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Second cohomology space of $Vect(\mathbb{R})$ acting on the space of bilinear bidifferential operators, vanishing on $\mathfrak{sl}(2)$

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Abstract. We consider the $Vect(\mathbb{R})$-module structure on the spaces of bilinear bidifferential operators acting on the spaces of weighted densities. We compute the second relative cohomology group of the Lie algebra $Vect(\mathbb{R})$ with coefficients in the space of bilinear bidifferential operators that act on tensor densities $\mathcal{D}_{\lambda,\nu,\mu}$, vanishing on the Lie algebra $\mathfrak{sl}(2)$.

This work is the simplest generalization of a result by Basdouri et al. [Second cohomology space of $\mathfrak{sl}(2)$ acting on the space of bilinear bidifferential operators], **Journal Revista de la Union Matematica Argentina (2020)**.

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1. Introduction

Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. We shall associate a cochain complex known a the **Chevalley-Eilenberg differential**. The n-th space of this complex will be denoted by $C^n(\mathfrak{g},\ M)$.
For $n>0$, it is the space of n-linear antisymmetric mappings of $\mathfrak{g}$ into M: they will be called n-cochains of $\mathfrak{g}$ with coefficients in M. The space of 0-cochains $C^0(\mathfrak{g},\ M)$ reduces to M. The differential δ^n will be defined by the following formula: for $c\in C^n(\mathfrak{g},\ M)$, the $(n+1)$-cochain $\delta^n(c)$ evaluated on $g_1, g_2, \cdots, g_{n+1}\in\mathfrak{g}$ gives:

$$\begin{aligned}\delta^n c(g_1,\dots,g_{n+1}) = &\sum_{1\le s<t\le n+1}(-1)^{s+t-1}c([g_s,g_t],g_1,\dots,\hat{g}_s,\dots,\hat{g}_t,\dots,g_{n+1})\\ &+\sum_{1\le s\le n+1}(-1)^s g_s\, c(g_1,\dots,\hat{g}_s,\dots,g_{n+1}).\end{aligned}$$

the notation $\hat{g}_i$ indicates that the i-th term is omitted.
We check that $\delta^{n+1}\circ\delta^n=0$, so we have a complex:

$$0\to C^0(\mathfrak{g},\ M)\to\cdots\to C^{n-1}(\mathfrak{g},\ M)\overset{\delta^{n-1}}{\to} C^n(\mathfrak{g},\ M)\to\cdots$$

We note by $H^n(\mathfrak{g},\ M)=\ker\delta^n/Im\delta^{n-1}$ the quotient space. This space is called the space of n-cohomology of $\mathfrak{g}$ with coefficients in M. We denote by:

$Z^n(\mathfrak{g},\ M)=\ker\delta_n$: the space of n-cocycles

$B^n(\mathfrak{g},\ M)=Im\delta_{n-1}$: the space of n-coboundaries.

For $M=\mathbb{R}$ (or $\mathbb{C}$) considered as a trivial module, we denote the cohomologies, in this case, $H^n(\mathfrak{g})$.

We shall now recall classical interpretations of cohomology spaces of low degrees.

• The space $H^0(\mathfrak{g},\ M)\simeq Inv_{\mathfrak{g}}(M):=\{m\in M;\ \forall\ X\in\mathfrak{g},\ X.m=0\}$.

• The space $H^1(\mathfrak{g},\ M)$ classifies derivations of $\mathfrak{g}$ with values in M modulo inner ones (see [6]). This result is particularly useful when $M=\mathfrak{g}$ with the adjoint representation.

• The space $\mathrm{H}^2(\mathfrak{g},M)$ classifies extensions of Lie algebra $\mathfrak{g}$ by M (see [24, 25]), i.e. short exact sequences of Lie algebras

$$0\to M\to\hat{\mathfrak{g}}\to\mathfrak{g}\to 0.$$

in which M is considered as an abelian Lie algebra. We shall mainly consider two particular cases of this situation which will be extensively studied in the sequel:

∘ If M is a trivial $\mathfrak{g}$-module (typically $M=\mathbb{R}$ or $\mathbb{C}$), $\mathrm{H}^2(\mathfrak{g},M)$ classifies central extensions modulo trivial ones. Recall that a central extension of $\mathfrak{g}$ by $\mathbb{R}$ produces a new Lie bracket on $\hat{\mathfrak{g}}=\mathfrak{g}\oplus M$ by setting

$$[(X,\lambda),(Y,\mu)]=([X,Y],c(X,Y)).$$

It is trival if the cocycle $c = dl$ is a coboundary of a 1-cochain l, in which case the map $(X, \lambda) \to (X, \lambda - l(X))$ yields a Lie isomorphism between $\hat{\mathfrak{g}}$ and $\mathfrak{g} \oplus M$ considered as a direct sum of Lie algebras.

∘ If $M = \mathfrak{g}$ with the adjoint representation, then $\mathrm{H}^2(\mathfrak{g},\ \mathfrak{g})$ classifies infinitesimal deformations modulo trivial ones. By definition, a (formal) series

$$(X, Y) \to \Phi_\lambda(X, Y) := [X, Y] + \lambda f_1(X, Y) + \lambda^2 f_2(X, Y) + \cdots \tag{1.1}$$

is a deformation of Lie bracket $[,]$ if Φ_λ is a Lie bracket for every λ, i.e. is an antisymmetric bilinear form in X, Y and satisfies Jacobi's identity. If one sets simply

$$[X, Y]_\lambda = [X, Y] + \lambda c(X, Y), \tag{1.2}$$

c being a 2-cochain with values in $\mathfrak{g}$ and λ being a scalar, then this bracket satisfies Jacobi identity modulo terms of order $O(\lambda^2)$ if and only if c is a 2-cocycle.

Let Vect($\mathbb{R}$) be the Lie algebra of all vector fields $X_h = h\frac{d}{dx}$, where $h \in \mathcal{C}^\infty(\mathbb{R})$ on $\mathbb{R}$. Consider the 1-parameter deformation of the Vect($\mathbb{R}$) action on $\mathcal{C}^\infty(\mathbb{R})$:

$$L^\lambda_{X_h}(f) = hf' + \lambda h' f,$$

where $f',\ h'$ are $\frac{df}{dx},\ \frac{dh}{dx}$. Denote by $\mathcal{F}_\lambda$ the Vect($\mathbb{R}$)-module structure on $\mathcal{C}^\infty(\mathbb{R})$ defined by L^λ for a fixed λ.

Each bilinear bidifferential operator A on $\mathbb{R}$ gives thus rise to a morphism from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to $\mathcal{F}_\mu$, for any $\lambda, \nu, \mu \in \mathbb{R}$, by $f dx^\lambda \otimes g dx^\nu \longmapsto A(f \otimes g) dx^\mu$.

$$A(f dx^\lambda \otimes g dx^\nu) = \sum_{k=0}^{m} \sum_{i+j=k} a_{i,j} f^i g^j dx^\mu$$

where the coefficients $a_{i,j}$ is constant.
The Lie algebra Vect($\mathbb{R}$) acts on the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$ as follows

$$X_h.A = L^\mu_{X_h} \circ A - A \circ L^{(\lambda,\nu)}_{X_h}, \tag{1.3}$$

where $L^{(\lambda,\nu)}_{X_h}$ is the Lie derivative on $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ defined by the Leibniz rule:

$$L^{(\lambda,\nu)}_{X_h}(f \otimes g) = L^\lambda_{X_h}(f) \otimes g + f \otimes L^\nu_{X_h}(g).$$

Bouarroudj, in [21], computes the cohomology space
$\mathrm{H}^2_{\mathrm{diff}}(Vect(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$ where $\mathrm{H}^1_{\mathrm{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered (see.e.g. [16, 19, 22, 24, 25, 26]).

I. Basdouri et al. in [1, 5, 10] had studied the classical and super case to deduce the integrability condition for the infinitesimal deformation, I. Basdouri et al. compute in the same way as in [6, 7, 9, 12, 13, 11, 18, 15] to find the first cohomology space $\mathrm{H}^1(\mathfrak{g}, \mathcal{D})$. Basdouri et al. compute in the same way as in [4, 2, 3, 14, 19, 17, 23] to find the second cohomology space $\mathrm{H}^1(\mathfrak{g}, \mathcal{D})$.

In this paper we compute here the second cohomology space
$\mathrm{H}^2_{\mathrm{diff}}(Vect(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu})$ of the Lie algebra $Vect(\mathbb{R})$ with coefficients in the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$, vanishing on the Lie algebra

$\mathfrak{sl}(2)$. Moreover, we give explicit formulae for non trivial 2-cocycles which generate these spaces.

2. Vect($\mathbb{R}$)-module structures on the space of bilinear bidifferential operators

The Lie algebra $\mathfrak{sl}(2)$ is realized as subalgebra of the Lie algebra V$ect(\mathbb{R})$:

$$\mathfrak{sl}(2) = Span(X_1 = \frac{d}{dx}, X_x = x\frac{d}{dx}, X_{x^2} = x^2\frac{d}{dx}). \tag{2.1}$$

corresponding to the fraction-linear transformations

$$x \mapsto \frac{ax+b}{cx+d}, \qquad ad - bc = 1.$$

A *projective structure* on $\mathbb{R}$ (or S^1) is given by an atlas with fraction-linear coordinate transformations (in other words, by an atlas such that the $\mathfrak{sl}(2)$-action (2.1) is well-defined).

The commutation relations are

$$[X_1, X_x] = X_1, \qquad [X_x, X_x] = 0, \qquad [X_1, X_1] = 0,$$
$$[X_1, X_{x^2}] = 2X_x, \qquad [X_x, X_{x^2}] = X_{x^2}, \qquad [X_{x^2}, X_{x^2}] = 0.$$

2.1. The space of tensor densities on $\mathbb{R}$.

Let V$ect(\mathbb{R})$ be the Lie algebra of vector fields on $\mathbb{R}$. Consider the 1-parameter deformation of the Vect($\mathbb{R}$) action on $\mathcal{C}^\infty(\mathbb{R})$:

$$L^\lambda_{X_h}(f) = hf' + \lambda h' f,$$

where f', h' are $\frac{df}{dx}$, $\frac{dh}{dx}$. Denote by $\mathcal{F}_\lambda$ the Vect($\mathbb{R}$)-module structure on $\mathcal{C}^\infty(\mathbb{R})$ defined by L^λ for a fixed λ. Geometrically, $\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in \mathcal{C}^\infty(\mathbb{R}) \right\}$ is the space of weighted densities of weight $\lambda \in \mathbb{R}$, so its elements can be represented as $f(x)dx^\lambda$, where $f(x)$ is a function and dx^λ is a formal (for a time being) symbol. This space coincides with the space of vector fields, functions and differential forms for $\lambda = -1$, 0 and 1, respectively.

The space $\mathcal{F}_\lambda$ is a V$ect(\mathbb{R})$-module for the action defined by

$$L^\lambda_{g\frac{d}{dx}}(f dx^\lambda) = (gf' + \lambda g' f) dx^\lambda. \tag{2.2}$$

2.2. The space of bilinear bidifferential operators as a Vect($\mathbb{R}$)-module.

We are interested in defining a cohomology of the Lie algebra Vect($\mathbb{R}$)with coefficients in the space of bilinear bidifferential operators $\mathcal{D}_{\lambda,\nu,\mu}$. The counterpart Vect($\mathbb{R}$)-modules of the space of linear differential operators is a classical object (see e.g. [26]).

Consider bilinear bidifferential operators that act on tensor densities:

$$A : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \longrightarrow \mathcal{F}_\mu. \tag{2.3}$$

The Lie algebra Vect($\mathbb{R}$) acts on the space of bilinear bidifferential operators as follows. For all $\phi \in \mathcal{F}_\lambda$ and for all $\psi \in \mathcal{F}_\nu$:

$$L^{\lambda,\nu,\mu}_X(A)(\phi, \psi) = L^\mu_X \circ A(\phi, \psi) - A(L^\lambda_X(\phi), \psi) - A(\phi, L^\nu_X(\psi)). \tag{2.4}$$

where L_X^{λ} is the action (2.2). We denote by $\mathcal{D}_{\lambda,\nu,\mu}$ the space of bilinear bidifferential operators (2.3) endowed with the defined Vect($\mathbb{R}$)-module structure (2.4).

3. The second differentiable cohomology space of $\mathfrak{sl}(2)$ acting on $\mathcal{D}_{\lambda,\upsilon,\mu}$

In this section, we investigate the second space differentiable cohomology of the Lie algebra $\mathfrak{sl}(2)$ with coefficients in the space of bilinear bidifferential operators that act on tensor densities $\mathcal{D}_{\lambda,\nu,\mu}$. Following Sofiane Bouarroudj, we give explicit expressions of the basis cocycles. To know, we consider only cochains that are given by differentiable maps.

Lemma 3.1. [20] *Any 2-cocycle vanishing on the Lie subalgebra $\mathfrak{sl}(2)$ of* Vect($\mathbb{R}$) *is $\mathfrak{sl}(2)$-invariant.*

3.1. Theorem.

Theorem 3.2. [14] *The second differentiable cohomology space of the $\mathfrak{sl}(2)$-module $\mathcal{D}_{\lambda,\upsilon,\mu}$ has the following structure:*

(1) *For $\mu - \lambda - \upsilon = 0$, then*

$$\mathrm{H}^2(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) = \mathbb{R}.$$

(2) *For $\mu - \lambda - \upsilon = k$, where k is a positive integer, then*

$$\mathrm{H}^2(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq \begin{cases} \mathbb{R}^4 & \textit{if } (\lambda,\mu) = (-\frac{s}{2}, -\frac{t}{2}), \textit{ where} \\ & 0 \leq s, k-s-2 < t \leq k-1, \\ \mathbb{R} & \textit{otherwise.} \end{cases} \tag{3.1}$$

(3) *For $\mu - \lambda - \upsilon = k$, where k is not a positive integer, then*

$$\mathrm{H}^2(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq 0.$$

3.2. Theorem.

Theorem 3.3. *In this section, we will investigate the second cohomology group of* Vect($\mathbb{R}$) *with values in $\mathcal{D}_{\lambda,\nu,\mu}$ vanishing on $\mathfrak{sl}(2)$*

(1) *The case when $k = 5$*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq \begin{cases} 0 & \textit{if } (\lambda,\mu) = (-1,0), (0,0) \\ \mathbb{R} & \textit{if } (\lambda,\mu) = (0,-1) \\ \mathbb{R}^2 & \textit{otherwise.} \end{cases}$$

(2) *The case when k =6*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq \begin{cases} \mathbb{R} & \textit{if } (\lambda,\mu) = (0,-\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2}) \\ \mathbb{R}^3 & \textit{if } (\nu,\lambda) = (0.0) \\ 0 & \textit{if otherwise} \end{cases}$$

(3) *The case when k = 7*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})\simeq\begin{cases}\mathbb{R} & \textit{if } (\lambda,\mu)=(0,-\frac{1}{2}),(-\frac{3}{2},-2).\\ \mathbb{R}^2 & \textit{if } (\lambda,\mu)=(0,0),(0,-1),(0,-\frac{3}{2}),\\ & (-\frac{1}{2},0),(0,-2),(-\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},-\frac{3}{2}),\\ & (-1,-1),(-1,-\frac{1}{2}),(-\frac{3}{2},0),(-\frac{3}{2},-\frac{1}{2}),\\ & (-1,0),(-2,0),(-2,-\frac{3}{2})\\ \\ \mathbb{R}^3 & \textit{if } (\lambda,\mu)=(-\frac{1}{2},-1),(-\frac{1}{2},-2),(-\frac{2}{2},-\frac{3}{2}),\\ & (-1,-2),(-2,-\frac{1}{2}),(-2,-1),(-2,-2)\\ \mathbb{R}^4 & \textit{if } (\lambda,\mu)=(-\frac{3}{2},-1),(-\frac{3}{2},-\frac{3}{2})\\ 0 & \textit{if otherwise}\end{cases}$$

(4) *The case when k = 8*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})\simeq\begin{cases}\mathbb{R} & \textit{if } (\lambda,\mu)=(-\frac{7}{2},0),(-3,0),(0,-\frac{7}{2}),\\ & (-3,0),(0,-\frac{1}{2})\\ 0 & \textit{if otherwise.}\end{cases}$$

