence of the stochastic component, this equation degenerates to an UDE

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dC_t,$$

proposed by Liu (2008).

Theorem 2.11. *(existence and uniqueness theorem,* [1]*): An uncertain stochastic differential equation*

$$dX_t = f\left(t, X_t\right) dt + g\left(t, X_t\right) dC_t + h\left(t, X_t\right) dW_t,$$

has a sample continuous unique solution if for some constant c, the functions $f\left(t, X_t\right)$*,* $g\left(t, X_t\right)$ *and* $h\left(t, X_t\right)$ *satisfy the linear growth and Lipschitz condition*

$$|f(t,x)| + |g(t,x)| + |h(t,x)| \leq c(1+|x|), \quad \forall x \in \mathbb{R}, t \geq 0,$$
$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| + |h(t,x) - h(t,y)| \leq c|x-y|, \quad \forall x, y \in \mathbb{R}, t \geq 0.$$

Definition 2.12. (stability, [1]): Let X_t and Y_t be any two solutions of an uncertain stochastic differential equation. If the two solutions satisfy the condition

$$\lim_{|X_0 - Y_0| \to 0} Ch\left\{|X_t - Y_t| > \epsilon, \forall t \geq 0\right\} = 0,$$

then an uncertain stochastic differential equation is said to be stable.

Theorem 2.13. *(stability theorem,* [1]*): If* a_t, b_t *and* c_t *are continuous functions satisfying*

$$\sup_{s \geq 0} \int_0^s a_t dt < +\infty, \int_0^{+\infty} |b_t|\, dt < +\infty, \int_0^{+\infty} |c_t|\, dt < +\infty,$$

then an uncertain stochastic differential equation

$$dX_t = a_t X_t dt + b_t X_t dC_t + c_t X_t dW_t,$$

is stable.

Definition 2.14. (Yao and Chen, [8]) The α-path $(0 < \alpha < 1)$ of an uncertain differential equation

$$dX_t = f\left(t, X_t\right) dt + g\left(t, X_t\right) dC_t,$$

with initial value X_0, is a deterministic function X_t^α with respect to t that solves the corresponding ordinary differential equation

$$dX_t^\alpha = f\left(t, X_t^\alpha\right) dt + \left|g\left(t, X_t^\alpha\right)\right| \Phi^{-1}(\alpha) dt,$$

where $\Phi^{-1}(\alpha)$ represents an inverse uncertainty distribution of the standard normal uncertain variable given by

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{\alpha}{1-\alpha}\right).$$

The solution X_t has inverse uncertainty distribution X_t^α [1].

3. Milstein Scheme with α-path

Consider the uncertain stochastic initial value problem $Y(0) = Y_0$:

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}t + g(Y_t)\mathrm{d}C_t + h(Y_t)\mathrm{d}W_t, \quad 0 \le t \le T.$$

The SDE with α-path is given by

$$\mathrm{d}Y_t^\alpha = f\left(Y_t^\alpha\right)\mathrm{d}t + \left|g\left(Y_t^\alpha\right)\right|\Phi^{-1}(\alpha)\mathrm{d}t + h\left(Y_t^\alpha\right)\mathrm{d}W_t, \quad Y^\alpha(0) = Y_0^\alpha. \tag{3.1}$$

To numerically solve the USDE, we first discretize the time interval by letting $\delta t = T/N$ for some positive integer N and $t_j = j\delta t$ [4]. $Y_{t_j}^\alpha$ is denoted by Y_j^α. The Milstein method for the SDE with α-path reads as

$$Y_j^\alpha = Y_{j-1}^\alpha + f\left(Y_{j-1}^\alpha\right)\delta t + \left|g\left(Y_{j-1}^\alpha\right)\right|\Phi^{-1}(\alpha)\delta t + h\left(Y_{j-1}^\alpha\right)\left(W_{t_j} - W_{t_{j-1}}\right)$$
$$+ \frac{1}{2}h\left(Y_{j-1}^\alpha\right)h'\left(Y_{j-1}^\alpha\right)\left(\left(W_{t_j} - W_{t_{j-1}}\right)^2 - \delta t\right),$$

for $j = 1, 2, \ldots, N$. Taking g = 0 results in the well-known stochastic case, where the system is assumed to have no uncertainty. Taking h = 0 yields the uncertainty case without randomness in the system.

Letting T be a fixed time, N be the number of time steps of length $\delta t = T/N$. The step size Δt is chosen as $\Delta t = R * \delta t$, with R $\ge$ 1. Following the algorithm below, the numerical solution is calculated, where M represents the number of simulated paths.

