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Some miscellaneous results of the Fibonacci sequence and the golden ratio

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Abstract. The Fibonacci sequence and the golden ratio for centuries due to their deep mathematical properties and diverse applications in theoretical and applied fields.This paper explores the mathematical relationships between these two concepts and their practical uses in different fields. We construct a power series using Fibonacci numbers and demonstrate that the radius of convergence of this series is equal to the golden ratio. Furthermore, we investigate the conditions under which three Fibonacci numbers can form a triangle and analyze the properties of such triangles. We also introduce the concept of pseudo-right-angled triangles and provide a characterization of these figures. Finally, we analyze and decompose the polynomial $x^n - F_n x - F_{n-1} = 0$, a relation in which the golden ratio emerges as a root.

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1. Introduction and Mathematical Background

The Golden Ratio, denoted as ϕ, is a mathematical constant that appears in many aspects of mathematics, nature, art, and architecture [6]. It is defined by the following equation:

$$\frac{a+b}{a} = \frac{a}{b} = \phi,$$

where a and b are two quantities such that $a > b$. This ratio reveals a natural relationship that is seen in various geometrical structures and mathematical phenomena.

The Fibonacci sequence is a numerical sequence that starts with the two numbers 0 and 1, and each subsequent number is the sum of the two preceding ones:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2.$$

This sequence is one of the most important sequences in mathematics and is observed in various natural systems, such as the growth of plants and biological structures.

The Golden Ratio and the Fibonacci sequence are closely related. In fact, the ratio of consecutive Fibonacci numbers approaches ϕ as the numbers increase. This connection has made the Golden Ratio particularly significant in mathematical analysis, as well as in the arts and architecture [13].

The appearance of the golden ratio as the radius of convergence of the Fibonacci-generated power series is not just a mathematical curiosity; it reflects the deep connection between discrete recursive sequences and continuous analytic structures. In particular, generating functions involving Fibonacci numbers often emerge in the study of dynamical systems, analytic combinatorics, and computational models. The fact that the golden ratio governs the convergence behavior behaviour reinforces its role as a natural scaling factor in recursive phenomena. This connection has been discussed in the context of generating functions and analytic combinatorics [5].

In recent years, various families of special polynomials have been developed and analyzed using different operational and algebraic frameworks. These approaches offer versatile methods to extend classical polynomial structures, including Hermite, Laguerre, and hybrid types, and connect them with functional transforms and spectral methods. Notably, studies such as Dattoli et al. on hybrid polynomials [2], and further developments in the context of Hermite-type polynomials linked to parabolic cylinder functions [1] provide broader perspectives on generalizations and their functional applications. Similarly, structural frameworks for such generalized polynomials and their role in number theory and approximation theory have been explored in [9]. These perspectives support the integration of our approach into a wider mathematical context involving weighted polynomial families and pseudo-right-angled triangle classifications.

In this article, we will explore the properties and applications of the Golden Ratio and the Fibonacci sequence and analyze their interrelation in the real world.

1.1. **Properties and Applications of the Golden Ratio.** The Golden Ratio, as mentioned in the introduction, is a mathematical constant that appears in various natural and human-made phenomena. This ratio has applications in several fields, including geometry, art, architecture, and biology.

i). **Geometrical Properties**: The Golden Ratio is particularly relevant in geometry, especially in relation to rectangles. Rectangles whose side lengths adhere to the Golden Ratio are known as *Golden Rectangles*. In such rectangles, when a square is removed, the remaining rectangle maintains the same Golden Ratio.

If we have a rectangle with dimensions $a \times b$, then:

$$\frac{a}{b} = \phi,$$

where $\phi \approx 1.618$ [3].

ii). **Algebraic Equation**: The Golden Ratio ϕ can be defined algebraically by the following equation:

$$\phi = \frac{1+\sqrt{5}}{2}.$$

This equation is a quadratic equation, which has only one positive root, and that root is the Golden Ratio.

iii). **Similar Ratios in Nature**: The Golden Ratio also appears in natural structures, such as flowers, tree branches, and even in the structure of galaxies [13].

iv). **Architecture**: One of the most famous applications of the Golden Ratio in architecture is found in historical structures such as the Great Pyramid of Giza and the temples of ancient Greece (Figure 1). These structures typically feature proportions that approximate the Golden Ratio.