(5) *The case when k =9*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})\simeq\begin{cases}\mathbb{R} & \textit{if } (\lambda,\mu)=(0,-\frac{1}{2}),(0,-1),(0,-\frac{3}{2}),(0,-2),\\ & (-2,-1),(0,-3),(-\frac{1}{2},0),(-\frac{1}{2},-\frac{1}{2}),(-\frac{1}{2},-1),\\ & (-\frac{1}{2},-\frac{5}{2}),(-1,0),(-1,-\frac{1}{2}),(-1,-1),(-1,-\frac{3}{2})\\ & (-2,-\frac{5}{2}),(-1,-3),(-\frac{3}{2},0),(-\frac{3}{2},-\frac{1}{2}),(-\frac{3}{2},-1),\\ & (-\frac{3}{2},-2),(-\frac{3}{2},-\frac{5}{2}),(-\frac{3}{2},-3),(-2,0),(-2,-\frac{1}{2})\\ & (-2,-\frac{3}{2}),(-2,-2),(-2,-\frac{5}{2}),(-2,-3),(-\frac{5}{2},0),\\ & (-\frac{5}{2},-1),(-3,-2),(-\frac{5}{2},-\frac{3}{2}),(-\frac{5}{2},-\frac{4}{2}),(-\frac{5}{2},-\frac{5}{2}),\\ & (-\frac{5}{2},-3),(-3,0),(-3,-\frac{3}{2}),(-3,-\frac{4}{2}),(-3,-\frac{5}{2}),\\ & (0,-\frac{5}{2}),(-\frac{1}{2},-2),(-1,-2),(-\frac{3}{2},-\frac{3}{2}),\\ & (-\frac{5}{2},-\frac{1}{2}),(-\frac{5}{2},-3),(-3,-3).\\ \mathbb{R}^2 & \textit{if } (\lambda,\mu)=(-3,-\frac{1}{2}),(-\frac{1}{2},-3)\\ 0 & \textit{if otherwise}\end{cases}$$

(6) *The case when $k = 10$*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq \begin{cases} \mathbb{R} & \text{if } (\lambda,\nu) = (0,-\frac{1}{2}), (0,-1), (0,-\frac{3}{2}), (0,-\frac{5}{2}), \\ & (0,-3), (0,-\frac{1}{2}), (0,-1), (0,-\frac{3}{2}), (0,-2), \\ & (0,-\frac{5}{2}), (0,-\frac{3}{2}), (0,-2), (0,-\frac{5}{2}), (-\frac{1}{2},0), \\ & (-\frac{3}{2},0), (-\frac{3}{2},-\frac{5}{2}), (-1,0), (-1,-3), (-2,0), \\ & (-2,-2), (-\frac{5}{2},0), (-\frac{5}{2},-\frac{3}{2}), (-3,0), (-3,-1), \\ & (-3,-\frac{7}{2}), (-3,-2), (-\frac{7}{2},-\frac{1}{2}) \\ \mathbb{R}^2 & \text{if } (\lambda,\nu) = (0,0), (0,-2), (-\frac{3}{2},-3), (\frac{3}{2},-\frac{7}{2}), \\ & (-1,-\frac{7}{2}), (-2,-\frac{5}{2}), (-2,-\frac{6}{2}), (-2,-\frac{7}{2}), \\ & (-\frac{5}{2},-3), (-\frac{5}{2},-\frac{7}{2}), (-3,-\frac{3}{2}), (-3,-3), \\ & (-3,-\frac{7}{2}), (-\frac{7}{2},-1), (-\frac{7}{2},-\frac{3}{2}), (-\frac{7}{2},-2), \\ & (-\frac{7}{2},-\frac{5}{2}), (-\frac{7}{2},-3), (-\frac{5}{2},-2), (-\frac{7}{2},-\frac{7}{2}) \\ 0 & \text{if otherwise} \end{cases}$$

(7) *The case when $k = 11$*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq \mathbb{R}^2$$

(8) *The case when k =12*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq \begin{cases} \mathbb{R} & \text{if } (\lambda,\mu) = (0,\frac{-3}{2}), (0,\frac{-5}{2}), (0,\frac{-7}{2}), \\ & (0,-4), (\frac{1}{108}(-389 \pm \sqrt{32737}), -1), \\ & (0,-3), (\frac{1}{108}(-389 \pm \sqrt{32737}) \\ & , \frac{1}{108}(-389 \pm \sqrt{32737})), (0,-2) \\ \mathbb{R}^2 & \text{if } (\lambda,\mu) = (0,0), \\ 0 & \text{if otherwise} \end{cases}$$

(9) *The case when $k = 13$*

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) \simeq 0$$

3.3. **Proof of Theorem (3.3).** The generic form of any such a differential operator is (here $X = f\frac{d}{dx}, y = g\frac{d}{dx} \in \mathrm{Vect}(\mathbb{R})$, $\phi(x) \in \mathcal{F}_\lambda$ and $\psi(x) \in \mathcal{F}_\nu$).

$$\Upsilon(X,Y)(\phi(x)\psi(x)) = \sum_{i+j+k+l=k+2} \alpha_{ijkl} X^{(i)} Y^{(j)} \phi(x)^{(k)} \psi(x)^{(l)}$$

Where $X^{(i)} = \frac{d^i X}{dx}$.

The invariance property with respect to the vector field X with arbitrary Y and Z implies that $\alpha'_{ijkl} = 0$, therefore α_{ijkl} are constants $\mu - \lambda - \upsilon = k$. Since this 2-cocycle vanishes on sl(2), the Lemma 3.1 implies that this 2-cocycle is $\mathfrak{sl}(2)$-invariant. The last statement means that the 2-cocycle is homogenous.

Besides, we have $\alpha_{ijkl}=0$ for all $i,j\in\{0.1.2\}$. To proof Theorem we proceed bye explaining our strategy. First,the dimension of the space of operators that satisfy the 2-cocycle condition. we get a linear system for α_{ijkl}. Second, taking into account these conditions, We will study all trivial 2-cocycles,we will determine different values of λ and ν for which the space of operators of the form δb. Gluing these bits of information together we deduce the dimension of the cohomology group $\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2),\mathcal{D}_{\lambda,\nu,\mu})$.

(1) The case when k = 5
The 2-cocycle has the form

$$\Upsilon(X,Y)(\phi(x)\psi(x))=\alpha_{4300}\phi(x)\psi(x)Y^{(3)}X^{(4)}+\alpha_{3400}\phi(x)\psi(x)X^{(3)}Y^{(4)}. \quad (3.2)$$

The 2-cocycle condition is always satisfied. According to these values, let us study the triviality of the 2-cocycle. A direct computation proves that

$$\begin{aligned}&\delta b_5^{\lambda,\nu}(X,Y)(\phi(x),\psi(x))=((1+2\nu)\beta_{312}+(1+2\lambda)\beta_{321}+2\beta_{411})\\&\left(Y''(x)X^{(3)}(x)-X''(x)Y^{(3)}(x)\right)\phi'(x)\psi'(x)+(\nu\beta_{321}+(3+3\lambda)\beta_{330}+2\beta_{420})\\&\left(Y''(x)X^{(3)}(x)-X''(x)Y^{(3)}(x)\right)\psi(x)\phi''(x)+(3\beta_{303}+3\nu\beta_{303}+\lambda\beta_{312}+2\beta_{402})\\&\left(Y''(x)X^{(3)}(x)-X''(x)Y^{(3)}(x)\right)\phi(x)\psi''(x)+(\nu\beta_{411}+(1+2\lambda)\beta_{420}+5\beta_{510})\\&\left(Y''(x)X^{(4)}-X''(x)Y^{(4)}(x)\right)\psi(x)\phi'(x)+((1+2\nu)\beta_{402}+\lambda\beta_{411}+5\beta_{501})\\&\left(Y''(x)X^{(4)}-X''(x)Y^{(4)}(x)\right)\phi(x)\psi'(x)+(\nu\beta_{303}+\lambda\beta_{330}-\nu\beta_{402}-\lambda\beta_{420}\\&-5\beta_{600})\left(-Y^{(3)}(x)X^{(4)}+X^{(3)}(x)Y^{(4)}(x)\right)\phi(x)\psi(x)+(\nu\beta_{501}+\lambda\beta_{510}+9\beta_{600})\\&\left(Y''(x)X^{(5)}(x)-X''(x)Y^{(5)}(x)\right)\phi(x)\psi(x).\end{aligned}$$

The space of solutions of the system above is zero-dimensional for $(\nu,\lambda)=(-1,0),(0,0)$ and one-dimensional for $(\nu,\lambda)=(0,-1)$. For all values of ν and λ one can easily prove that the equation $\delta b_5^{\lambda,\nu}=0$ has no solutions the system is tow-dimensional.

- if $\nu=-1,\lambda=0$ the constant β_{411} and β_{510} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_5^{\lambda,\nu}$ to our 2-cocycle (3.2). Hence, the cohomology group is zero-dimensional. On the other hand,
  $\beta_{600}=0,\ \beta_{402}=0,\ \beta_{501}=0,\ \beta_{420}=\beta_{411}-5\beta_{510},$
  $\beta_{312}=3\beta_{330}+4\beta_{411}-10\beta_{510},\ \beta_{321}=3\beta_{330}+2\beta_{411}-10\beta_{510}.$

- if $\nu=0,\lambda=0$, the constant β_{303} and β_{510} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_5^{\lambda,\nu}$ to our 2-cocycle (3.2). Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{600}=0,\ \beta_{402}=-\frac{3}{2}\beta_{303},\ \beta_{501}=\frac{3}{10}\beta_{303},\ \beta_{330}=\frac{10}{3}\beta_{510},$
$\beta_{420}=-5\beta_{510},\ \beta_{321}=-\beta_{312}-2\beta_{411}.$

- if $\nu=0,\lambda=-1$, the constant β_{330} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_5^{\lambda,\nu}$ to our 2-cocycle (3.2). Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{600}=0,\ \beta_{510}=0,\ \beta_{501}=\frac{1}{5}\left(-\beta_{402}+\beta_{411}\right),$
  $\beta_{420}=0,\ \beta_{312}=3\beta_{303}+2\beta_{402},\ \beta_{321}=3\beta_{303}+2\beta_{402}+2\beta_{411}.$

- if $\lambda=-1$ and$\nu\neq 0,\nu\neq -1,\nu\neq -\frac{1}{2}$, in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{303}=-\frac{2}{\nu(1+3\nu+2\nu^2)}\left(\nu\beta_{411}+15\beta_{600}\right),\ \beta_{402}=\frac{1}{\nu(1+2\nu)}(\nu\beta_{411}$
  $-5\beta_{510}+45\beta_{600}),\ \beta_{501}=\frac{1}{5}\left(-\beta_{402}-2\nu\beta_{402}+\beta_{411}\right),$
  $\beta_{420}=\nu\beta_{411}+5\beta_{510},\ \beta_{312}=3\beta_{303}+3\nu\beta_{303}+2\beta_{402},$
  $\beta_{321}=3\beta_{303}+9\nu\beta_{303}+6\nu^2\beta_{303}+2\beta_{402}+4\nu\beta_{402}+2\beta_{411}.$

- if $\lambda=-\frac{1}{2}$, and $\nu\neq 0,\nu\neq -1,\nu\neq -\frac{1}{2}$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{303}=\frac{-1}{\nu(1+3\nu+2\nu^2)}(\nu\beta_{411}+30\beta_{600}),\ \beta_{510}=-\frac{1}{5}\nu\beta_{411},$
  $\beta_{402}=\frac{-1}{2(1+2\nu)}(3\beta_{303}+9\nu\beta_{303}+6\nu^2\beta_{303}+\beta_{411}),$
  $\beta_{501}=\frac{1}{10}\left(3\beta_{303}+9\nu\beta_{303}+6\nu^2\beta_{303}+2\beta_{411}\right),$
  $\beta_{312}=2\left(3\beta_{303}+3\nu\beta_{303}+2\beta_{402}\right),\ \beta_{321}=\frac{-1}{2\nu}(3\beta_{330}+4\beta_{420}).$

- if $\lambda=0$, and$\nu\neq 0,\nu\neq -1,\nu\neq -\frac{1}{2}$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{303}=-\frac{30}{\nu(1+3\nu+2\nu^2)}\beta_{600},\ \beta_{402}=-\frac{3}{2}\left(\beta_{303}+\nu\beta_{303}\right),$
  $\beta_{501}=\frac{3}{10}\left(\beta_{303}+3\nu\beta_{303}+2\nu^2\beta_{303}\right),\ \beta_{420}=-\nu\beta_{411}-5\beta_{510},$
  $\beta_{312}=\frac{1}{\nu(1+2\nu)}(3\beta_{330}-4\nu\beta_{411}-10\beta_{510}),$
  $\beta_{321}=-\beta_{312}-2\nu\beta_{312}-2\beta_{411}.$

- if $\nu=-1,\ \lambda=-1$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,
  $\beta_{411}=15\beta_{600},\ \beta_{402}=-5\left(\beta_{510}-6\beta_{600}\right),\ \beta_{501}=-\beta_{510}+9\beta_{600},$
  $\beta_{420}=5\left(\beta_{510}-3\beta_{600}\right),\ \beta_{312}=-10\left(\beta_{510}-6\beta_{600}\right),$
  $\beta_{321}=10\left(\beta_{510}-3\beta_{600}\right).$

- if $\nu=-1,\ \lambda=-\frac{1}{2}$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{411}=30\beta_{600},\ \beta_{510}=6\beta_{600},\ \beta_{402}=15\beta_{600},\ \beta_{501}=6\beta_{600},$
  $\beta_{312}=60\beta_{600},\ \beta_{321}=\frac{1}{2}\left(3\beta_{330}+4\beta_{420}\right).$

- if $\nu=-\frac{1}{2},\ \lambda=-1$ in this case we have we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{411}=30\beta_{600},\ \beta_{510}=6\beta_{600},\ \beta_{501}=6\beta_{600},\ \beta_{420}=15\beta_{600},$
  $\beta_{312}=\frac{1}{2}\left(3\beta_{303}+4\beta_{402}\right),\ \beta_{321}=60\beta_{600}.$

- if $\nu=-\frac{1}{2},\ \lambda=-\frac{1}{2}$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{600}=0,\ \beta_{411}=0,\ \beta_{510}=0,\ \beta_{501}=0,$
  $\beta_{312}=3\beta_{303}+4\beta_{402},\ \beta_{321}=3\beta_{330}+4\beta_{420}.$

- if $\nu=0,\lambda=-\frac{1}{2}$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional.On the other hand,

  $\beta_{600}=0,\ \beta_{510}=0,\ \beta_{402}=\frac{1}{2}\left(-3\beta_{303}-\beta_{411}\right),$
  $\beta_{420}=-\frac{3}{4}\beta_{330},\ \beta_{501}=\frac{1}{10}\left(3\beta_{303}+2\beta_{411}\right),\ \beta_{312}=-2\beta_{411}.$

- if $\lambda=0$ and $(\nu=-\frac{1}{2}\|\nu=0)$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{600}=0,\ \beta_{402}=-\frac{3}{2}\left(\beta_{303}+\nu\beta_{303}\right),\ \beta_{501}=\frac{3}{10}\left(\beta_{303}+2\nu\beta_{303}\right),$
  $\beta_{330}=\frac{2}{3}\left(2\nu\beta_{411}+5\beta_{510}\right),\ \beta_{420}=-\nu\beta_{411}-5\beta_{510},$
  $\beta_{321}=-\beta_{312}-2\nu\beta_{312}-2\beta_{411}.$

- if $\nu(1+2\nu)\neq 0$ and $\lambda\left(1+3\lambda+2\lambda^2\right)\neq 0$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x))=0$. Hence, the cohomology group is tow-dimensional.On the other hand,

  $\beta_{402}=\frac{1}{\nu(1+2\nu)}(-\lambda\nu\beta_{411}+5\lambda\beta_{510}+45\beta_{600}),\ \beta_{420}=\frac{-1}{1+2\lambda}(\nu\beta_{411}$
  $+5\beta_{510}),\ \beta_{501}=\frac{1}{5}\left((-1-2\nu)\beta_{402}-\lambda\beta_{411}\right),\ \beta_{312}=\frac{-1}{\lambda}(3\beta_{303}$
  $+3\nu\beta_{303}+2\beta_{402}),\ \beta_{330}=\frac{1}{\lambda(1+3\lambda+2\lambda^2)}(\nu(-1-3\nu)\beta_{303}-2\nu^3\beta_{303}$
  $+2\lambda\nu\beta_{411}-30\beta_{600}),\ \beta_{321}=\frac{-1}{\nu}(3\beta_{330}+3\lambda\beta_{330}+2\beta_{420}).$

- if $\nu = -\frac{1}{2}, \lambda \neq 0$ and $1+3\lambda+2\lambda^2 \neq 0$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x)) = 0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $$\begin{aligned}
  &\beta_{510} = \tfrac{-1}{10\lambda}(\lambda\beta_{411} + 90\beta_{600}),\ \beta_{321} = 2\left(3\beta_{330} + 3\lambda\beta_{330} + 2\beta_{420}\right),\\
  &\beta_{501} = -\tfrac{1}{5}\lambda\beta_{411}, \beta_{330} = \tfrac{2}{3(1+3\lambda+2\lambda^2)}(-\beta_{411} + 5\beta_{510}),\\
  &\beta_{420} = \tfrac{1}{2(1+2\lambda)}(\beta_{411} - 10\beta_{510}), \beta_{312} = \tfrac{-1}{2\lambda}(3\beta_{303} + 4\beta_{402}).
  \end{aligned}$$