Step 1. Set $\alpha = 0.05$ and $j = 1$.
Step 2. Simulate increments of Brownian motion by $dW = sqrt(dt) * randn(M, N)$.
Step 3. Solve the SDE with $alpha$-path (3.1) using the Milstein scheme.
Step 4. Increase j by 1 and repeat steps 2 and 3 to the maximum value of N.
Step 5. Calculate $\mathbb{E}\left(Y_N^\alpha\right)$ by $\frac{\sum_{j=1}^M Y_N^\alpha}{M}$.
Step 6. The process is repeated for α values $0.05, 0.10, 0.15 \ldots 0.95$. This provides us with the inverse uncertainty distribution of Y_T.

4. Numerical Experiment

Matenda and Chikodza [6] proposed an uncertain stochastic stock model for the stock price S_t and bond price dQ_t:

$$\begin{cases} dQ_t = rQ_t dt, \\ dS_t = \mu S_t dt + \sigma_2 S_t dC_t + \sigma_1 S_t dW_t, \end{cases}$$

where r is the riskless interest rate, μ represents the drift of the stock, σ_2 is the uncertain diffusion and σ_1 is volatility of the stock.
Let $T = 1, \mu = 0.06, S_0 = 40, \sigma_1 = 0.29, \sigma_2 = 0.32, N = 100$. We estimate the distribution of the stock price S_t applying the Milstein

scheme and compare it with the exact solution, fourth-order Runge-Kutta (RK4) method, proposed by [1] and the Euler method of [7]. The corresponding α-path is

$$\mathrm{d}S_t^\alpha = \mu S_t^\alpha \mathrm{d}t + |\sigma_2 S_t^\alpha|\, \Phi^{-1}(\alpha)\mathrm{d}t + \sigma_1 S_t^\alpha \mathrm{d}W_t, \qquad (4.1)$$

which is an SDE whose solution S_t is a stochastic process and $\Phi^{-1}(\alpha)$ is deterministic and represents the inverse uncertainty distribution of the normal uncertain variable given by: $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$. Dropping $\sigma_1 S_t dW_t$ leaves us with the Liu's stock model proposed by Liu (2013), $dS_t = \mu S_t dt + \sigma_2 S_t dC_t$ and also, eliminating $\sigma_2 S_t dC_t$ reduces to the Black-Scholes (1973) stock model, $dS_t = \mu S_t dt + \sigma_1 S_t dW_t$.

α-path	Stock Price			
	Analytical method	EM method	RK4 method	Milstein scheme
0.05	25.265	24.524	24.617	24.861
0.15	31.276	29.580	30.474	31.754
0.20	33.258	31.292	32.406	33.978
0.25	34.990	32.805	34.093	34.915
0.30	36.576	34.206	35.638	36.490
0.35	38.079	35.547	37.103	39.220
0.40	39.541	36.864	38.528	40.148
0.45	40.996	38.185	39.945	40.167
0.50	42.473	39.539	41.385	42.283
0.55	44.004	40.954	42.876	43.846
0.60	45.623	42.465	44.454	44.635
0.65	47.375	44.116	46.161	47.303
0.70	49.322	45.970	48.058	49.679
0.75	51.558	48.124	50.236	51.232
0.80	54.242	50.744	52.852	54.246
0.85	57.680	54.155	56.201	57.339
0.90	62.586	59.126	60.981	62.626
0.95	71.404	68.362	69.574	71.545

TABLE 1. Comparison of stock prices at $T = 1$.

We numerically solve the Equation (4.1) using the Milstein method in MATLAB, with the results presented in Table 1. In comparison to other methods, the Milstein scheme produces accurate stock price estimates. We also calculate the error of the Milstein method, and Figure 1 shows that, compared to the Euler method [7], Milstein achieves a lower error and a strong order of convergence of 1, which is higher than that of the Euler method.