In particular, the Great Pyramid of Giza is believed to have dimensions based on the Golden Ratio. The ratio of its height to half the base length closely approximates $\phi \approx 1.618$. More precisely, this ratio can be expressed as $\sqrt{\phi^2 + 1}$, which mathematically links the pyramid's slope angle to the golden number.

Similarly, the Parthenon in Athens, a symbol of ancient Greek architecture, is often cited as a canonical example. The ratio of facade height to width, as well as internal spatial divisions, exhibit approximations of ϕ. These proportional relationships, observed in both external dimensions and column spacing, suggest a deliberate or naturally emergent use of the Golden Ratio in classical design.

Such geometric and numerical evidence strengthens the hypothesis that the Golden Ratio was more than an aesthetic principle.

The following image illustrates these applications of the Golden Ratio in architecture, showcasing both the Great Pyramid of Giza (on the left) and the Parthenon (on the right). The Golden Ratio proportions are highlighted on both structures, indicating how this mathematical principle has been applied to create harmonious and aesthetically pleasing designs in ancient architecture [8, 10].

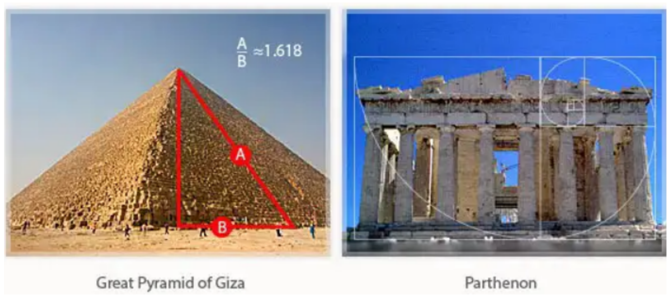

FIGURE 1. This image shows the application of the Golden Ratio in two iconic architectural structures: the Great Pyramid of Giza (left) and the Parthenon (right). The proportions of these structures approximately follow the Golden Ratio. (Image source: *https://connect2grp.medium.com/java-understanding-the-golden-ration-phi-19fef244f81a*)

v). **The Golden Ratio in Art** Leonardo da Vinci's "The Last Supper" (Figure 2) is a significant example of how the Golden Ratio plays a crucial role in the composition of the artwork. This famous painting demonstrates how mathematical principles, particularly the Golden Ratio, were applied by the artist to achieve balance and harmony in the arrangement of figures and architectural elements.

In the provided image, we can observe various rectangular divisions, highlighted in different colors, which are based on the Golden Ratio. These divisions are not coincidental but represent the careful placement of the figures and key elements of the scene. The placement of Jesus at the center, as well as the proportional arrangement of the surrounding apostles, follows the Golden Ratio to create an aesthetically pleasing and harmonious visual effect.

The use of the Golden Ratio is also evident in the overall proportions of the room and the arches above. The division of space within the room, the distribution of figures, and the positioning of the background elements such as the windows and ceiling are all designed to reflect the Golden Ratio, enhancing the sense of balance and symmetry [6].

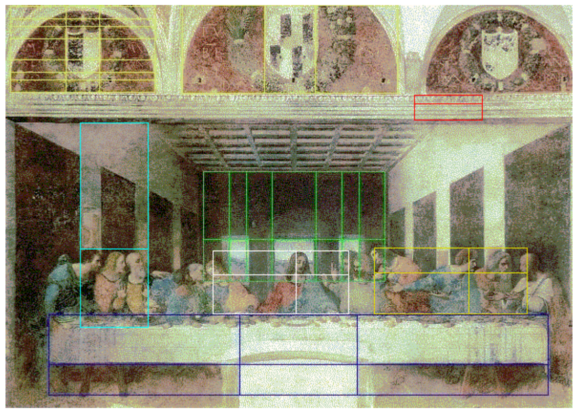

FIGURE 2. An example of Leonardo da Vinci's "The Last Supper," where the proportions and composition follow the Golden Ratio. (Image source: https://www.goldennumber.net/art-composition-design/)

1.2. **Main Properties of the Fibonacci Sequence.** The Fibonacci sequence is one of the most important and well-known numerical sequences, which has applications in mathematics, nature, art, and architecture. This sequence is generally composed of a series of numbers where each number is the sum of the two preceding ones. The sequence is defined as follows:

$$F_0 = 0, \quad F_1 = 1,$$

and for $n \geq 2$:

$$F_n = F_{n-1} + F_{n-2}.$$

i). **Recurrence relation :** As seen in the definition of the Fibonacci sequence, each number (except for the first two) is the sum of the two previous numbers [13]. This property is especially useful in many mathematical problems and computer algorithms. For example, by using this relation, we can easily calculate any number in the sequence.