- if $\nu = 0, \lambda \neq 0$ and $1+3\lambda+2\lambda^2 \neq 0$ in this case we have $\Upsilon(X,Y)(\phi(x)\psi(x)) = 0$. Hence, the cohomology group is tow-dimensional. On the other hand,

  $$\begin{aligned}
  &\beta_{510} = -\tfrac{9}{\lambda}\beta_{600}, \beta_{501} = \tfrac{1}{5}\left(-\beta_{402} - \lambda\beta_{411}\right),\ \beta_{330} = \tfrac{10}{3(1+3\lambda+2\lambda^2)}\beta_{510},\\
  &\beta_{420} = -\tfrac{3}{2}\left(\beta_{330} + \lambda\beta_{330}\right),\ \beta_{312} = \tfrac{1}{\lambda}(-3\beta_{303} - 2\beta_{402}),\\
  &\beta_{321} = \tfrac{-1}{1+2\lambda}(\beta_{312} + 2\beta_{411}).
  \end{aligned}$$

(2) The case when k =6
The 2-cocycle has the form

$$\begin{aligned}
\Upsilon_6(X,Y)(\phi(x),\psi(x)) = &\\
&\alpha_{4310}\psi(x)\phi'(x)Y^{(3)}(x)X^{(4)}(x) + \alpha_{4301}\phi(x)\psi'(x)Y^{(3)}(x)X^{(4)}(x)\\
&+ \alpha_{3410}\psi(x)\phi'(x)X^{(3)}(x)Y^{(4)}(x) + \alpha_{3401}\phi(x)\psi'(x)X^{(3)}(x)Y^{(4)}(x)\\
&+ \alpha_{5300}\phi(x)\psi(x)X^{(3)}(x)Y^{(5)}(x) + \alpha_{3500}\phi(x)\psi(x)X^{(3)}(x)Y^{(5)}(x).
\end{aligned} \tag{3.3}$$

The 2-cocycle condition is equivalent to the following system

$$\alpha_{3500} = \frac{1}{5}\left(5\alpha_{5300} - \nu\alpha_{3401} + \nu\alpha_{4301} - \lambda\alpha_{3410} + \lambda\alpha_{4310}\right) \tag{3.4}$$

According to these values, let us study the triviality of the 2-cocycle . A direct computation proves the space of solutions of the system above is three-dimensional for $(\nu,\lambda) = (0.0)$, one-dimensional for $(\nu,\lambda) = (0,-\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2})$ and zero-dimensional for $(0,-\frac{3}{2}), (0,-1), (-\frac{1}{2},0)$, $(-\frac{1}{2},-1), (-1,0), (-\frac{1}{2},-\frac{3}{2}), (\nu-\frac{1}{2},-1), (-1,-\frac{1}{2}), (-1,-1), (-1,-\frac{3}{2})$, $(-\frac{3}{2},0), (-\frac{3}{2},-\frac{1}{2}), (-\frac{3}{2},-1), (-\frac{3}{2},-\frac{3}{2})$.

- $\nu = 0,\ \lambda = 0$ then the constant β_{403},β_{430} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is three-dimensional. On the other hand,

  $$\begin{aligned}
  &\beta_{421} = -\tfrac{1}{2}(\beta_{322} + 3\beta_{331}),\ \beta_{412} = -\tfrac{1}{2}(3\beta_{313} - \beta_{322}),\\
  &\beta_{340} = \tfrac{1}{3}\beta_{430},\ \beta_{520} = -\tfrac{3}{5}\beta_{430},\ \beta_{304} = -\tfrac{1}{3}\beta_{403},
  \end{aligned}$$

$\beta_{511} = \frac{1}{10}(3\beta_{313} + 2\beta_{322} + 3\beta_{331}),\ \beta_{502} = -\frac{3}{5}\beta_{403},$
$\beta_{610} = \frac{1}{15}\beta_{430},\ \beta_{601} = \frac{1}{15}\beta_{403},\ \beta_{700} = 0.$

- $\nu = 0,\ \lambda = -\frac{1}{2}$ then the constant $\beta_{430}, \beta_{331}, \beta_{322}, \beta_{313}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{421} = -\frac{1}{4}(2\beta_{322} - 3\beta_{331}),\ \beta_{412} = -\frac{3}{2}\beta_{313},\ \beta_{340} = \frac{1}{2}\beta_{430},$
  $\beta_{304} = \frac{1}{12}(\beta_{313} - 4\beta_{403}),\ \beta_{511} = \frac{3}{10}\beta_{313},\ \beta_{520} = -\frac{3}{10}\beta_{430},$
  $\beta_{502} = -\frac{3}{20}(\beta_{313} - 4\beta_{403}),\ \beta_{601} = \frac{1}{30}(\beta_{313} + 2\beta_{403}),$
  $\beta_{610} = 0,\ \beta_{700} = 0$

- $\nu = 0,\ \lambda = -\frac{3}{2}$ then the constant $\beta_{430}, \beta_{313}, \beta_{322}, \beta_{331}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421} = -\frac{1}{4}(2\beta_{322} + 3\beta_{331}),\ \beta_{412} = -\frac{3}{2}\beta_{313} + \beta_{322},$
  $\beta_{304} = \frac{1}{4}\beta_{313} - \frac{1}{3}\beta_{403},\ \beta_{511} = \frac{1}{10}(3\beta_{313} - 4\beta_{322} + 3\beta_{331}),$
  $\beta_{502} = -\frac{1}{20}(9\beta_{313} - 6\beta_{322} + 12\beta_{403}),\ \beta_{601} = \frac{1}{20}(2\beta_{313}$
  $- 2\beta_{322} + \beta_{331} + \frac{4}{3}\beta_{403}),\ \beta_{430} = 0,\ \beta_{520} = 0,\ \beta_{610} = 0,\ \beta_{700} = 0.$

- $\nu = 0,\ \lambda = -1$ then the constant $\beta_{430}, \beta_{313}, \beta_{322}, \beta_{331}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421} = -\frac{1}{2}\beta_{322},\ \beta_{412} = -\frac{1}{2}(3\beta_{313} + \beta_{322}),\ \beta_{340} = \beta_{430},$
  $\beta_{304} = \frac{1}{6}(\beta_{313} - 2\beta_{403}),\ \beta_{511} = \frac{1}{10}(3\beta_{313} - 2\beta_{322}),$
  $\beta_{502} = -\frac{1}{10}(3\beta_{313} + \beta_{322} - 6\beta_{403}),\ \beta_{601} = \frac{1}{30}(2\beta_{313} - \beta_{322} + 2\beta_{403}),$
  $\beta_{520} = 0, \beta_{610} = 0,\ \beta_{700} = 0.$

- $\nu = -\frac{1}{2},\ \lambda = 0$ then the constant $\beta_{430}, \beta_{313}, \beta_{322}, \beta_{331}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421} = -\frac{3}{2}\beta_{331},\ \beta_{412} = -\frac{1}{4}(3\beta_{313} + 2\beta_{322}),\ \beta_{304} = -\frac{1}{2}\beta_{403},$
  $\beta_{511} = \frac{3}{10}\beta_{331},\ \beta_{502} = -\frac{3}{10}\beta_{403},\ \beta_{340} = \frac{1}{12}(\beta_{331} + 4\beta_{430}),$
  $\beta_{520} = -\frac{1}{20}(3\beta_{331} + 12\beta_{430}),\ \beta_{610} = \frac{1}{30}(\beta_{331} + 2\beta_{430}),$
  $\beta_{601} = 0, \beta_{700} = 0.$

- $\nu = -\frac{1}{2},\ \lambda = -\frac{1}{2}$ then the constant $\beta_{313}, \beta_{331}, \beta_{403}, \beta_{430}$ can be chosen in such a way that once adding the trivial 2-cocycle

$\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is one-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{421} = -\tfrac{3}{4}\beta_{331},\ \beta_{412} = -\tfrac{3}{4}\beta_{313},\ \beta_{340} = \tfrac{1}{8}(\beta_{331} + 4\beta_{430}),\\
&\beta_{304} = \tfrac{1}{8}(\beta_{313} + 4\beta_{403}),\ \beta_{520} = -\tfrac{1}{40}(3\beta_{331} + 12\beta_{430}),\\
&\beta_{502} = -\tfrac{1}{40}(3\beta_{313} + 12\beta_{403}),\ \beta_{511} = 0,\ \beta_{610} = 0,\\
&\beta_{601} = 0,\ \beta_{700} = 0.
\end{aligned}$$

- $\nu = -\frac{1}{2},\ \lambda = -1$ then the constant $\beta_{313}, \beta_{322}, \beta_{331}, \beta_{403}, \beta_{430}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{412} = -\tfrac{1}{4}(3\beta_{313} - 2\beta_{322}),\ \beta_{340} = \tfrac{1}{4}\beta_{331} + \beta_{430},\\
&\beta_{304} = \tfrac{1}{4}(\beta_{313} - 2\beta_{403}),\ \beta_{502} = -\tfrac{1}{20}(3\beta_{313} - 2\beta_{322} + 6\beta_{403}),\\
&\beta_{511} = 0,\ \beta_{520} = 0,\ \beta_{610} = 0,\ \beta_{601} = 0, \beta_{421} = 0, \beta_{700} = 0.
\end{aligned}$$

- $\nu = -1,\ \lambda = 0$ then the constant $\beta_{403}, \beta_{313}, \beta_{322}, \beta_{331}, \beta_{430}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{421} = \tfrac{1}{2}(\beta_{322} - 3\beta_{331}),\ \beta_{412} = -\tfrac{1}{2}\beta_{322},\ \beta_{304} = -\beta_{403},\\
&\beta_{340} = \tfrac{1}{6}(\beta_{331} + 2\beta_{430}),\ \beta_{511} = \tfrac{1}{10}(3\beta_{331} - 2\beta_{322}),\\
&\beta_{520} = \tfrac{1}{10}(\beta_{322} - 3\beta_{331} - 6\beta_{430}),\ \beta_{610} = \tfrac{1}{30}(2\beta_{331} - \beta_{322} + 2\beta_{430}),\\
&\beta_{502} = 0,\ \beta_{601} = 0, \beta_{700} = 0.
\end{aligned}$$

- $\nu = -\frac{1}{2},\ \lambda = -\frac{3}{2}$ then the constant $\beta_{313}, \beta_{322}, \beta_{331}, \beta_{340}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{421} = \tfrac{3}{4}\beta_{331},\ \beta_{412} = -\tfrac{3}{4}\beta_{313} + \beta_{322},\ \beta_{511} = \tfrac{3}{10}\beta_{331},\\
&\beta_{610} = \tfrac{1}{60}\beta_{331},\ \beta_{601} = \tfrac{1}{20}\beta_{331}, \beta_{304} = \tfrac{1}{8}(3\beta_{313} - 4\beta_{403}),\\
&\beta_{430} = -\tfrac{1}{4}\beta_{331},\ \beta_{502} = \tfrac{3}{40}(4\beta_{322} - \beta_{313} - 4\beta_{403}), \beta_{700} = \tfrac{1}{280}\beta_{331},\\
&\beta_{520} = 0.
\end{aligned}$$

- $\nu = -\frac{1}{2},\ \lambda = -1$ then the constant $\beta_{313}, \beta_{322}, \beta_{331}, \beta_{340}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{412} = \tfrac{1}{4}(2\beta_{322} - 3\beta_{313}),\ \beta_{340} = \tfrac{1}{4}\beta_{331} + \beta_{430},\\
&\beta_{304} = \tfrac{1}{4}(\beta_{313} - 2\beta_{403}),\ \beta_{502} = \tfrac{1}{20}(2\beta_{322} - 3\beta_{313} - 6\beta_{403}),
\end{aligned}$$

$\beta_{511}=0, \beta_{520}=0,\ \beta_{421}=0,\ \beta_{610}=0,\ \beta_{601}=0,\ \beta_{700}=0.$

- $\nu=-1,\ \lambda=-\frac{1}{2}$ then the constant $\beta_{313},\beta_{322},\beta_{331},\beta_{340},\beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421}=\frac{1}{4}(2\beta_{322}-3\beta_{331}),\ \beta_{340}=\frac{1}{4}(\beta_{331}+2\beta_{430}),$
  $\beta_{304}=\frac{1}{4}\beta_{313}-\beta_{403},\ \beta_{520}=\frac{1}{20}(2\beta_{322}-3\beta_{331}-6\beta_{430}),$
  $\beta_{412}=0,\ \beta_{511}=0,\ \beta_{502}=0,\ \beta_{610}=0,\ \beta_{601}=0,\ \beta_{700}=0.$

- $\nu=-1,\ \lambda=-1$ then the constant $\beta_{313},\beta_{322},\beta_{331},\beta_{340},\beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421}=\frac{1}{2}\beta_{322},\ \beta_{412}=\frac{1}{2}\beta_{322},\ \beta_{340}=\frac{1}{2}\beta_{331}+\beta_{430},$
  $\beta_{304}=\frac{1}{2}\beta_{313}-\beta_{403},\ \beta_{511}=\frac{1}{5}\beta_{322},\ \beta_{520}=\frac{1}{10}\beta_{322},$
  $\beta_{502}=\frac{1}{10}\beta_{322},\beta_{610}=\frac{1}{30}\beta_{322},\ \beta_{601}=\frac{1}{30}\beta_{322},\ \beta_{700}=\frac{1}{210}\beta_{322}.$

- $\nu=-1,\ \lambda=-\frac{3}{2}$ then the constant $\beta_{313},\beta_{322},\beta_{331},\beta_{340},\beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421}=\frac{1}{4}(2\beta_{322}+3\beta_{331}),\ \beta_{412}=\beta_{322},\ \beta_{430}=-\frac{1}{2}\beta_{331},$
  $\beta_{304}=\frac{3}{4}\beta_{313}-\beta_{403},\ \beta_{511}=\frac{1}{10}(4\beta_{322}+3\beta_{331}),$
  $\beta_{520}=\frac{1}{10}\beta_{322},\beta_{502}=\frac{3}{10}\beta_{322},\ \beta_{610}=\frac{1}{30}(2\beta_{322}+\beta_{331}),$
  $\beta_{601}=\frac{1}{20}(2\beta_{322}+\beta_{331}),\ \beta_{700}=\frac{1}{140}(2\beta_{322}+\beta_{331}).$

- $\nu=-\frac{3}{2},\ \lambda=0$ then the constant $\beta_{313},\beta_{322},\beta_{331},\beta_{340},\beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{421}=\beta_{322}-\frac{3}{2}\beta_{331},\ \beta_{412}=\frac{1}{4}(3\beta_{313}-2\beta_{322}),$
  $\beta_{340}=\frac{1}{4}\beta_{331}+\frac{1}{3}\beta_{430},\ \beta_{511}=\frac{1}{10}(3\beta_{313}-4\beta_{322}+3\beta_{331}),$
  $\beta_{520}=\frac{1}{20}(6\beta_{322}-9\beta_{331}-12\beta_{430}),\ \beta_{610}=\frac{1}{20}(\beta_{313}-2\beta_{322}+2\beta_{331}$
  $+12\beta_{430}),\ \beta_{403}=0,\ \beta_{502}=0,\ \beta_{601}=0,\ \beta_{700}=0.$

- $\nu=-\frac{3}{2},\ \lambda=-\frac{1}{2}$ then the constant $\beta_{313},\beta_{322},\beta_{331},\beta_{340},\beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{421} = \beta_{322} - \tfrac{3}{4}\beta_{331},\ \beta_{412} = \tfrac{3}{4}\beta_{313},\ \beta_{403} = \tfrac{1}{4}\beta_{313},\\
&\beta_{340} = \tfrac{1}{8}(3\beta_{331} + 4\beta_{430}),\ \beta_{511} = \tfrac{3}{10}\beta_{313},\ \beta_{601} = \tfrac{1}{20}\beta_{313},\\
&\beta_{610} = \tfrac{1}{20}\beta_{313},\ \beta_{502} = \tfrac{3}{20}\beta_{313},\ \beta_{700} = \tfrac{1}{140}\beta_{313},\\
&\beta_{520} = \tfrac{1}{40}(12\beta_{322} - 9\beta_{331} - 12\beta_{430}).
\end{aligned}$$

- $\nu = -\frac{3}{2},\ \lambda = -1$ then the constant $\beta_{313}, \beta_{322}, \beta_{331}, \beta_{340}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{421} = \beta_{322},\ \beta_{412} = \tfrac{1}{4}(3\beta_{313} + 2\beta_{322}),\ \beta_{403} = \tfrac{1}{2}\beta_{313},\\
&\beta_{340} = \tfrac{3}{4}\beta_{331} + \beta_{430},\ \beta_{511} = \tfrac{1}{10}(3\beta_{313} + 4\beta_{322}),\ \beta_{520} = \tfrac{3}{10}\beta_{322},\\
&\beta_{502} = \tfrac{1}{10}(3\beta_{313} + \beta_{322}),\ \beta_{610} = \tfrac{1}{20}(\beta_{313} + 2\beta_{322}),\\
&\beta_{601} = \tfrac{1}{10}\beta_{313} + \tfrac{1}{15}\beta_{322},\ \beta_{700} = \tfrac{1}{70}(\beta_{313} + \beta_{322}).
\end{aligned}$$