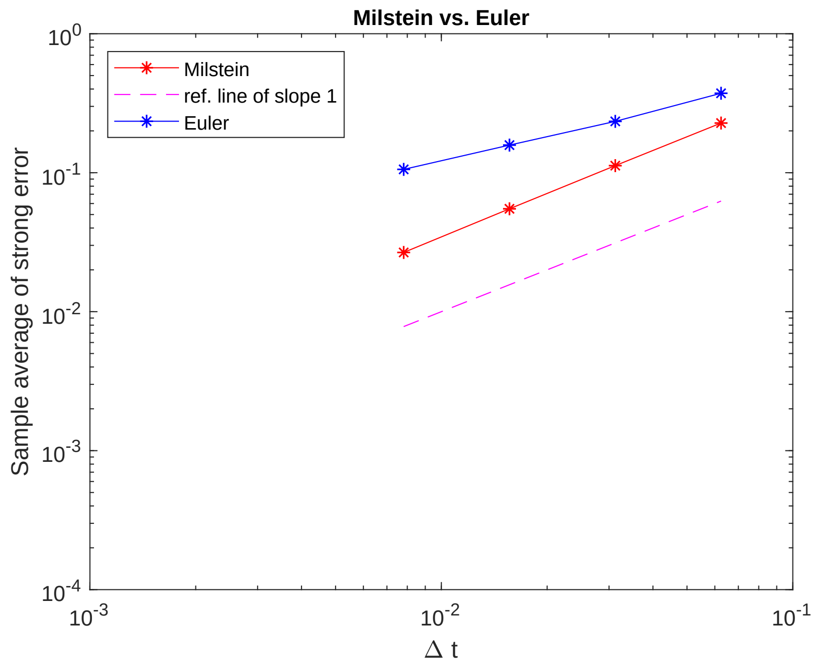

Figure 1. Comparison of Milstein vs. Euler errors.

Uncertainty distribution curves for the stock price and its enlarged section is shown here to compare the methods and the results show that the Milstein scheme is numerically exact enough for the estimation.

5. Conclusion

This paper presents chance space and its associated concepts as fundamental tools for defining USDEs (2.1), a novel class of differential equations that integrate both randomness and belief degree uncertainty. The Milstein scheme is utilized to approximate the stock price dynamics in an uncertain market. Numerical results indicate that, in comparison to other approaches, the Milstein scheme provides greater accuracy and produces lower errors.

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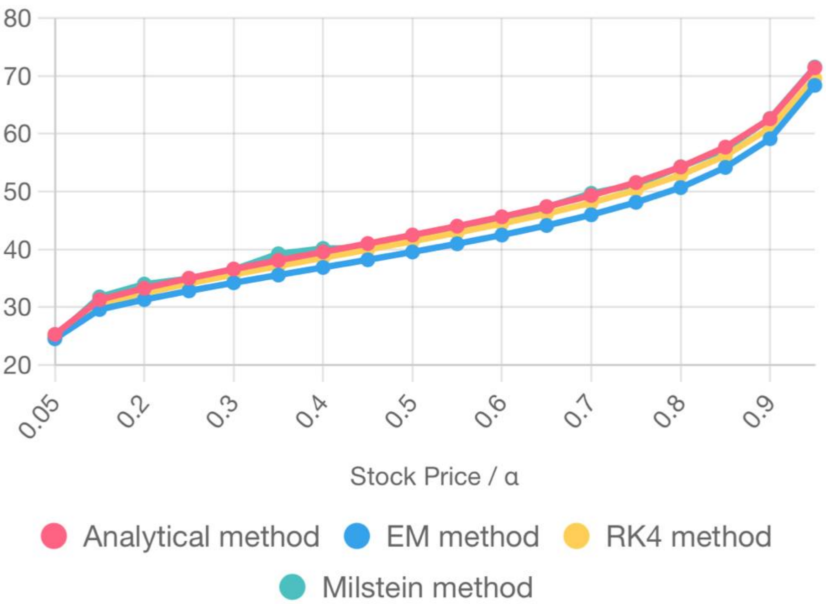

FIGURE 2. Uncertainty distribution curves for the stock price.

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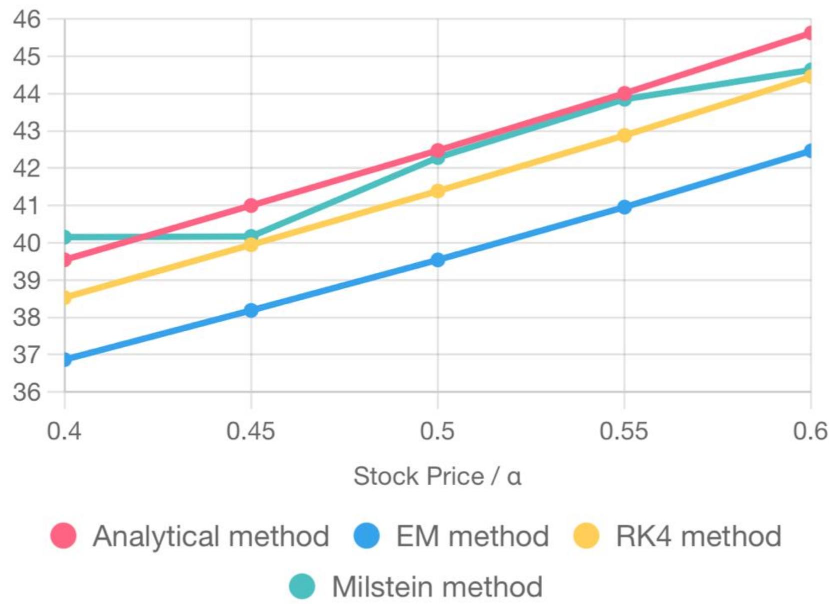

FIGURE 3. Enlarged section of Figure 2.