For example, to calculate F_5 in the Fibonacci sequence, we first calculate the previous values:

$$F_2 = F_1 + F_0 = 1 + 0 = 1,$$
$$F_3 = F_2 + F_1 = 1 + 1 = 2,$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3,$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5.$$

Therefore, $F_5 = 5$.

ii). **Golden Ratio:** One of the interesting properties of the Fibonacci sequence is that the ratio of two consecutive numbers in the sequence, when we approach larger numbers, tends to a constant value. This constant value is known as the Golden Ratio, denoted by ϕ, and is approximately equal to 1.618.

For example, if we consider $F_5 = 5$ and $F_6 = 8$, the ratio between them is:

$$\frac{F_6}{F_5} = \frac{8}{5} = 1.6.$$

Now, if we consider $F_7 = 13$:

$$\frac{F_7}{F_6} = \frac{13}{8} = 1.625.$$

As we can see, this ratio continuously gets closer to the Golden Ratio.

iii). **Presence of the Fibonacci Sequence in Nature:** Another fascinating property of the Fibonacci sequence is its occurrence in nature. In many plants, flowers, leaves, and even the structures of animal bodies, the Fibonacci sequence appears. This natural property creates a link between mathematics and nature.

iv). **Combinatorial Properties:** The Fibonacci sequence also has interesting properties in combinatorics [6, 11, 12]. For example, the number of ways to climb a staircase with n steps, where at each step one can either take a single step or jump two steps, is directly related to the Fibonacci sequence. Specifically, the number of ways to reach the n-th step is equal to F_n.

This combinatorial aspect could be used to interpret the triangular constructions: the number of such triangle configurations based on Fibonacci sides corresponds to partitions of a given number, reminiscent of tiling problems and recursive sequences [11, 12].

For example, for 5 steps, the number of distinct ways to climb them is $F_5 = 5$. In fact, this problem can be solved using the recurrence relation of the Fibonacci sequence:

$$W_n = W_{n-1} + W_{n-2},$$

where W_n is the number of ways to climb n steps, and it is clearly equivalent to the Fibonacci sequence.

2. Main results

In the following theorem, we show that the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$, whose general term is the inverse of the nth term of the Fibonacci sequence, is a convergent series.

Theorem 2.1. *Let $\{F_n\}_{n=1}^{\infty}$ be a Fibonacci sequence, then the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$ is convergent and*

$$0 < \sum_{n=1}^{\infty} \frac{1}{F_n} \leq 2\sqrt{5}.$$

Proof. Suppose that $\{F_n\}_{n=1}^{\infty}$ is a Fibonacci sequence and consider the series $\sum_{n=1}^{\infty} \dfrac{1}{F_n}$ and applying the ratio test, one computes the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{F_{n+1}}}{\frac{1}{F_n}} \right| = \lim_{n\to\infty} \left| \frac{F_n}{F_{n+1}} \right| = \frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{\sqrt{5}-1}{2},$$

since this limit is less than 1, the series converges. On the other hand, for $n \geq 1$ we have

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \right),$$

and if n is an odd number so

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \right) \geq \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

To bound the series, note that $F_n \geq \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$. Thus,

$$\frac{1}{F_n} \leq \sqrt{5} \left(\frac{2}{1+\sqrt{5}} \right)^n = \sqrt{5} \cdot r^n, \text{ where } r = \frac{2}{1+\sqrt{5}} < 1,$$

which ensures convergence and gives the upper bound via the geometric series formula.