- $\nu = -\frac{3}{2},\ \lambda = -\frac{3}{2}$ then the constant $\beta_{313}, \beta_{322}, \beta_{331}, \beta_{340}, \beta_{403}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_6^{\lambda,\nu}$ to our 2-cocycle (3.3). Hence, the cohomology group is zero-dimensional. On the other hand,

$$\begin{aligned}
&\beta_{421} = \beta_{322} + \tfrac{3}{4}\beta_{331},\ \beta_{412} = \tfrac{3}{4}\beta_{313} + \beta_{322},\ \beta_{430} = -\tfrac{3}{4}\beta_{331},\\
&\beta_{403} = \tfrac{3}{4}\beta_{313},\ \beta_{511} = \tfrac{1}{10}(3\beta_{313} + 8\beta_{322} + 3\beta_{331}),\\
&\beta_{520} = \tfrac{3}{10}\beta_{322},\ \beta_{502} = \tfrac{1}{20}(9\beta_{313} + 6\beta_{322}),\\
&\beta_{610} = \tfrac{1}{20}(\beta_{313} + 4\beta_{322} + \beta_{331}),\ \beta_{601} = \tfrac{1}{20}(3\beta_{313} + 4\beta_{322} + \beta_{331}),\\
&\beta_{700} = \tfrac{1}{280}(6\beta_{313} + 12\beta_{322} + 3\beta_{331}).
\end{aligned}$$

(3) The case when k = 7
The 2-cocycle condition is equivalent to the following system

$$\begin{aligned}
&\alpha_{3510} = \tfrac{1}{5}\nu\alpha_{4311}\\
&\alpha_{3501} = \tfrac{1}{5}\left(\lambda\alpha_{4311} + 5\alpha_{5301}\right)\\
&\alpha_{3600} = \tfrac{1}{45}\left(-2\lambda\nu\alpha_{4311} - 10\alpha_{4500} + 10\alpha_{5400} + 45\alpha_{6300}\right).
\end{aligned}$$

Let us study the triviality of this 2-cocycle. A direct computation proves the space of solutions of the system above is four-dimensional for $(\nu,\lambda) = (-\frac{3}{2},-1), (-\frac{3}{2},-\frac{3}{2})$, three-dimensional for $(\nu,\lambda) = (-\frac{1}{2},-1)$, $(-\frac{1}{2},-2), (-1,-\frac{3}{2}), (-1,-2), (-2,-\frac{1}{2}), (-2,-1), (-2,-2)$, tow-dimensional for $(\nu,\lambda) = (0,0), (0,-1), (0,-\frac{3}{2}), (0,-2), (-\frac{1}{2},0)$, $(-\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},-\frac{3}{2}), (-1,0), (-1,-1), (-1,-\frac{1}{2}), (-\frac{3}{2},0), (-\frac{3}{2},-\frac{1}{2})$, $(-2,0), (-2,-\frac{3}{2})$ and one-dimensional for $(0,-\frac{1}{2}), (-\frac{3}{2},-2)$.

- $\nu = 0,\ \lambda = 0$ then the constant $\beta_{431}, \beta_{800}, \beta_{413}, \beta_{332}, \beta_{404}, \beta_{323}$, β_{440} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422} = -\frac{1}{2}(3\beta_{323} - 3\beta_{332} + 3\beta_{800}),\ \beta_{341} = -\frac{1}{6}(\beta_{332} - 2\beta_{431}),$
$\beta_{314} = -\frac{1}{6}(\beta_{323} - 2\beta_{413} - \beta_{800}),\ \beta_{521} = \frac{3}{10}(\beta_{323} + \beta_{332} - 2\beta_{431}$
$+ \beta_{800}),\ \beta_{512} = \frac{3}{10}(\beta_{323} + \beta_{332} - 2\beta_{413} + \beta_{800}),$
$\beta_{530} = -\frac{6}{5}\beta_{440},\ \beta_{503} = -\frac{6}{5}\beta_{404},\ \beta_{350} = -\frac{1}{5}\beta_{440},$
$\beta_{305} = -\frac{1}{5}\beta_{404},\ \beta_{611} = -\frac{1}{15}(\beta_{323} - \beta_{332} + \beta_{413} + \beta_{431} - \beta_{800}),$
$\beta_{620} = \frac{2}{5}\beta_{440},\ \beta_{602} = \frac{2}{5}\beta_{404},\ \beta_{710} = -\frac{1}{35}\beta_{440},\ \beta_{701} = -\frac{1}{35}\beta_{404}.$

- $\nu = 0,\ \lambda = -\frac{1}{2}$ then the constant $\beta_{332}, \beta_{323}, \beta_{323}, \beta_{440}, \beta_{413}, \beta_{800}, \beta_{431}, \beta_{404}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional. On the other hand,

$\beta_{422} = -\frac{3}{2}(\beta_{323} + 2\beta_{332} + \beta_{800}),\ \beta_{341} = -\frac{1}{4}(\beta_{332} + 2\beta_{431}),$
$\beta_{314} = -\frac{1}{3}\beta_{413},\ \beta_{521} = \frac{3}{20}(2\beta_{323} + \beta_{332} - \beta_{431} + 2\beta_{800}),$
$\beta_{512} = -\frac{3}{5}\beta_{413},\ \beta_{530} = -\frac{4}{5}\beta_{440},\ \beta_{503} = -\frac{1}{10}(12\beta_{404} - \beta_{413}),$
$\beta_{350} = -\frac{4}{15}\beta_{440},\ \beta_{305} = -\frac{1}{5}\beta_{404} - \frac{1}{60}\beta_{413},$
$\beta_{611} = \frac{1}{15}\beta_{413},\ \beta_{620} = \frac{2}{15}\beta_{440},\ \beta_{602} = \frac{1}{15}(6\beta_{404} - \beta_{413}),$
$\beta_{701} = -\frac{1}{35}\beta_{404} + \frac{1}{140}\beta_{413}, \beta_{710} = 0.$

- $\nu = 0,\ \lambda = -1$ then the constant $\beta_{431}, \beta_{404}, \beta_{440}, \beta_{323}, \beta_{800}$ $, \beta_{332}, \beta_{413}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422} = -\frac{3}{2}(\beta_{323} + \beta_{800}),\ \beta_{530} = -\frac{2}{5}\beta_{440},\ \beta_{341} = -\frac{1}{2}\beta_{332}$
$- \beta_{431}, \beta_{314} = \frac{1}{6}(\beta_{323} - 2\beta_{413} + \beta_{800}),\ \beta_{521} = \frac{3}{10}(\beta_{323} + \beta_{800}),$
$\beta_{512} = -\frac{3}{10}(\beta_{323} + 2\beta_{413} + \beta_{800}),\ \beta_{503} = \frac{1}{5}(\beta_{413} - 6\beta_{404}),$
$\beta_{350} = -\frac{2}{5}\beta_{440},\ \beta_{305} = \frac{1}{60}(\beta_{323} - 12\beta_{404} - 2\beta_{413} + \beta_{800}),$
$\beta_{611} = \frac{1}{15}(\beta_{323} + \beta_{413} + \beta_{800}),\ \beta_{602} = -\frac{1}{30}(\beta_{323} + 14\beta_{404} - 4\beta_{413}$
$-\beta_{800}),\ \beta_{701} = \frac{1}{140}(\beta_{323} - 4\beta_{404} + 2\beta_{413} + \beta_{800}),\ \beta_{620} = 0,\ \beta_{710} =$
$0.$

- $\nu = 0,\ \lambda = -\frac{3}{2}$ then the constant $\beta_{413}, \beta_{341}, \beta_{323}, \beta_{440}, \beta_{800}, \beta_{404}, \beta_{332}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422} = \frac{3}{4}(\beta_{332} - 2\beta_{800} - \beta_{323}),\ \beta_{350} = -\frac{4}{5}\beta_{440},\ \beta_{431} = -\frac{1}{2}\beta_{332},$
$\beta_{701} = \frac{1}{140}(3\beta_{323} - 2\beta_{332} - 4\beta_{404} + 3\beta_{413} + 3\beta_{800}),$
$\beta_{314} = \frac{1}{3}(\beta_{323} - \beta_{413} + \beta_{800}),\ \beta_{521} = \frac{3}{10}(\beta_{323} - \beta_{332} + \beta_{800}),$
$\beta_{512} = \frac{3}{10}(\beta_{332} - 2\beta_{323} - 2\beta_{800}),\ \beta_{503} = \frac{3}{10}(\beta_{413} - 4\beta_{404}),$
$\beta_{305} = \frac{1}{20}(\beta_{323} - 4\beta_{404} - \beta_{413} + \beta_{800}),\ \beta_{611} = \frac{1}{30}(4\beta_{323} - 3\beta_{332}$

$+2\beta_{413}+4\beta_{800}),\ \beta_{602}=\frac{1}{20}(-2\beta_{323}+\beta_{332}+8\beta_{404}-4\beta_{413}-2\beta_{800}),$
$\beta_{530}=0,\ \beta_{620}=0,\ \beta_{710}=0.$

- $\nu=0,\ \lambda=-2$ then the constant $\beta_{431},\beta_{323},\beta_{332},\beta_{350},\beta_{413},$ β_{800},β_{404} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422}=\frac{3}{2}(\beta_{332}-\beta_{323}-\beta_{800}),\ \beta_{341}=\frac{1}{2}\beta_{332}+\beta_{431},$
$\beta_{314}=\frac{1}{2}\beta_{323}-\frac{1}{3}\beta_{413}+\frac{1}{2}\beta_{800},\ \beta_{521}=\frac{3}{10}(\beta_{323}-\beta_{332}$
$+2\beta_{431}+\beta_{800}),\ \beta_{512}=\frac{9}{10}(\beta_{332}-\beta_{323}-\frac{2}{3}\beta_{413}-\beta_{800}),$
$\beta_{503}=\frac{1}{30}(3\beta_{323}-6\beta_{404}-2\beta_{413}+3\beta_{800}),$
$\beta_{305}=\frac{1}{10}(\beta_{323}-2\beta_{404}+\beta_{800}-\frac{2}{3}\beta_{413}),$
$\beta_{611}=\frac{1}{15}(3\beta_{323}-3\beta_{332}+\beta_{413}+3\beta_{431}+3\beta_{800}),\ \beta_{602}=\frac{1}{15}(3\beta_{332}$
$+6\beta_{404}-3\beta_{323}-3\beta_{800}-4\beta_{413}),\ \beta_{701}=\frac{3}{70}(\beta_{323}-\beta_{332}-2\beta_{404}$
$+2\beta_{413}+2\beta_{431}+\beta_{800}),\ \beta_{440}=0,\ \beta_{530}=0,\ \beta_{710}=0,\ \beta_{620}=0.$

- $\nu=-\frac{1}{2},\ \lambda=0$ then the constant $\beta_{431},\beta_{413},\beta_{332},\beta_{323},\beta_{800},$ β_{404},β_{440} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422}=-\frac{3}{4}(\beta_{323}+2\beta_{332}+\beta_{800})\ \beta_{503}=-\frac{4}{5}\beta_{404},,\ \beta_{341}=-\frac{1}{3}\beta_{431},$
$\beta_{314}=-\frac{1}{4}(\beta_{323}+2\beta_{413}+\beta_{800}),\ \beta_{521}=-\frac{3}{5}\beta_{431},\ \beta_{611}=\frac{1}{15}\beta_{431},$
$\beta_{512}=\frac{3}{20}(\beta_{323}+2\beta_{332}-2\beta_{413}+\beta_{800}),\ \beta_{530}=\frac{1}{10}\beta_{431}-\frac{6}{5}\beta_{440},$
$\beta_{350}=-\frac{1}{60}\beta_{431}-\frac{1}{5}\beta_{440},\ \beta_{305}=-\frac{4}{15}\beta_{404},$
$\beta_{620}=\frac{1}{15}(6\beta_{440}-\beta_{431}),\ \beta_{602}=\frac{2}{15}\beta_{404},$
$\beta_{710}=\frac{1}{140}(\beta_{431}-4\beta_{440}),\ \beta_{701}=0.$

- $\nu=-\frac{1}{2},\ \lambda=-\frac{1}{2}$ then the constant $\beta_{332},\beta_{323},\beta_{413},\beta_{440},\beta_{404},$ β_{800},β_{431} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_5^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422}=-\frac{3}{4}(\beta_{323}+\beta_{332}+\beta_{800}),\ \beta_{341}=-\frac{1}{2}\beta_{431},\ \beta_{314}=-\frac{1}{2}\beta_{413},$
$\beta_{521}=-\frac{3}{10}\beta_{431},\ \beta_{512}=-\frac{3}{10}\beta_{413},\ \beta_{530}=\frac{1}{10}(\beta_{431}-8\beta_{440}),$
$\beta_{503}=\frac{1}{10}(\beta_{413}-8\beta_{404}),\ \beta_{350}=-\frac{1}{30}(\beta_{431}+8\beta_{440}),$
$\beta_{305}=\frac{1}{30}(8\beta_{404}+\beta_{413}),\ \beta_{620}=\frac{1}{30}(4\beta_{440}-\beta_{431}),$
$\beta_{602}=\frac{1}{30}(4\beta_{404}-\beta_{413}),\ \beta_{611}=0,\ \beta_{710}=0,\beta_{701}=0.$

- $\nu=-\frac{1}{2},\ \lambda=-1$ then the constant $\beta_{413},\beta_{440},\beta_{800},\beta_{431},\beta_{404},\beta_{323}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_5^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is three-dimensional. On the other hand,

$\beta_{422} = -\frac{3}{4}(\beta_{323} - \beta_{800})$ $\beta_{341} = -\beta_{431},$ $\beta_{314} = \frac{1}{4}(\beta_{323} - 2\beta_{413} + \beta_{800}),$ $\beta_{512} = -\frac{3}{20}(\beta_{323}+2\beta_{413}+\beta_{800}),$ $\beta_{530} = \frac{1}{10}(\beta_{431}-4\beta_{440}),$ $\beta_{503} = \frac{1}{5}(\beta_{413} - 4\beta_{404}),$ $\beta_{350} = -\frac{1}{10}(4\beta_{440} + \beta_{431}),$ $\beta_{305} = \frac{1}{30}(\beta_{323} - 8\beta_{404} - 2\beta_{413} + \beta_{800}),$ $\beta_{602} = \frac{1}{60}(8\beta_{404} - \beta_{323} - 4\beta_{413} - \beta_{800}),$ $\beta_{710} = 0,$ $\beta_{521} = 0,$ $\beta_{611} = 0,$ $\beta_{620} = 0,$ $\beta_{701} = 0.$

- $\nu = -\frac{1}{2},\ \lambda = -\frac{3}{2}$ then the constant $\beta_{440}, \beta_{413}, \beta_{404}, \beta_{323}, \beta_{341}, \beta_{800}, \beta_{332}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{422} = -\frac{3}{4}(\beta_{332} - \beta_{323} - \beta_{800}),$ $\beta_{314} = \frac{1}{2}(\beta_{323} - \beta_{413} + \beta_{800}),$ $\beta_{512} = \frac{3}{10}(\beta_{332} - \beta_{323} - \beta_{413} - \beta_{800}),$ $\beta_{503} = \frac{1}{10}(3\beta_{413} - 8\beta_{404}),$ $\beta_{350} = \frac{1}{5}(\beta_{341} - 4\beta_{440}),$ $\beta_{305} = \frac{1}{10}(\beta_{323} - \beta_{413} + \beta_{800}) - \frac{4}{15}\beta_{404},$ $\beta_{602} = \frac{1}{20}(\beta_{332} - \beta_{323} - 2\beta_{413} - \beta_{800}) + \frac{2}{15}\beta_{404},$ $\beta_{521} = 0,$ $\beta_{530} = 0,$ $\beta_{611} = 0,$ $\beta_{620} = 0,$ $\beta_{431} = 0,$ $\beta_{710} = 0,$ $\beta_{701} = 0.$

- $\nu = -\frac{1}{2},\ \lambda = -2$ then the constant $\beta_{332}, \beta_{350}, \beta_{323}, \beta_{800}, \beta_{413}, \beta_{404}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is three-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{4}(2\beta_{332} - \beta_{323} - \beta_{800}),$ $\beta_{314} = \frac{3}{4}(\beta_{323} - 2\beta_{413} + \beta_{800}),$ $\beta_{512} = \frac{3}{20}(6\beta_{332} - 3\beta_{323} - 2\beta_{413} - 3\beta_{800}),$ $\beta_{503} = \frac{2}{5}(\beta_{413} - 2\beta_{404}),$ $\beta_{305} = \frac{1}{15}(3\beta_{323} - 4\beta_{404} - 2\beta_{413} + 3\beta_{800}),$ $\beta_{602} = \frac{1}{10}(2\beta_{332} - \beta_{323} - \beta_{800}) + \frac{1}{15}(2\beta_{404} - 2\beta_{413}),$ $\beta_{431} = 0,$ $\beta_{341} = 0,$ $\beta_{440} = 0,$ $\beta_{521} = 0,$ $\beta_{530} = 0,$ $\beta_{611} = 0,$ $\beta_{620} = 0,$ $\beta_{710} = 0, \beta_{701} = 0.$

- $\nu = -1,\ \lambda = 0$ then the constant $\beta_{440}, \beta_{431}, \beta_{413}, \beta_{323}, \beta_{404}, \beta_{332}, \beta_{800}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{422} = -\frac{3}{2}\beta_{332},$ $\beta_{341} = \frac{1}{6}(\beta_{332} - 2\beta_{431}),$
  $\beta_{314} = -\frac{1}{2}(\beta_{323} - 2\beta_{413} + \beta_{800}),$ $\beta_{512} = \frac{3}{10}\beta_{332},$ $\beta_{503} = -\frac{2}{5}\beta_{404},$ $\beta_{521} = -\frac{3}{10}(\beta_{332}+2\beta_{431}),$ $\beta_{530} = \frac{1}{5}(\beta_{431}-6\beta_{440}),$ $\beta_{305} = -\frac{2}{5}\beta_{404},$ $\beta_{350} = \frac{1}{60}(\beta_{332} - 2\beta_{431} - 12\beta_{440}),$ $\beta_{611} = \frac{1}{15}(\beta_{332} + \beta_{431}),$
  $\beta_{620} = \frac{1}{30}(12\beta_{440} - \beta_{332} - 4\beta_{431}),$
  $\beta_{710} = \frac{1}{140}(\beta_{332} + 2\beta_{431} - 4\beta_{440}),$ $\beta_{602} = 0,$ $\beta_{701} = 0$

- $\nu = -1,\ \lambda = -\frac{1}{2}$ then the constant $\beta_{323}, \beta_{413}, \beta_{440}, \beta_{431}, \beta_{800}, \beta_{404}, \beta_{332}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

$\beta_{422} = -\frac{3}{4}\beta_{332},\ \beta_{341} = \frac{1}{4}(\beta_{332} - 2\beta_{431}),\ \beta_{314} = -\beta_{413},$
$\beta_{521} = -\frac{3}{20}(\beta_{332} + 2\beta_{431}),\ \beta_{530} = \frac{1}{5}(\beta_{431} - 4\beta_{440}),$
$\beta_{503} = \frac{1}{10}(\beta_{413} - 4\beta_{404}),\ \beta_{350} = \frac{1}{30}(\beta_{332} - 2\beta_{431} - 8\beta_{440}),$
$\beta_{305} = -\frac{1}{10}(4\beta_{404} - \beta_{413}),\ \beta_{620} = \frac{1}{60}(8\beta_{440} - \beta_{332} - 4\beta_{431}),$
$\beta_{512} = 0,\ \beta_{611} = 0,\ \beta_{602} = 0,\ \beta_{710} = 0,\ \beta_{701} = 0.$

- $\nu = -1,\ \lambda = -1$ then the constant $\beta_{431}, \beta_{413}, \beta_{440}, \beta_{404}, \beta_{800}, \beta_{323}, \beta_{332}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{341} = \frac{1}{2}\beta_{332} - \beta_{431},\ \beta_{314} = \frac{1}{2}(\beta_{323} - 2\beta_{413} + \beta_{800}),$
  $\beta_{530} = \frac{1}{5}(\beta_{431} - 2\beta_{440}),\ \beta_{503} = \frac{1}{5}(\beta_{413} - 2\beta_{404}),$
  $\beta_{350} = \frac{1}{10}(\beta_{332} - 2\beta_{431} - 4\beta_{440}),\ \beta_{305} = \frac{1}{10}(\beta_{323} - 4\beta_{404}$
  $-2\beta_{413}+\beta_{800}),\ \beta_{422} = 0,\ \beta_{521} = 0,\ \beta_{512} = 0,\ \beta_{611} = 0,\ \beta_{620} = 0,$
  $\beta_{602} = 0,\ \beta_{710} = 0,\ \beta_{701} = 0.$

- $\nu = -1,\ \lambda = -\frac{3}{2}$ then the constant $\beta_{440}, \beta_{404}, \beta_{341}, \beta_{323}, \beta_{800}, \beta_{413}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is three-dimensional. On the other hand,

  $\beta_{314} = \beta_{323} - \beta_{413} + \beta_{800},\ \beta_{503} = \frac{1}{10}(3\beta_{413} - 4\beta_{404}),$
  $\beta_{350} = \frac{2}{5}(\beta_{341} - 2\beta_{440}),\ \beta_{305} = \frac{1}{10}(3\beta_{323} - 4\beta_{404} - 3\beta_{413} + 3\beta_{800}),$
  $\beta_{332} = 0,\ \beta_{422} = 0,\ \beta_{431} = 0,\ \beta_{521} = 0,\ \beta_{512} = 0,$
  $\beta_{530} = 0,\ \beta_{611} = 0,\ \beta_{620} = 0,\ \beta_{602} = 0,\ \beta_{710} = 0,\ \beta_{701} = 0$

- $\nu = -1,\ \lambda = -2$ then the constant $\beta_{332}, \beta_{350}, \beta_{404}, \beta_{323}, \beta_{413}, \beta_{800}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is three-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{2}\beta_{332},\ \beta_{431} = -\frac{1}{2}\beta_{332}, \beta_{341} = -\beta_{332},\ \beta_{440} = -\frac{1}{2}\beta_{332},$
  $\beta_{314} = \frac{3}{2}(\beta_{323} + \beta_{800}) - \beta_{413}, \beta_{512} = \frac{9}{10}\beta_{332}, \beta_{530} = -\frac{3}{10}\beta_{332},$
  $\beta_{503} = \frac{2}{5}(\beta_{413} - \beta_{404}), \beta_{305} = \frac{1}{5}(\beta_{323} - 2\beta_{404} - 2\beta_{413} + 3\beta_{800}),$
  $\beta_{611} = \frac{1}{10}\beta_{332},\ \beta_{620} = -\frac{1}{10}\beta_{332},\ \beta_{602} = \frac{1}{5}\beta_{332},$
  $\beta_{710} = -\frac{1}{70}\beta_{332},\ \beta_{701} = \frac{1}{35}\beta_{332},\ \beta_{521} = 0,.$

- $\nu = -\frac{3}{2},\ \lambda = 0$ then the constant $\beta_{440}, \beta_{314}, \beta_{323}, \beta_{404}, \beta_{431}, \beta_{332}, \beta_{800}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{4}(\beta_{323} - 2\beta_{332} + \beta_{800}), \beta_{413} = -\frac{1}{2}(\beta_{323} + \beta_{800}),$
  $\beta_{341} = \frac{1}{3}(\beta_{332} - \beta_{431}),\ \beta_{521} = \frac{3}{10}(\beta_{323} - 2\beta_{332} - 2\beta_{431} + \beta_{800}),$

$\beta_{512} = \frac{3}{10}(\beta_{332} - \beta_{323} - \beta_{800}), \beta_{530} = \frac{3}{10}(\beta_{431} - 12\beta_{440}),$
$\beta_{350} = \frac{1}{20}(\beta_{332} - \beta_{431} - 4\beta_{440}),\ \beta_{305} = -\frac{4}{5}\beta_{404},\ \beta_{611} = \frac{1}{15}(2\beta_{332} + \beta_{431}) - \frac{1}{10}(\beta_{323} + \beta_{800}),\ \beta_{620} = \frac{1}{20}(\beta_{323} - 2\beta_{332} - 4\beta_{431} + 8\beta_{440} + \beta_{800}),\ \beta_{710} = \frac{1}{140}(3\beta_{332} - 2\beta_{323} + 3\beta_{431} - 4\beta_{440} - 2\beta_{800}),$
$\beta_{503} = 0, \beta_{602} = 0, \beta_{701} = 0.$

- $\nu = -\frac{3}{2},\ \lambda = -\frac{1}{2}$ then the constant $\beta_{404}, \beta_{431}, \beta_{323}, \beta_{800}, \beta_{440}, \beta_{332}, \beta_{314}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{4}(\beta_{323} - \beta_{332} + \beta_{800}),\ \beta_{341} = \frac{1}{2}(\beta_{332} - \beta_{431}),$
  $\beta_{521} = \frac{3}{10}(\beta_{323} - \beta_{332} - \beta_{431} + \beta_{800}),$
  $\beta_{530} = \frac{1}{10}(3\beta_{431} - 8\beta_{440}),\ \beta_{350} = \frac{1}{10}(\beta_{332} - \beta_{431}) - \frac{4}{15}\beta_{440},$
  $\beta_{305} = \frac{1}{5}(\beta_{314} - 4\beta_{404}),\ \beta_{620} = \frac{1}{20}(\beta_{323} - \beta_{332} - 2\beta_{431} + \beta_{800}) + \frac{2}{15}\beta_{440},\ \beta_{413} = 0, \beta_{512} = 0,\ \beta_{503} = 0,\ \beta_{611} = 0,$
  $\beta_{602} = 0, \beta_{710} = 0, \beta_{701} = 0.$

- $\nu = -\frac{3}{2},\ \lambda = -1$ then the constant $\beta_{440}, \beta_{431}, \beta_{404}, \beta_{332}, \beta_{314}$, can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is foor-dimensional. On the other hand,

  $\beta_{800} = -\beta_{323},\ \beta_{341} = \beta_{332} - \beta_{431},$
  $\beta_{530} = \frac{1}{10}(3\beta_{431} - 4\beta_{440}),\ \beta_{350} = \frac{1}{10}(3\beta_{332} - 3\beta_{431} - 4\beta_{440}),$
  $\beta_{305} = \frac{2}{5}(\beta_{314} - 2\beta_{404}),$
  $\beta_{422} = 0, \beta_{413} = 0, \beta_{521} = 0, \beta_{512} = 0, \beta_{503} = 0,\ \beta_{611} = 0,$
  $\beta_{620} = 0,\ \beta_{602} = 0,\ \beta_{710} = 0,\ \beta_{701} = 0$

- $\nu = -\frac{3}{2},\ \lambda = -\frac{3}{2}$ then the constant $\beta_{440}, \beta_{404}, \beta_{341}, \beta_{332}, \beta_{314}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is for-dimensional. On the other hand,

  $\beta_{800} = -\beta_{323} - \beta_{332},\ \beta_{431} = \beta_{332},\ \beta_{413} = -\beta_{332},\ \beta_{521} = \frac{3}{10}\beta_{332},$
  $\beta_{512} = -\frac{3}{10}\beta_{332},\ \beta_{530} = \frac{3}{10}\beta_{332}, \beta_{602} = -\frac{1}{10}\beta_{332},\ \beta_{503} = -\frac{3}{10}\beta_{332},$
  $\beta_{350} = \frac{1}{5}(3\beta_{341} - 4\beta_{440}),\ \beta_{305} = \frac{1}{5}(3\beta_{314} - 4\beta_{404}),\ \beta_{620} = \frac{1}{10}\beta_{332},$
  $\beta_{710} = \frac{1}{70}\beta_{332},\ \beta_{701} = -\frac{1}{70}\beta_{332},\ \beta_{422} = 0,\ \beta_{611} = 0,$

- $\nu = -\frac{3}{2},\ \lambda = -2$ then the constant$\beta_{404}, \beta_{323}, \beta_{350}, \beta_{314}, \beta_{332}, \beta_{800}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is three-dimensional. On the other hand

$\beta_{422} = \frac{3}{4}(\beta_{323} + 2\beta_{332} + \beta_{800}),\ \beta_{431} = -\beta_{323} - \beta_{332} - \beta_{800},$
$\beta_{413} = \frac{3}{2}(\beta_{323} + \beta_{800}),\ \beta_{341} = -\beta_{323} - 2\beta_{332} - \beta_{800},$
$\beta_{440} = -\frac{3}{4}(\beta_{323} + 2\beta_{332} + \beta_{800}),\ \beta_{521} = -\frac{3}{10}(\beta_{323} + \beta_{800}),$
$\beta_{512} = \frac{9}{10}(\beta_{323} + \beta_{332} + \beta_{800}),\ \beta_{530} = -\frac{3}{10}(2\beta_{323}$
$+ 3\beta_{332} + 2\beta_{800}),\ \beta_{503} = \frac{3}{5}(\beta_{323} + \beta_{800}),\ \beta_{305} = \frac{4}{5}(\beta_{314} - \beta_{404}),$
$\beta_{611} = \frac{1}{10}(\beta_{323} + 2\beta_{332} + \beta_{800}),\ \beta_{620} = -\frac{1}{4}(\beta_{323} + \beta_{800}) - \frac{3}{10}\beta_{332},$
$\beta_{602} = \frac{1}{10}(3\beta_{323} + 2\beta_{332} + 3\beta_{800}),$
$\beta_{710} = -\frac{3}{70}(\beta_{323} - \beta_{332} - \beta_{800}),\ \beta_{701} = \frac{2}{35}(\beta_{323} + \beta_{332} + \beta_{800}).$

- $\nu = -2,\ \lambda = 0$ then the constant$\beta_{431}, \beta_{332}, \beta_{323}, \beta_{305}, \beta_{800}, \beta_{440}$, β_{413} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{2}(\beta_{323} - \beta_{332} + \beta_{800}),\ \beta_{341} = \frac{1}{2}\beta_{332} - \frac{1}{3}\beta_{431},$
  $\beta_{314} = \frac{1}{2}(\beta_{323} + 2\beta_{413}\beta_{800}), \beta_{521} = \frac{9}{10}(\beta_{323} - \beta_{332} + \beta_{800} - \frac{2}{3}\beta_{431}),$
  $\beta_{512} = \frac{3}{10}(-\beta_{323} + \beta_{332} + 2\beta_{413} - \beta_{800}),$
  $\beta_{530} = \frac{2}{5}(\beta_{431} - 3\beta_{440}),\ \beta_{350} = \frac{1}{10}(\beta_{332} - 2\beta_{440}) - \frac{1}{15}\beta_{431},$
  $\beta_{611} = \frac{1}{15}(3\beta_{332} - 3\beta_{323} + 3\beta_{413} + \beta_{431} - 3\beta_{800}),$
  $\beta_{620} = \frac{1}{15}(3\beta_{323} - 3\beta_{332} - 4\beta_{431} + 6\beta_{440} + 3\beta_{800}),$
  $\beta_{710} = \frac{1}{70}(3\beta_{332} - 3\beta_{323} + 2\beta_{413} + 2\beta_{431} - 2\beta_{440} - 3\beta_{800}),$
  $\beta_{404} = 0,\ \beta_{503} = 0,\ \beta_{602} = 0,\ \beta_{701} = 0$

- $\nu = -2,\ \lambda = -\frac{1}{2}$ then the constant$\beta_{431}, \beta_{332}, \beta_{305}, \beta_{800}, \beta_{323}, \beta_{440}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is three-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{4}(2\beta_{323} - \beta_{332} + 2\beta_{800}),\ \beta_{341} = \frac{1}{4}(3\beta_{332} - 2\beta_{431}),$
  $\beta_{521} = \frac{3}{20}(6\beta_{323} - 3\beta_{332} - 6\beta_{431} + 6\beta_{800}),$
  $\beta_{530} = \frac{2}{5}(\beta_{431} - 2\beta_{440}),\ \beta_{350} = \frac{1}{15}(3\beta_{332} - 2\beta_{431} - 4\beta_{440}),$
  $\beta_{620} = \frac{1}{10}(2\beta_{323} - \beta_{332} + 2\beta_{800}) - \frac{2}{15}(\beta_{431} - \beta_{440}),\ \beta_{413} = 0,$
  $\beta_{314} = 0,\ \beta_{404} = 0,\ \beta_{512} = 0,\ \beta_{503} = 0,\ \beta_{611} = 0,$
  $\beta_{602} = 0,\ \beta_{710} = 0,\ \beta_{701} = 0.$

- $\nu = -2,\ \lambda = -1$ then the constant$\beta_{440}, \beta_{431}, \beta_{305}, \beta_{332}, \beta_{800}, \beta_{323}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is three-dimensional. On the other hand,

  $\beta_{422} = \frac{3}{2}(\beta_{323} + \beta_{800}),\ \beta_{413} = -\frac{1}{2}(\beta_{323} - \beta_{800}),$
  $\beta_{341} = \frac{3}{2}(\beta_{332} - \beta_{431}),\ \beta_{314} = -\beta_{323} - \beta_{800},\ \beta_{530} = \frac{2}{5}(\beta_{431} - \beta_{440}),$
  $\beta_{404} = -\frac{1}{2}(\beta_{323} + \beta_{800}), \beta_{521} = \frac{9}{10}(\beta_{323} + \beta_{800}), \beta_{503} = -\frac{3}{10}(\beta_{323}$
  $- \beta_{800}),\ \beta_{350} = \frac{1}{5}(3\beta_{332} - 2\beta_{431} - 2\beta_{440}), \beta_{611} = \frac{1}{10}(\beta_{323} + \beta_{800}),$