Since $F_n < F_{n+1}$, for every $n \geq 2$ then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{F_n} &= \frac{1}{F_1} + \frac{1}{F_3} + \frac{1}{F_5} + \cdots + \frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_6} + \cdots \\
&\leq \frac{1}{F_1} + \frac{1}{F_3} + \frac{1}{F_5} + \cdots + \frac{1}{F_1} + \frac{1}{F_3} + \frac{1}{F_5} + \cdots \\
&= 2\left(\frac{1}{F_1} + \frac{1}{F_3} + \frac{1}{F_5} + \cdots\right) \\
&\leq 2\sqrt{5}\left(\left(\frac{\sqrt{5}-1}{2}\right) + \left(\frac{\sqrt{5}-1}{2}\right)^3 + \left(\frac{\sqrt{5}-1}{2}\right)^5 + \cdots\right) \\
&= 2\sqrt{5}\frac{\frac{\sqrt{5}-1}{2}}{1-\left(\frac{\sqrt{5}-1}{2}\right)^2} \\
&= 2\sqrt{5}.
\end{aligned}$$

According to the above and also that the members of the Fibonacci sequence are positive, consequently,

$$0 < \sum_{n=1}^{\infty} \frac{1}{F_n} \leq 2\sqrt{5},$$

and this completes the proof. □

From Theorem 2.1, it can be concluded that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{F_n}$ is equal to $\dfrac{1+\sqrt{5}}{2}$, which is the golden ratio.

Corollary 2.2. *Let $x_0 \in \mathbb{R}$ and $\{F_n\}_{n=1}^{\infty}$ be a Fibonacci sequence, then the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{F_n}$ is equal to $\dfrac{1+\sqrt{5}}{2}$, which is the golden ratio.*

The following theorem shows under what conditions the three members of the Fibonacci sequence can be the length of the three sides of a triangle.

Theorem 2.3. *Let A, B, and C are the three vertices of a triangle and a, b, and c are three members of the Fibonacci sequence such that*

$$\begin{cases} BC = a, \\ AC = b, \\ AB = c, \end{cases}$$

then the following are true.

(1) *If a, b, and c are three distinct members of the Fibonacci sequence, then $\overset{\Delta}{ABC}$ does not form a triangle.*

(2) *If $a = b = F_n$ and $c = F_{n+1}$, where in $n \geq 2$, then $\overset{\Delta}{ABC}$ forms an isosceles triangle*

(3) *If $a = F_n$ and $b = c = F_{n+1}$, then $\overset{\Delta}{ABC}$ forms an isosceles triangle*

Proof. (1) Suppose that $a = F_l$, $b = F_m$ and $c = F_n$, where $l < m < n$, then $a < b < c$, so we have

$$a + b = F_l + F_m \leq F_{m-1} + F_m = F_{m+1} \leq F_n = c,$$

therefore $a + b \leq c$, consequently $\overset{\Delta}{ABC}$ does not form a triangle.

(2) Let's assume that $\overset{\Delta}{ABC}$ is an isosceles triangle with sides $a = b = F_n$ and $c = F_{n+1}$, then

$$\begin{cases} a + b = F_n + F_n > F_n + F_{n-1} = F_{n+1} = c, \\ a + c = F_n + F_{n+1} > F_n = b, \\ b + c = F_n + F_{n+1} > F_n = a, \end{cases}$$

so $\overset{\Delta}{ABC}$ forms an isosceles triangle.

(3) Let's assume that $\overset{\Delta}{ABC}$ is an isosceles triangle with sides $a = F_n$ and $b = c = F_{n+1}$, then

$$\begin{cases} a + b = F_n + F_{n+1} > F_{n+1} = c, \\ a + c = F_n + F_{n+1} > F_{n+1} = b, \\ b + c = F_{n+1} + F_{n+1} > F_n = a, \end{cases}$$

so $\overset{\Delta}{ABC}$ forms an isosceles triangle.

□

Now, if we denote the isosceles triangles made in theorem 2.3 then the following theorems hold for these isosceles triangles generated by the Fibonacci sequence.

Theorem 2.4. *Let $A_n\overset{\Delta}{B_n}C_n$ be an isosceles triangle generated by the Fibonacci sequence such that $A_nC_n = B_nC_n = F_n$, and $A_nB_n = F_{n+1}$ where in $n \geq 2$, then*

(1) *The angle $\widehat{C_n}$ is always an obtuse angle.*

(2) $\lim_{n\to\infty} \cos \widehat{C_n} = \cos 108°$.

Proof. (1) It is enough to show that $\cos\widehat{C_n} < 0$. For $n \geq 2$ we have

$$\begin{aligned}\cos\widehat{C_n} &= \frac{F_n^2 + F_n^2 - F_{n+1}^2}{2F_nF_n} \\ &= \frac{2F_n^2 - (F_n + F_{n-1})^2}{2F_n^2} \\ &= \frac{F_n^2 - 2F_nF_{n-1} - F_{n-1}^2}{2F_n^2} \\ &= \frac{F_n(F_n - 2F_{n-1}) - F_{n-1}^2}{2F_n^2}.\end{aligned}$$

Since $F_n - 2F_{n-1} < 0$ and $-F_{n-1}^2 < 0$, then $\cos\widehat{C_n} < 0$ and Hence, the angle $\widehat{C_n}$ is an obtuse angle.

(2)

$$\begin{aligned}\lim_{n\to\infty}\cos\widehat{C_n} &= \lim_{n\to\infty}\frac{2F_n^2 - F_{n+1}^2}{2F_n^2} \\ &= \frac{1}{2}\lim_{n\to\infty}\left(2 - \frac{F_{n+1}^2}{F_n^2}\right) \\ &= \frac{1}{2}\left(2 - (\frac{1+\sqrt{5}}{2})^2\right) \\ &= \frac{1-\sqrt{5}}{4} \\ &= \cos 108^\circ.\end{aligned}$$

□

Theorem 2.5. *Let* $A_n\overset{\Delta}{B_n}C_n$ *be an isosceles triangle generated by the Fibonacci sequence such that* $A_nB_n = F_n$, *and* $A_nC_n = B_nC_n = F_{n+1}$, *then*

(1) *The angle* $\widehat{C_n}$ *is always an acute angle.*

(2) $\lim_{n\to\infty}\cos\widehat{C_n} = \cos 36^\circ$.

Proof. (1) It is enough to show that $\cos\widehat{C_n} > 0$. For every $n \in \mathbb{N}$ we have

$$\cos\widehat{C_n} = \frac{F_{n+1}^2 + F_{n+1}^2 - F_n^2}{2F_{n+1}F_{n+1}} > \frac{2F_{n+1}^2 - F_{n+1}^2}{2F_{n+1}^2} = \frac{F_{n+1}^2}{2F_{n+1}^2} = \frac{1}{2},$$

so the angle $\widehat{C_n}$ is an acute angle.

(2)

$$\begin{aligned}
\lim_{n\to\infty} \cos \widehat{C_n} &= \lim_{n\to\infty} \frac{2F_{n+1}^2 - F_n^2}{2F_{n+1}^2} \\
&= \frac{1}{2} \lim_{n\to\infty} \left(2 - \frac{F_n^2}{F_{n+1}^2}\right) \\
&= \frac{1}{2} \left(2 - (\frac{2}{1+\sqrt{5}})^2\right) \\
&= \frac{1+\sqrt{5}}{4} = \cos 36^\circ.
\end{aligned}$$

□

Lemma 2.6. *Let $\overset{\triangle}{ABC}$ be a right-angled triangle, if the sides of the triangle $\overset{\triangle}{ABC}$ form a geometric sequence, then the ratio of the chord to the smaller side of the right angle is equal to the golden number.*

Proof. Suppose that $\begin{cases} BC = a \\ AC = b \\ AB = c \end{cases}$ and $\widehat{A} = 90^\circ$, then we have according to the given conditions

$$a^2 = b^2 + c^2,$$

and

$$b^2 = ac,$$

so

$$(\frac{a}{c})^2 - \frac{a}{c} - 1 = 0,$$

because a, b and c are positive numbers, therefore

$$\frac{a}{c} = \frac{1+\sqrt{5}}{2}.$$

□

In this part, we define the concept of pseudo-right-angled triangle.

Definition 2.7. Let's consider the triangle $\overset{\triangle}{ABC}$ so that $\begin{cases} BC = a \\ AC = b \\ AB = c \end{cases}$. We call this triangle pseudo-Pythagorean or pseudo-right-angled if

$$a^2 = mb^2 + nc^2,$$

where $m, n \in \mathbb{R}$. In a special case, if $m = n = 1$, then the triangle $\overset{\Delta}{ABC}$ is the right triangle. If m and n are chosen from the Fibonacci sequence, in this case, we call the triangle $\overset{\Delta}{ABC}$ Fibonacci pseudo-Pythagorean.