$\beta_{620}=\frac{1}{5}(\beta_{323}+\beta_{800}),\ \beta_{602}=-\frac{1}{10}(\beta_{323}+\beta_{800}),$
$\beta_{710}=\frac{1}{35}(\beta_{323}+\beta_{800}),\ \beta_{701}=-\frac{1}{70}(\beta_{323}+\beta_{800}),\ \beta_{512}=0.$

- $\nu=-2,\ \lambda=-\frac{3}{2}$ then the constant$\beta_{440},\beta_{341},\beta_{332},\beta_{305},\beta_{323},$ β_{800} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is tow-dimensional. On the other hand,

  $\beta_{422}=\frac{3}{4}(2\beta_{323}+\beta_{332}+2\beta_{800}),\ \beta_{431}=\frac{3}{2}\beta_{332},$
  $\beta_{413}=-\beta_{323}-\beta_{332}-\beta_{800},\ \beta_{314}=-2\beta_{323}-\beta_{332}-2\beta_{800},$
  $\beta_{404}=-\frac{3}{4}(2\beta_{323}+\beta_{332}-2\beta_{800}),\ \beta_{521}=\frac{9}{10}(\beta_{323}+\beta_{332}+\beta_{800}),$
  $\beta_{512}=-\frac{3}{10}\beta_{332},\beta_{530}=\frac{3}{5}\beta_{332},\ \beta_{503}=-\frac{9}{10}(\beta_{323}-\frac{2}{3}\beta_{332}-\beta_{800}),$
  $\beta_{350}=\frac{4}{5}(\beta_{341}-\beta_{440}),\beta_{611}=\frac{1}{10}(2\beta_{323}+\beta_{332}+2\beta_{800}),$
  $\beta_{620}=\frac{1}{10}(2\beta_{323}+3\beta_{332}+2\beta_{800}),\ \beta_{602}=-\frac{3}{10}(\beta_{323}-\beta_{800})$
  $-\frac{1}{4}\beta_{332},\ \beta_{710}=\frac{2}{35}(\beta_{323}+\beta_{332}+\beta_{800}),$
  $\beta_{701}=-\frac{3}{70}(\beta_{323}+\beta_{332}+\beta_{800})$

- $\nu=-2,\ \lambda=-2$ then the constant$\beta_{431},\beta_{350},\beta_{323},\beta_{305},\beta_{800},$ β_{332} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_7^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is three-dimensional.On the other hand,

  $\beta_{422}=\frac{3}{2}(\beta_{323}+\beta_{332}+\beta_{800}),\ \beta_{413}=-\frac{3}{2}(\beta_{323}+\beta_{332}+\frac{2}{3}\beta_{431}$
  $+\beta_{800}),\ \beta_{341}=-\frac{3}{2}\beta_{332}+\beta_{431},\ \beta_{440}=-\frac{3}{2}\beta_{332}+\beta_{431},$
  $\beta_{314}=-3\beta_{323}-\frac{3}{2}\beta_{332}-\beta_{431}-3\beta_{800},\ \beta_{404}=-3\beta_{323}-\frac{3}{2}\beta_{332}$
  $-\beta_{431}-3\beta_{800},\ \beta_{521}=\frac{3}{10}(3\beta_{323}+3\beta_{332}+2\beta_{431}+3\beta_{800}),$
  $\beta_{512}=-\frac{3}{5}\beta_{431},\ \beta_{530}=\frac{1}{5}(4\beta_{431}-3\beta_{332}),$
  $\beta_{503}=-\frac{1}{5}(\beta_{323}+6\beta_{332}+4\beta_{431}+9\beta_{800}),$
  $\beta_{611}=\frac{3}{10}(\beta_{323}+\beta_{332}+3\beta_{800}),$
  $\beta_{620}=\frac{1}{5}(\beta_{323}+2\beta_{431}+\beta_{800}),\ \beta_{602}=-\frac{1}{5}(\beta_{323}+2\beta_{332}+2\beta_{431}$
  $+3\beta_{800}),\ \beta_{710}=\frac{3}{70}(2\beta_{323}+\beta_{332}+2\beta_{431}+2\beta_{800}),$
  $\beta_{701}=-\frac{3}{70}(2\beta_{323}+\beta_{332}+2\beta_{431}+2\beta_{800}).$

(4) The case when $k=8$
The 2-cocycle condition is equivalent to the following system where $\alpha_{ijkl}=-\alpha_{jikl}$
$\alpha_{3511}=\frac{1}{5}\left((1+2\nu)\alpha_{4312}+(1+2\lambda)\alpha_{4321}\right),$
$\alpha_{3520}=\frac{1}{5}\left(\nu\alpha_{4321}+3(1+\lambda)\alpha_{4330}\right),$
$\alpha_{3502}=\frac{1}{5}\left(3(1+\nu)\alpha_{4303}+\lambda\alpha_{4312}\right),$
$\alpha_{4600}=\frac{1}{9}\left(\nu\alpha_{5401}+\lambda\alpha_{5410}\right),$
$\alpha_{6310}=\frac{1}{45}(\nu(1+2\nu)\alpha_{4312}+2(1+2\lambda)\nu\alpha_{4321}+3\left(1+3\lambda+2\lambda^2\right)\alpha_{4330}$
$-10\alpha_{5410}),$
$\alpha_{6301}=\frac{1}{45}(3\left(1+3\nu+2\nu^2\right)\alpha_{4303}+2\lambda(1+2\nu)\alpha_{4312}+\lambda(1+2\lambda)\alpha_{4321}$

$-10\alpha_{5401}),$
$\alpha_{3700}=\frac{1}{630}(3\nu(1+3\nu+2\nu^2)\alpha_{4303}+3\lambda\nu(1+2\nu)\alpha_{4312}+3\lambda(1+2\lambda)\nu\alpha_{4321}$
$+3\lambda(1+3\lambda+2\lambda^2)\alpha_{4330}-20\nu\alpha_{5401}-20\lambda\alpha_{5410}).$

Let us study the triviality of the 2-cocycle . A direct computation proves the space of solutions of the system above is one-dimensional for $(\nu,\lambda)=(-\frac{7}{2},0),(-3,0),(0,-\frac{7}{2}),(-3,0),(0,-\frac{1}{2})$ zero-dimensional for $(\nu,\lambda)=(-1,-1),(-1,-\frac{3}{2}),(-1,-2),(-1,-\frac{5}{2}),(-\frac{3}{2},-1),(-\frac{3}{2},-\frac{3}{2}),$ $(-\frac{3}{2},-\frac{5}{2}),(-\frac{3}{2},-2),(-2,-1),(-2,-\frac{3}{2}),(-2,-2),(-2,-\frac{5}{2}),(-\frac{5}{2},-1),$ $(-\frac{5}{2},-\frac{3}{2}),(-\frac{5}{2},-2),(-\frac{5}{2},-\frac{5}{2}).$

- $\lambda=-1,\ \nu=-1$ the constants $\beta_{333},\ \beta_{342},\ \beta_{324},\ \beta_{351},\ \beta_{315}$, β_{360}, β_{306} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414}=\frac{1}{2}(\beta_{324}-5\beta_{315}),\ \beta_{441}=\frac{1}{2}(\beta_{342}-5\beta_{351}),\ \beta_{423}=-\beta_{324},$
  $\beta_{432}=-\beta_{342},\ \beta_{450}=\frac{1}{2}(\beta_{351}-9\beta_{360}),\ \beta_{405}=\frac{1}{2}(\beta_{315}-9\beta_{306}),$
  $\beta_{513}=\beta_{315}-\frac{2}{5}\beta_{324},\ \beta_{531}=-\frac{2}{5}\beta_{342}+\beta_{351},\ \beta_{540}=\frac{1}{10}(45\beta_{342}$
  $-10\beta_{351}+9\beta_{360}),\ \beta_{504}=\frac{1}{10}(45\beta_{306}-10\beta_{315}+\beta_{324}),$
  $\beta_{630}=-\frac{1}{15}\beta_{342}+\frac{1}{3}\beta_{351}-\beta_{360},\ \beta_{603}=-\beta_{306}+\frac{1}{3}\beta_{315}-\frac{1}{15}\beta_{324},$
  $\beta_{522}=0,\ \beta_{612}=0,\ \beta_{621}=0,\ \beta_{711}=0,\ \beta_{702}=0,$
  $\beta_{720}=0,\ \beta_{810}=0,\ \beta_{801}=0,\ \beta_{900}=0$

- $\lambda=0,\ \nu=-\frac{1}{2}$ the constants $\beta_{342},\beta_{324},\beta_{351},\beta_{315},\beta_{360},\beta_{306}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{414}=-\frac{1}{4}(15\beta_{315}+2\beta_{324}),\ \beta_{441}=-5\beta_{351},\ \beta_{423}=-2\beta_{324}$
  $-\frac{3}{2}\beta_{333},\ \beta_{432}=-\frac{3}{4}\beta_{333}-3\beta_{342},\ \beta_{450}=\frac{1}{4}(\beta_{351}-30\beta_{360}),$
  $\beta_{405}=-6\beta_{306},\ \beta_{531}=6\beta_{351},\ \beta_{513}=3\beta_{315}+\frac{4}{5}\beta_{324}+\frac{3}{10}\beta_{333},$
  $\beta_{522}=\frac{3}{5}\beta_{324}+\frac{9}{10}\beta_{333}+\frac{9}{5}\beta_{342},\ \beta_{540}=-\beta_{351}+15\beta_{360},$
  $\beta_{612}=-\frac{1}{2}\beta_{315}-\frac{1}{5}\beta_{324}-\frac{3}{20}\beta_{333}-\frac{1}{5}\beta_{342},\ \beta_{621}=-2\beta_{351},$
  $\beta_{630}=\beta_{351}-10\beta_{360},\ \beta_{603}=-4\beta_{306},\ \beta_{711}=\frac{1}{7}\beta_{351},$
  $\beta_{702}=\frac{3}{7}\beta_{306},\ \beta_{720}=-\frac{2}{7}\beta_{351}+\frac{15}{7}\beta_{360},\ \beta_{810}=\frac{1}{56}\beta_{351}-\frac{3}{28}\beta_{360},$
  $\beta_{504}=9\beta_{306},\ \beta_{801}=0,\ \beta_{900}=0.$

- $\lambda=-\frac{7}{2},\ \nu=0$, the constants $\beta_{342},\beta_{315},\beta_{333},\beta_{351},\beta_{324},\beta_{306}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{414}=-5\beta_{315}+3\beta_{324},\ \beta_{441}=-\frac{1}{2}\beta_{342}+\frac{15}{4}\beta_{351},$
  $\beta_{423}=-3\beta_{324}+\frac{15}{4}\beta_{333},\ \beta_{432}=-\frac{3}{2}\beta_{333}+4\beta_{342},\ \beta_{450}=3\beta_{360},$

$\beta_{405} = -\frac{15}{2}\beta_{306} + \frac{7}{4}\beta_{315},\ \beta_{513} = 6\beta_{315} - \frac{36}{5}\beta_{324} + \frac{9}{2}\beta_{333},$
$\beta_{531} = \frac{3}{10}\beta_{333} - \frac{8}{5}\beta_{342} + 6\beta_{351},\ \beta_{522} = \frac{9}{5}\beta_{324} - \frac{9}{2}\beta_{333} + 6\beta_{342},$
$\beta_{540} = \frac{9}{2}\beta_{360},\ \beta_{504} = 15\beta_{306} - 7\beta_{315} + \frac{21}{10}\beta_{324},$
$\beta_{612} = -2\beta_{315} + \frac{18}{5}\beta_{324} - \frac{9}{2}\beta_{333} + 4\beta_{342},$
$\beta_{621} = -\frac{1}{5}\beta_{324} + \frac{3}{4}\beta_{324} - 2\beta_{342} + 5\beta_{351},$
$\beta_{630} = 4\beta_{360},\ \beta_{603} = -10\beta_{306} + 7\beta_{315} - \frac{21}{5}\beta_{324} + \frac{7}{4}\beta_{333},$
$\beta_{711} = \frac{1}{7}\beta_{315} - \frac{12}{35}\beta_{324} + \frac{9}{14}\beta_{333} - \frac{8}{7}\beta_{342} + \frac{15}{7}\beta_{351},$
$\beta_{702} = \frac{15}{7}\beta_{306} - 2\beta_{315} + \frac{9}{5}\beta_{324} - \frac{3}{2}\beta_{333} + \beta_{342},\ \beta_{720} = \frac{15}{7}\beta_{360},$
$\beta_{810} = \frac{9}{14}\beta_{360},\ \beta_{801} = -\frac{3}{28}\beta_{306} + \frac{1}{8}\beta_{315} - \frac{3}{20}\beta_{324} + \frac{3}{16}\beta_{333} - \frac{1}{4}\beta_{342} + \frac{3}{8}\beta_{351},\ \beta_{900} = \frac{1}{12}\beta_{360}.$

- $\lambda = -3,\ \nu = 0$ the constants $\beta_{315}, \beta_{333}, \beta_{342}, \beta_{324}, \beta_{306}, \beta_{360}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{414} = -5\beta_{315} + \frac{5}{2}\beta_{324},\ \beta_{441} = -\frac{1}{2}\beta_{342} + \frac{5}{2}\beta_{351},$
  $\beta_{423} = -3\beta_{324} + 3\beta_{333},\ \beta_{432} = -\frac{3}{2}\beta_{333} + 3\beta_{342},\ \beta_{450} = \frac{3}{2}\beta_{360},$
  $\beta_{405} = -\frac{15}{2}\beta_{306} + \frac{3}{2}\beta_{315},\ \beta_{513} = 6\beta_{315} - 6\beta_{324} + 3\beta_{333},$
  $\beta_{531} = \frac{3}{10}\beta_{333} - \frac{6}{5}\beta_{342} + 3\beta_{351},\ \beta_{522} = \frac{1}{5}(9\beta_{324} - 18\beta_{333} + 18\beta_{342}),$
  $\beta_{540} = \frac{3}{2}\beta_{360},\ \beta_{504} = 15\beta_{306} - 6\beta_{315} + \frac{3}{2}\beta_{324},\ \beta_{612} = -2\beta_{315} + 3\beta_{324} - 3\beta_{333} + 2\beta_{342},\ \beta_{621} = -\frac{1}{5}\beta_{324} + \frac{3}{5}\beta_{333} - \frac{6}{5}\beta_{342} + 2\beta_{351},$
  $\beta_{630} = \beta_{360},\ \beta_{603} = -10\beta_{306} + 6\beta_{315} - 3\beta_{324} + \beta_{333},$
  $\beta_{711} = \frac{1}{7}(\beta_{315} - 2\beta_{324} + 3\beta_{333} - 4\beta_{342} + 5\beta_{351}),\ \beta_{702} = \frac{1}{7}(15\beta_{306} - 12\beta_{315} + 9\beta_{324} - 6\beta_{333} + 3\beta_{342}),\ \beta_{720} = \frac{3}{7}\beta_{360},\ \beta_{810} = \frac{3}{28}\beta_{360},$
  $\beta_{801} = \frac{3}{28}(\beta_{315} - \beta_{306} - \beta_{324} + \beta_{333} - \beta_{342} + \beta_{351}),\ \beta_{900} = \frac{1}{84}\beta_{360}.$

- $\lambda = 0,\ \nu = -\frac{7}{2}$ the constants $\beta_{324}, \beta_{351}, \beta_{333}, \beta_{315}, \beta_{342}, \beta_{360}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{414} = \frac{15}{4}\beta_{315} - \frac{1}{2}\beta_{324},\ \beta_{441} = 3\beta_{342} - 5\beta_{351}, \beta_{423} = 4\beta_{324} - \frac{3}{2}\beta_{333},$
  $\beta_{432} = \frac{15}{4}\beta_{333} - 3\beta_{342},\ \beta_{450} = \frac{7}{4}\beta_{351} - \frac{15}{2}\beta_{360},\ \beta_{405} = 3\beta_{306},$
  $\beta_{513} = 6\beta_{315} - \frac{8}{5}\beta_{324} + \frac{3}{10}\beta_{333},\ \beta_{531} = \frac{9}{2}\beta_{333} - \frac{36}{5}\beta_{342} + 6\beta_{351},$
  $\beta_{522} = 6\beta_{324} - \frac{9}{2}\beta_{333} + \frac{9}{5}\beta_{342}, \beta_{540} = \frac{21}{10}\beta_{342} - 7\beta_{351} + 15\beta_{360},$
  $\beta_{504} = \frac{9}{2}\beta_{306},\ \beta_{612} = 5\beta_{315} - 2\beta_{324} + \frac{3}{4}\beta_{333} - \frac{1}{5}\beta_{342},$
  $\beta_{621} = 4\beta_{324} - \frac{9}{2}\beta_{333} + \frac{18}{5}\beta_{342} - 2\beta_{351}, \beta_{603} = 4\beta_{306},$
  $\beta_{630} = \frac{7}{4}\beta_{333} - \frac{21}{5}\beta_{342} + 7\beta_{351} - 10\beta_{360}, \beta_{702} = \frac{15}{7}\beta_{306},$
  $\beta_{711} = \frac{15}{7}\beta_{315} - \frac{8}{7}\beta_{324} + \frac{9}{14}\beta_{333} - \frac{12}{35}\beta_{342} + \frac{1}{7}\beta_{351},$
  $\beta_{720} = \beta_{324} - \frac{3}{2}\beta_{333} + \frac{9}{5}\beta_{342} - 2\beta_{351} + \frac{15}{7}\beta_{360},$
  $\beta_{810} = \frac{3}{8}\beta_{315} - \frac{1}{4}\beta_{324} + \frac{3}{16}\beta_{333} - \frac{3}{20}\beta_{342} + \frac{1}{8}\beta_{351} - \frac{3}{28}\beta_{360},$
  $\beta_{801} = \frac{9}{14}\beta_{306},\ \beta_{900} = \frac{1}{12}\beta_{306}.$

- $\lambda=-3,\ \nu=0$ the constants $\beta_{315},\beta_{333},\beta_{342},\beta_{324},\beta_{306},\beta_{360}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional. On the other hand,

  $\beta_{414}=-5\beta_{315}+\frac{5}{2}\beta_{324},\ \beta_{441}=-\frac{1}{2}\beta_{342}+\frac{5}{2}\beta_{351},$
  $\beta_{423}=-3\beta_{324}+3\beta_{333},\beta_{432}=-\frac{3}{2}\beta_{333}+3\beta_{342},\ \beta_{450}=\frac{3}{2}\beta_{360},$
  $\beta_{405}=-\frac{15}{2}\beta_{306}+\frac{3}{2}\beta_{315},\beta_{513}=6\beta_{315}-6\beta_{324}+3\beta_{333},$
  $\beta_{531}=\frac{3}{10}\beta_{333}-\frac{6}{5}\beta_{342}+3\beta_{351},\ \beta_{522}=\frac{9}{5}\beta_{324}-\frac{18}{5}\beta_{333}+\frac{18}{5}\beta_{342},$
  $\beta_{540}=\frac{3}{2}\beta_{360},\ \beta_{504}=15\beta_{306}-6\beta_{315}+\frac{3}{2}\beta_{324},\ \beta_{630}=\beta_{360},$
  $\beta_{612}=-2\beta_{315}+3\beta_{324}-3\beta_{333}+2\beta_{342},\ \beta_{621}=-\frac{1}{5}\beta_{324}+\frac{3}{5}\beta_{333}$
  $-\frac{6}{5}\beta_{342}+2\beta_{351},\ \beta_{603}=-10\beta_{306}+6\beta_{315}-3\beta_{324}+\beta_{333},$
  $\beta_{711}=\frac{1}{7}(\beta_{315}-2\beta_{324}+3\beta_{333}-4\beta_{342}+5\beta_{351}),\ \beta_{702}=\frac{1}{7}(15\beta_{306}$
  $-12\beta_{315}+9\beta_{324}-6\beta_{333}+3\beta_{342}),\ \beta_{720}=\frac{3}{7}\beta_{360},\ \beta_{810}=\frac{3}{28}\beta_{360},$
  $\beta_{801}=\frac{1}{28}(-3\beta_{306}+3\beta_{315}-3\beta_{324}+3\beta_{333}-3\beta_{342}+3\beta_{351}),$
  $\beta_{900}=\frac{1}{84}\beta_{360}.$

- $\lambda=-1,\ \nu=-\frac{3}{2}$ the constants $\beta_{324},\beta_{342},\beta_{351},\beta_{315},\beta_{333},\beta_{360},\beta_{306}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand ,

  $\beta_{414}=-\frac{1}{4}(5\beta_{315}+2\beta_{324}),\ \beta_{441}=\beta_{342}-\frac{5}{2}\beta_{351},$
  $\beta_{432}=\frac{3}{4}\beta_{333}-\beta_{342},\ \beta_{450}=\frac{3}{4}\beta_{351}-\frac{9}{2}\beta_{360},$
  $\beta_{405}=-3\beta_{306}+\frac{1}{2}\beta_{315},\ \beta_{531}=\frac{3}{10}\beta_{333}-\frac{4}{5}\beta_{342}+\beta_{351},$
  $\beta_{540}=\frac{1}{10}(3\beta_{342}-15\beta_{351}+45\beta_{360}),\ \beta_{504}=\frac{1}{10}(15\beta_{306}-5\beta_{315}$
  $+\beta_{324}),\ \beta_{630}=\frac{1}{20}\beta_{333}-\frac{1}{5}\beta_{342}+\frac{1}{2}\beta_{351}-\beta_{360},\ \beta_{603}=0,$
  $\beta_{711}=0,\ \beta_{702}=0,\ \beta_{720}=0,\ \beta_{810}=0,\ \beta_{801}=0,$
  $\beta_{612}=0,\ \beta_{621}=0,\ \beta_{900}=0,\ \beta_{513}=0,\ \beta_{423}=0,\ \beta_{522}=0,$

- $\lambda=-1,\ \nu=-2$ the constants $\beta_{315},\beta_{351},\beta_{342},\beta_{333},\beta_{360},\beta_{306},\beta_{324}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414}=\frac{1}{2}\beta_{324},\ \beta_{441}=\frac{3}{2}\beta_{342}-\frac{5}{2}\beta_{351},\ \beta_{423}=\beta_{324},$
  $\beta_{432}=\frac{3}{2}\beta_{333}-\beta_{342},\ \beta_{450}=\beta_{351}-\frac{9}{2}\beta_{360},\ \beta_{405}=\frac{1}{2}(\beta_{315}$
  $-3\beta_{306}),\ \beta_{513}=\frac{2}{5}\beta_{324},\ \beta_{531}=\frac{9}{10}\beta_{333}-\frac{6}{5}\beta_{342}+\beta_{351},$
  $\beta_{522}=\frac{3}{5}\beta_{324},\ \beta_{540}=\frac{3}{5}\beta_{342}-2\beta_{351}+\frac{9}{2}\beta_{360},\ \beta_{504}=\frac{1}{10}\beta_{324},$
  $\beta_{612}=\frac{1}{5}\beta_{324},\ \beta_{621}=\frac{1}{5}\beta_{324},\ \beta_{630}=\frac{1}{5}\beta_{333}-\frac{2}{5}\beta_{342}+\frac{2}{3}\beta_{351}$
  $-\beta_{360},\ \beta_{603}=\frac{1}{15}\beta_{324},\ \beta_{711}=\frac{2}{35}\beta_{324},\ \beta_{702}=\frac{1}{35}\beta_{324},$
  $\beta_{720}=\frac{1}{35}\beta_{324},\ \beta_{810}=\frac{1}{140}\beta_{324},\beta_{801}=\frac{1}{140}\beta_{324},\ \beta_{900}=\frac{1}{126}\beta_{324}.$

- $\lambda=-1,\ \nu=-\frac{5}{2}$ the constants $\beta_{342},\beta_{351},\beta_{333},\beta_{360},\beta_{306},\beta_{315},\beta_{324}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414}=\frac{1}{4}(5\beta_{315}+4\beta_{324}),\ \beta_{441}=2\beta_{342}-\frac{5}{2}\beta_{351},\ \beta_{423}=2\beta_{324},$
  $\beta_{432}=\frac{9}{4}\beta_{333}-\beta_{342},\ \beta_{450}=\frac{1}{4}(5\beta_{351}-18\beta_{360}),\ \beta_{405}=\frac{1}{2}\beta_{315},$
  $\beta_{513}=\beta_{315}+\frac{4}{5}\beta_{324},\ \beta_{531}=\frac{1}{5}(9\beta_{333}-8\beta_{342}+5\beta_{351}),$
  $\beta_{522}=\frac{9}{5}\beta_{324},\ \beta_{540}=\beta_{342}-\frac{5}{2}\beta_{351}+\frac{9}{2}\beta_{360},\beta_{621}=\frac{4}{5}\beta_{324},$
  $\beta_{504}=\frac{1}{10}(5\beta_{315}+\beta_{324}),\ \beta_{612}=\frac{1}{2}\beta_{315}+\frac{3}{5}\beta_{324},\ \beta_{630}=\frac{1}{2}\beta_{333}$
  $-\frac{2}{3}\beta_{342}+\frac{5}{6}\beta_{351}-\beta_{360},\ \beta_{603}=\frac{1}{15}(5\beta_{315}+2\beta_{324}),$
  $\beta_{711}=\frac{1}{35}(5\beta_{315}+8\beta_{324}),\ \beta_{702}=\frac{1}{35}(5\beta_{315}+3\beta_{324}),$
  $\beta_{720}=\frac{1}{7}\beta_{324},\ \beta_{810}=\frac{1}{56}(\beta_{315}+2\beta_{324}),\ \beta_{801}=\frac{1}{28}\beta_{315}+\frac{1}{35}\beta_{324},$
  $\beta_{900}=\frac{1}{252}(\beta_{315}+\beta_{324}).$

- $\lambda=-\frac{3}{2},\ \nu=-1$ the constants $\beta_{342},\beta_{324},\beta_{315},\beta_{351},\beta_{333},\beta_{306},\beta_{360}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414}=-\frac{5}{2}\beta_{315}+\beta_{324},\ \beta_{441}=\frac{1}{4}(2\beta_{342}-5\beta_{351}),\ \beta_{423}=-\beta_{324}$
  $+\frac{3}{4}\beta_{333},\ \beta_{450}=\frac{1}{2}\beta_{351}-3\beta_{360},\ \beta_{405}=\frac{1}{4}(3\beta_{315}-18\beta_{306}),$
  $\beta_{513}=\frac{1}{10}(10\beta_{315}-8\beta_{324}+3\beta_{333}),\ \beta_{540}=\frac{1}{10}(\beta_{342}-5\beta_{351}$
  $+15\beta_{360}),\ \beta_{504}=\frac{1}{10}(18\beta_{306}-15\beta_{315}+3\beta_{324}),\ \beta_{603}=\frac{1}{20}(-20\beta_{306}$
  $+10\beta_{315}-4\beta_{324}+\beta_{333}),\ \beta_{432}=0,\ \beta_{531}=0,\ \beta_{522}=0,$
  $\beta_{630}=0,\ \beta_{612}=0,\ \beta_{621}=0,\ \beta_{711}=0,\ \beta_{702}=0,$
  $\beta_{720}=0,\ \beta_{810}=0,\ \beta_{801}=0,\ \beta_{900}=0.$

- $\lambda=-\frac{3}{2},\ \nu=-\frac{3}{2}$ the constants $\beta_{342},\beta_{324},\beta_{315},\beta_{351},\beta_{360},\beta_{306},\beta_{333}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414}=-\frac{5}{4}\beta_{315}+\beta_{324},\ \beta_{441}=\beta_{342}-\frac{5}{4}\beta_{351},\ \beta_{423}=\frac{3}{4}\beta_{333},$
  $\beta_{432}=\frac{3}{4}\beta_{333},\ \beta_{450}=\frac{3}{4}\beta_{351}-3\beta_{360},\ \beta_{405}=-3\beta_{306}+\frac{3}{4}\beta_{315},$
  $\beta_{513}=\frac{3}{10}\beta_{333},\ \beta_{531}=\frac{3}{10}\beta_{333},\ \beta_{522}=\frac{9}{20}\beta_{333},\ \beta_{540}=\frac{3}{10}\beta_{342}$
  $-\frac{3}{4}\beta_{351}+\frac{3}{2}\beta_{360},\ \beta_{504}=\frac{3}{2}\beta_{306}-\frac{3}{4}\beta_{315}+\frac{3}{10}\beta_{324},\ \beta_{612}=\frac{3}{20}\beta_{333},$
  $\beta_{621}=\frac{3}{20}\beta_{333},\ \beta_{630}=\frac{1}{20}\beta_{333},\ \beta_{603}=\frac{1}{20}\beta_{333},\ \beta_{711}=\frac{3}{70}\beta_{333},$
  $\beta_{702}=\frac{3}{140}\beta_{333},\ \beta_{720}=\frac{3}{140}\beta_{333},\ \beta_{810}=\frac{3}{560}\beta_{333},\ \beta_{801}=\frac{3}{560}\beta_{333},$
  $\beta_{900}=\frac{1}{1680}\beta_{333}$

- $\lambda=-\frac{3}{2},\ \nu=-\frac{5}{2}$ the constants $\beta_{342},\beta_{351},\beta_{306},\beta_{360},\beta_{333},\beta_{315},\beta_{324}$ can be chosen in such a way that once adding the trivial 2-cocycle

$\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{414} = \frac{5}{4}\beta_{315} + \beta_{324},\ \beta_{441} = 2\beta_{342} - \frac{5}{4}\beta_{351},\ \beta_{432} = \frac{9}{4}\beta_{333},$
$\beta_{423} = 2\beta_{324} + \frac{3}{4}\beta_{333},,\ \beta_{450} = \frac{5}{4}\beta_{351} - 3\beta_{360},\ \beta_{405} = \frac{3}{4}\beta_{315},$
$\beta_{513} = \frac{1}{10}(10\beta_{315} + 16\beta_{324} + 3\beta_{333}),\ \beta_{531} = \frac{9}{5}\beta_{333},\ \beta_{522} = \frac{9}{5}\beta_{324}$
$+ \frac{27}{20}\beta_{333},\ \beta_{540} = \frac{1}{4}(4\beta_{342} - 5\beta_{351} + 6\beta_{360}),\ \beta_{504} = \frac{3}{4}\beta_{315} + \frac{3}{10}\beta_{324},$
$\beta_{612} = \frac{1}{2}\beta_{315} + \frac{6}{5}\beta_{324} + \frac{9}{20}\beta_{333},\ \beta_{621} = \frac{1}{10}(20\beta_{324} + 9\beta_{333}),$
$\beta_{630} = \frac{1}{2}\beta_{333},\ \beta_{603} = \frac{1}{20}(10\beta_{315} + 8\beta_{324} + \beta_{333}),\ \beta_{711} = \frac{1}{35}(5\beta_{315}$
$+ 16\beta_{324} + 9\beta_{333}),\ \beta_{702} = \frac{1}{140}(30\beta_{315} + 36\beta_{324} + 9\beta_{333}),$
$\beta_{720} = \frac{1}{14}(2\beta_{324} + 3\beta_{333}),\ \beta_{810} = \frac{1}{56}(\beta_{315} + 4\beta_{324} + 3\beta_{333}),$
$\beta_{801} = \frac{3}{56}\beta_{315} + \frac{3}{35}\beta_{324} + \frac{9}{280}\beta_{333},\ \beta_{900} = \frac{1}{168}(\beta_{315} + 2\beta_{324} + \beta_{333}).$

- $\lambda = -\frac{3}{2},\ \nu = -2$ the constants $\beta_{342}, \beta_{315}, \beta_{351}, \beta_{306}, \beta_{360}, \beta_{333}, \beta_{324}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand ,

$\beta_{414} = \beta_{324},\ \beta_{441} = \frac{1}{4}(6\beta_{342} - 5\beta_{351}),\ \beta_{423} = \beta_{324} + \frac{3}{4}\beta_{333},$
$\beta_{432} = \frac{3}{2}\beta_{333},\ \beta_{450} = \beta_{351} - 3\beta_{360},\ \beta_{405} = \frac{3}{4}(\beta_{315} - 2\beta_{306}),$
$\beta_{513} = \frac{4}{5}\beta_{324} + \frac{3}{10}\beta_{333},\ \beta_{531} = \frac{9}{10}\beta_{333},\ \beta_{522} = \frac{3}{5}(2\beta_{324} + 3\beta_{333}),$
$\beta_{540} = \frac{3}{5}\beta_{342} - \beta_{351} + \frac{3}{2}\beta_{360},\ \beta_{504} = \frac{3}{10}\beta_{324},\ \beta_{612} = \frac{1}{10}(4\beta_{324}$
$+3\beta_{333}),\ \beta_{621} = \frac{1}{20}(4\beta_{324} + 9\beta_{333}),\ \beta_{630} = \frac{1}{5}\beta_{333},\ \beta_{603} = \frac{1}{20}(4\beta_{324}$
$+ \beta_{333}),\ \beta_{711} = \frac{1}{70}(8\beta_{324} + 9\beta_{333}),\ \beta_{702} = \frac{3}{70}(2\beta_{324} + \beta_{333}),$
$\beta_{720} = \frac{1}{35}(\beta_{324} + 3\beta_{333}),\ \beta_{810} = \frac{1}{140}(2\beta_{324} + 3\beta_{333}),$
$\beta_{801} = \frac{1}{560}(12\beta_{324} + 9\beta_{333}),\ \beta_{900} = \frac{1}{420}(\beta_{324} + \beta_{333}).$