The concept of pseudo-right-angled triangles provides a generalization of the classical Pythagorean theorem, where the sum of the squares of two sides is linearly weighted by constants m and n. When m and n are selected from the Fibonacci sequence, this yields what we call Fibonacci pseudo-Pythagorean triangles. Such generalizations are not merely theoretical: they may arise in computational geometry, particularly in mesh generation algorithms where flexible triangle constraints are needed, or in optimization problems with geometric cost functions. Further, this idea aligns with extended notions of distance in weighted graph representations of geometric structures, where triangle side relations are modified based on data-driven weights. Below we characterize the pseudo-right-angled triangles.

Theorem 2.8. *If two medians of a triangle are perpendicular to each other, then the triangle is pseudo-right-angled.*

Proof. Let $\triangle ABC$ be a triangle with medians AE, BD, and CF drawn from vertices A, B, and C to the midpoints of the opposite sides BC, AC, and AB, respectively. Denote the midpoints as:

$$\begin{cases} E : \text{midpoint of } BC, \\ D : \text{midpoint of } AC, \\ F : \text{midpoint of } AB. \end{cases}$$

Assume that medians BD and CF are perpendicular and intersect at point G, the centroid of triangle $\triangle ABC$. Figure 3 illustrates this configuration, highlighting the perpendicularity of the medians at the centroid.

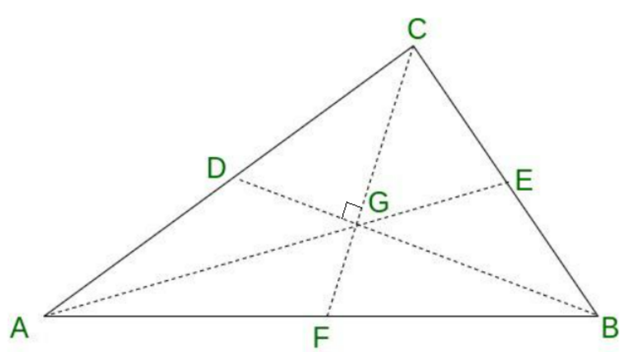

Figure 3.

Let the sides of the triangle be denoted by:

$$\begin{cases} BC = a, \\ AC = b, \\ AB = c, \end{cases} \quad \text{and the medians as:} \quad \begin{cases} BD = d, \\ CF = f, \\ AE = e. \end{cases}$$

From the well-known formula for the length of a median in a triangle, we have:

$$d^2 = \frac{2a^2 + 2c^2 - b^2}{4},$$
$$f^2 = \frac{2a^2 + 2b^2 - c^2}{4},$$
$$e^2 = \frac{2b^2 + 2c^2 - a^2}{4}.$$

Since the centroid divides each median in a $2:1$ ratio, we get:

$$BG = \frac{2}{3}d,$$
$$CG = \frac{2}{3}f.$$

By the Pythagorean Theorem applied to triangle $\triangle BGC$ (since $\angle BGC = 90^\circ$), we have:

$$(BG)^2 + (CG)^2 = (BC)^2 = a^2,$$

so:

$$\left(\frac{2}{3}d\right)^2 + \left(\frac{2}{3}f\right)^2 = a^2,$$

which simplifies to:

$$\frac{4}{9}(d^2 + f^2) = a^2 \quad \Rightarrow \quad 4(d^2 + f^2) = 9a^2.$$

Now substitute the values of d^2 and f^2:

$$4\left(\frac{2a^2 + 2c^2 - b^2}{4} + \frac{2a^2 + 2b^2 - c^2}{4}\right) = 9a^2,$$
$$(2a^2 + 2c^2 - b^2) + (2a^2 + 2b^2 - c^2) = 9a^2,$$
$$4a^2 + b^2 + c^2 = 9a^2 \quad \Rightarrow \quad b^2 + c^2 = 5a^2.$$

This is a generalized Pythagorean relation (not $a^2 + b^2 = c^2$), and hence the triangle satisfies the identity:

$$b^2 + c^2 = 5a^2,$$

which confirms that $\triangle ABC$ is a *pseudo-right-angled* triangle, as defined earlier.

$\square$

The following result is an interesting answer to the question under which conditions the converse of Theorem 2.8 is true.

Corollary 2.9. *Let* $\overset{\Delta}{ABC}$ *be a pseudo-right-angled such that* $b^2 + c^2 = 5a^2$, *then the two medians of this triangle are perpendicular.*

Corollary 2.10. *If two medians of a triangle are perpendicular to each other, then the third median is* $\frac{3}{2}$ *of the side facing to it.*

Proof. In the proof of Theorem 2.8, we saw that $e^2 = \dfrac{2b^2 + 2c^2 - a^2}{4}$, so $e^2 = \dfrac{9a^2}{4}$ consequently, $e = \dfrac{3a}{2}$, and the proof is complete. □

In the following theorem, we decompose the polynomial $x^n - F_n x - F_{n-1} = 0$, whose golden number is one of its roots.

Theorem 2.11. *Let* $\{F_n\}_{n=1}^{\infty}$ *be a Fibonacci sequence, then the polynomial* $x^n - F_n x - F_{n-1}$ *where* $n \geq 2$ *is decomposed as follows*

$$x^n - F_n x - F_{n-1} = (x^2 - x - 1)\left(\sum_{j=1}^{n-1} F_j x^{n-1-j}\right). \tag{2.1}$$

Proof. We will prove by induction that (2.1) holds for all $n \geq 2$. For the base case, we can clearly see that Equation (2.1) holds for $n = 2$. For the inductive step, assume that Equation (2.1) holds for $n = k$, meaning that

$$x^k - F_k x - F_{k-1} = (x^2 - x - 1)\left(\sum_{j=1}^{k-1} F_j x^{k-1-j}\right).$$

We will prove that Equation (2.1) holds for $n = k + 1$. To see this, note that

$$\begin{aligned}
x^{k+1} - F_{k+1}x - F_k &= x^{k+1} + x^k - x^k - F_k x - F_{k-1}x - F_{k-1} - F_{k-2} \\
&= (x^{k+1} - x^k - F_{k-1}x - F_{k-2}) + (x^k - F_k x - F_{k-1}) \\
&= (x^{k+1} - x^k - x^{k-1} + x^{k-1} - F_{k-1}x - F_{k-2}) + (x^k - F_k x - F_{k-1}) \\
&= x^{k-1}(x^2 - x - 1) + (x^{k-1} - F_{k-1}x - F_{k-2}) + (x^k - F_k x - F_{k-1}) \\
&= (x^2 - x - 1)\left(x^{k-1} + \sum_{j=1}^{k-2} F_j x^{k-2-j} + \sum_{j=1}^{k-1} F_j x^{k-1-j}\right) \\
&= (x^2 - x - 1)\left(\sum_{j=1}^{k} F_j x^{k-j}\right).
\end{aligned}$$

We've arrived at statement Equation (2.1), which we know is true and completing the induction. □

Potential Applications

The concept of pseudo-right-angled triangles provides a generalization of classical triangle relations and may be relevant in applied geometry, particularly in physics-based simulations where strict right-angle constraints are relaxed but geometric consistency is still required. For example, in finite element mesh generation for curved surfaces, approximate angular relations governed by weighted side squares (as in pseudo-right-angled configurations) could improve stability in irregular domains.

On the other hand, the polynomial decomposition $x^n - F_n x - F_{n-1} = (x^2 - x - 1) \cdot P_{n-2}(x)$ can be interpreted within the framework of recursive filters in signal processing, where Fibonacci-based recurrence models appear in digital systems. These polynomials may also serve as basis functions in spectral methods for solving differential equations, particularly in Fibonacci-like discretization schemes or grid refinement algorithms.

Conclusion

In this article, we investigated several important properties of the Fibonacci sequence and its connection to convergence of series and geometric properties. Specifically, we demonstrated that the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$, where the general term is the reciprocal of the n-th Fibonacci number, converges. Furthermore, we found an upper bound for the sum of this series, which is given by $2\sqrt{5}$. We also explored the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-x_0)^n}{F_n}$, establishing that it is equal to the golden ratio $\frac{1+\sqrt{5}}{2}$. Finally, we examined the conditions under which the Fibonacci numbers can represent the sides of a triangle. Specifically, we showed that when the sides of the triangle are Fibonacci numbers, certain configurations result in a valid triangle, including isosceles triangles. These results not only shed light on the mathematical properties of the Fibonacci sequence but also provide a connection between number theory, series convergence, and geometry. Future work could explore further generalizations and applications of these findings in both theoretical and practical contexts.

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