- $\lambda = -2,\ \nu = -1$ the constants $\beta_{351}, \beta_{315}, \beta_{324}, \beta_{333}, \beta_{306}, \beta_{360}, \beta_{342}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{414} = \frac{1}{2}(3\beta_{324} - 5\beta_{315}),\ \beta_{441} = \frac{1}{2}\beta_{342},\ \beta_{423} = -\beta_{324} + \frac{3}{2}\beta_{333},$
$\beta_{432} = \beta_{342},\ \beta_{450} = \frac{1}{2}(\beta_{351} - 3\beta_{360}),\ \beta_{405} = -\frac{9}{2}\beta_{306} + \beta_{315},$
$\beta_{513} = \frac{1}{10}(10\beta_{315} - 12\beta_{324} + 9\beta_{333}),\ \beta_{531} = \frac{2}{5}\beta_{342},\ \beta_{522} = \frac{3}{5}\beta_{342},$
$\beta_{540} = \frac{1}{10}\beta_{342},\ \beta_{504} = \frac{9}{2}\beta_{306} - 2\beta_{315} + \frac{3}{5}\beta_{324},\ \beta_{612} = \frac{1}{5}\beta_{342},$
$\beta_{621} = \frac{1}{5}\beta_{342},\ \beta_{630} = \frac{1}{15}\beta_{342},\ \beta_{603} = -\beta_{306} + \frac{2}{3}\beta_{315} - \frac{2}{5}\beta_{324}$
$+ \frac{1}{5}\beta_{333}, \beta_{711} = \frac{2}{35}\beta_{342},\ \beta_{702} = \frac{1}{35}\beta_{342},\ \beta_{720} = \frac{1}{35}\beta_{342},$
$\beta_{810} = \frac{1}{140}\beta_{342},\ \beta_{801} = \frac{1}{140}\beta_{342},\ \beta_{900} = \frac{1}{126}\beta_{342}.$

- $\lambda = -2,\ \nu = -\frac{3}{2}$ the constants $\beta_{324}, \beta_{315}, \beta_{351}, \beta_{360}, \beta_{306}, \beta_{342}, \beta_{333}$ can be chosen in such a way that once adding the trivial 2-cocycle

$\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{414} = -\frac{5}{4}\beta_{315} + \frac{3}{2}\beta_{324},\ \beta_{441} = \beta_{342},\ \beta_{423} = \frac{3}{2}\beta_{333},$
$\beta_{432} = \frac{3}{4}\beta_{333} + \beta_{342},\ \beta_{450} = \frac{3}{4}(\beta_{351} - 2\beta_{360}),\ \beta_{405} = -3\beta_{306}$
$+ \beta_{315},\ \beta_{513} = \frac{9}{10}\beta_{333},\ \beta_{531} = \frac{1}{10}(3\beta_{333} + 8\beta_{342}), \beta_{540} = \frac{3}{10}\beta_{342},$
$\beta_{522} = \frac{3}{10}(3\beta_{333} + 2\beta_{342}),\ \beta_{504} = \frac{3}{2}\beta_{306} - \beta_{315} + \frac{3}{5}\beta_{324},$
$\beta_{612} = \frac{1}{20}(9\beta_{333} + 4\beta_{342}),\ \beta_{621} = \frac{1}{10}(3\beta_{333} + 5\beta_{342}),$
$\beta_{630} = \frac{1}{20}(\beta_{333} + 4\beta_{342}),\ \beta_{603} = \frac{1}{5}\beta_{333},\ \beta_{711} = \frac{1}{70}(9\beta_{333}$
$+ 8\beta_{342}),\ \beta_{702} = \frac{1}{35}(3\beta_{333} + 2\beta_{342}),\ \beta_{720} = \frac{3}{70}(\beta_{333} + 2\beta_{342}),$
$\beta_{810} = \frac{1}{560}(9\beta_{333} + 12\beta_{342}),\ \beta_{801} = \frac{1}{140}(3\beta_{333} + 2\beta_{342}),$
$\beta_{900} = \frac{1}{420}(\beta_{333} + \beta_{342}).$

- $\lambda = -2,\ \nu = -2$ the constants $\beta_{315}, \beta_{351}, \beta_{360}, \beta_{306}, \beta_{342}, \beta_{324}, \beta_{333}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{414} = \frac{3}{2}\beta_{324},\ \beta_{540} = \frac{3}{5}\beta_{342},\ \beta_{441} = \frac{3}{2}\beta_{342},\ \beta_{423} = \beta_{324}$
$+ \frac{3}{2}\beta_{333}, \beta_{432} = \frac{3}{2}\beta_{333} + \beta_{342},\ \beta_{450} = \beta_{351} - \frac{3}{2}\beta_{360},$
$\beta_{405} = -\frac{3}{2}\beta_{306} + \beta_{315},\ \beta_{513} = \frac{3}{10}(4\beta_{324} + 3\beta_{333}),$
$\beta_{531} = \frac{3}{10}(3\beta_{333} + 4\beta_{342}),\ \beta_{522} = \frac{3}{5}(\beta_{324} + 3\beta_{333} + \beta_{342}),$
$\beta_{504} = \frac{3}{5}\beta_{324},\ \beta_{612} = \frac{1}{10}(6\beta_{324} + 9\beta_{333} + 2\beta_{342}),$
$\beta_{621} = \frac{1}{10}(2\beta_{324} + 9\beta_{333} + 6\beta_{342}),\ \beta_{630} = \frac{1}{5}(\beta_{333} + 2\beta_{342}),$
$\beta_{603} = \frac{1}{5}(2\beta_{324} + \beta_{333}), \beta_{711} = \frac{1}{70}(12\beta_{324} + 27\beta_{333} + 12\beta_{342}),$
$\beta_{702} = \frac{1}{35}(6\beta_{324} + 6\beta_{333} + \beta_{342}),\ \beta_{720} = \frac{1}{35}(\beta_{324} + 6\beta_{333} + 6\beta_{342}),$
$\beta_{810} = \frac{3}{140}(\beta_{324} + 3\beta_{333} + 2\beta_{342}),\ \beta_{801} = \frac{3}{140}(2\beta_{324}$
$+ 9\beta_{333} + \beta_{342}),\ \beta_{900} = \frac{1}{210}(\beta_{324} + 2\beta_{333} + \beta_{342}).$

- $\lambda = -2,\ \nu = -\frac{5}{2}$ the constants $\beta_{351}, \beta_{306}, \beta_{360}, \beta_{342}, \beta_{315}, \beta_{333}, \beta_{324}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{414} = \frac{1}{4}(5\beta_{315} + 6\beta_{324}),\ \beta_{441} = 2\beta_{342},\ \beta_{423} = 2\beta_{324} + \frac{3}{2}\beta_{333},$
$\beta_{432} = \frac{9}{4}\beta_{333} + \beta_{342},\ \beta_{450} = \frac{1}{4}(5\beta_{351} - 3\beta_{360}),\ \beta_{405} = \beta_{315},$
$\beta_{513} = \beta_{315} + \frac{12}{5}\beta_{324} + \frac{9}{10}\beta_{333},\ \beta_{531} = \frac{1}{5}(9\beta_{333} + 8\beta_{342}),$
$\beta_{522} = \frac{3}{5}(3\beta_{324} + 6\beta_{333} + \beta_{342}),\ \beta_{540} = \beta_{342},$
$\beta_{504} = \beta_{315} + \frac{3}{5}\beta_{324},\ \beta_{612} = \frac{1}{2}\beta_{315} + \frac{9}{5}\beta_{324} + \frac{27}{20}\beta_{333} + \frac{1}{5}\beta_{342},$
$\beta_{621} = \frac{1}{5}(4\beta_{324} + 9\beta_{333} + 4\beta_{342}),\ \beta_{630} = \frac{1}{2}\beta_{333} + \frac{2}{3}\beta_{342},$
$\beta_{603} = \frac{2}{3}\beta_{315} + \frac{4}{5}\beta_{324} + \frac{1}{5}\beta_{333},\ \beta_{711} = \frac{1}{35}(5\beta_{315} + 24\beta_{324}$
$+ 27\beta_{333} + 8\beta_{342}),\ \beta_{702} = \frac{1}{35}(10\beta_{315} + 18\beta_{324} + 9\beta_{333} + \beta_{342}),$
$\beta_{720} = \frac{1}{7}(\beta_{324} + 3\beta_{333} + 2\beta_{342}), \beta_{801} = \frac{1}{14}\beta_{315} + \frac{6}{35}\beta_{324} + \frac{9}{70}\beta_{333}$
$+ \frac{1}{35}\beta_{342},\ \beta_{810} = \frac{1}{56}(\beta_{315} + 6\beta_{324} + 9\beta_{333} + 4\beta_{342}),$

$\beta_{900} = \frac{1}{126}(\beta_{315} + 3\beta_{324} + 3\beta_{333} + \beta_{342}.$

- $\lambda = -\frac{5}{2},\ \nu = -1$ the constants $\beta_{324}, \beta_{315}, \beta_{333}, \beta_{306}, \beta_{360}, \beta_{351}, \beta_{342}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414} = 2\beta_{324} - \frac{5}{2}\beta_{315},\ \beta_{441} = \frac{1}{4}(2\beta_{342} + 5\beta_{351}),$
  $\beta_{423} = \frac{9}{4}\beta_{333} - \beta_{324},\ \beta_{432} = 2\beta_{342},\ \beta_{450} = \frac{1}{2}\beta_{351},$
  $\beta_{405} = \frac{1}{4}(-18\beta_{306} + 5\beta_{315}),\ \beta_{513} = \beta_{315} - \frac{8}{5}\beta_{324} + \frac{9}{5}\beta_{333},$
  $\beta_{531} = \frac{4}{5}\beta_{342} + \beta_{351},\ \beta_{522} = \frac{9}{5}\beta_{342},\ \beta_{540} = \frac{1}{10}(\beta_{342} + 5\beta_{351}),$
  $\beta_{504} = \frac{9}{2}\beta_{306} - \frac{5}{2}\beta_{315} + \beta_{324},\ \beta_{621} = \frac{3}{5}\beta_{342} + \frac{1}{2}\beta_{351},$
  $\beta_{630} = \frac{1}{15}(2\beta_{342} + 5\beta_{351}),\ \beta_{603} = \frac{5}{6}\beta_{315} - \beta_{306} - \frac{2}{3}\beta_{324} + \frac{1}{2}\beta_{333},$
  $\beta_{711} = \frac{1}{35}(8\beta_{342} + 5\beta_{351}),\ \beta_{702} = \frac{1}{7}\beta_{342},\ \beta_{720} = \frac{1}{35}(3\beta_{342}$
  $+ 5\beta_{351}),\ \beta_{810} = \frac{1}{35}\beta_{342} + \frac{1}{28}\beta_{351},\ \beta_{801} = \frac{1}{56}(2\beta_{342} + \beta_{351}),$
  $\beta_{612} = \frac{4}{5}\beta_{342},\ \beta_{900} = \frac{1}{252}(\beta_{342} + \beta_{351}).$

- $\lambda = -\frac{5}{2},\ \nu = -\frac{3}{2}$ the constants $\beta_{324}, \beta_{315}, \beta_{360}, \beta_{306}, \beta_{333}, \beta_{351}, \beta_{342}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414} = -\frac{5}{4}\beta_{315} + 2\beta_{324},\ \beta_{441} = \beta_{342} + \frac{5}{4}\beta_{351},\ \beta_{423} = \frac{9}{4}\beta_{333},$
  $\beta_{432} = \frac{3}{4}\beta_{333} + 2\beta_{342},\ \beta_{450} = \frac{3}{4}\beta_{351}, \beta_{405} = -3\beta_{306} + \frac{5}{4}\beta_{315},$
  $\beta_{513} = \frac{9}{5}\beta_{333},\ \beta_{531} = \frac{3}{10}\beta_{333} + \frac{8}{5}\beta_{342} + \beta_{351}, \beta_{522} = \frac{9}{20}(3\beta_{333}$
  $+ 4\beta_{342}),\ \beta_{540} = \frac{3}{10}\beta_{342} + \frac{3}{4}\beta_{351},\ \beta_{504} = \frac{1}{4}(6\beta_{306} - 5\beta_{315}$
  $+4\beta_{324}),\ \beta_{612} = \frac{1}{10}(9\beta_{333}+8\beta_{342}), \beta_{621} = \frac{9}{20}\beta_{333}+\frac{6}{5}\beta_{342}+\frac{1}{2}\beta_{351},$
  $\beta_{630} = \frac{1}{20}\beta_{333} + \frac{2}{5}\beta_{342} + \frac{1}{2}\beta_{351},$
  $\beta_{603} = \frac{1}{2}\beta_{333}, \beta_{711} = \frac{1}{35}(9\beta_{333} + 16\beta_{342} + 5\beta_{351}),$
  $\beta_{702} = \frac{1}{14}(3\beta_{333} + 2\beta_{342}),\ \beta_{720} = \frac{3}{140}(3\beta_{333} + 12\beta_{342} + 10\beta_{351}),$
  $\beta_{810} = \frac{9}{280}\beta_{333} + \frac{3}{35}\beta_{342} + \frac{3}{56}\beta_{351},\ \beta_{801} = \frac{1}{56}(3\beta_{333} + 4\beta_{342}$
  $+ \beta_{351}),\ \beta_{900} = \frac{1}{168}(\beta_{333} + 2\beta_{342} + \beta_{351}).$

- $\lambda = -\frac{5}{2},\ \nu = -2$ the constants $\beta_{315}, \beta_{360}, \beta_{306}, \beta_{324}, \beta_{351}, \beta_{333}, \beta_{342}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

  $\beta_{414} = 2\beta_{324},\ \beta_{441} = \frac{3}{2}\beta_{342} + \frac{5}{4}\beta_{351},\ \beta_{423} = \beta_{324} + \frac{9}{4}\beta_{333},$
  $\beta_{432} = \frac{3}{2}\beta_{333} + 2\beta_{342},\ \beta_{450} = \beta_{351},\ \beta_{405} = \frac{1}{4}(5\beta_{315} - 6\beta_{306}),$
  $\beta_{513} = \frac{1}{5}(8\beta_{324} + 9\beta_{333}),\ \beta_{531} = \frac{1}{10}(9\beta_{333} + 24\beta_{342} + 10\beta_{351}),$
  $\beta_{522} = \frac{1}{10}(6\beta_{324} + 27\beta_{333} + 18\beta_{342}),\ \beta_{540} = \frac{3}{5}\beta_{342} + \beta_{351},$
  $\beta_{504} = \beta_{324},\ \beta_{612} = \frac{1}{5}(4\beta_{324} + 9\beta_{333} + 4\beta_{342}),\ \beta_{621} = \frac{1}{5}\beta_{324}$
  $+ \frac{27}{20}\beta_{333} + \frac{9}{5}\beta_{342} + \frac{1}{2}\beta_{351},\ \beta_{630} = \frac{1}{5}\beta_{333} + \frac{4}{5}\beta_{342} + \frac{2}{3}\beta_{351},$

$\beta_{603}=\frac{2}{3}\beta_{324}+\frac{1}{2}\beta_{333},\ \beta_{711}=\frac{1}{35}(8\beta_{324}+27\beta_{333}+24\beta_{342}+5\beta_{351}),$
$\beta_{702}=\frac{1}{7}(2\beta_{324}+3\beta_{333}+\beta_{342}),\ \beta_{720}=\frac{1}{35}(\beta_{324}+9\beta_{333}$
$+18\beta_{342}+10\beta_{351}),\ \beta_{810}=\frac{1}{70}(2\beta_{324}+9\beta_{333}+12\beta_{342}+6\beta_{351}),$
$\beta_{801}=\frac{1}{56}(4\beta_{324}+9\beta_{333}+6\beta_{342}+\beta_{351}),$
$\beta_{900}=\frac{1}{126}(\beta_{324}+3\beta_{333}+3\beta_{342}+\beta_{351}).$

- $\lambda=-\frac{5}{2},\ \nu=-\frac{5}{2}$ the constants $\beta_{360},\beta_{306},\beta_{333},\beta_{351},\beta_{315},\beta_{342},\beta_{324}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_8^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is zero-dimensional. On the other hand,

$\beta_{414}=\frac{5}{4}\beta_{315}+2\beta_{324},\ \beta_{450}=\frac{5}{4}\beta_{351},\ \beta_{441}=2\beta_{342}+\frac{5}{4}\beta_{351},$
$\beta_{423}=2\beta_{324}+\frac{9}{4}\beta_{333},\ \beta_{432}=\frac{9}{4}\beta_{333}+2\beta_{342},\ \beta_{405}=\frac{5}{4}\beta_{315},$
$\beta_{513}=\frac{1}{5}(5\beta_{315}+16\beta_{324}+9\beta_{333}),\ \beta_{531}=\frac{1}{5}(9\beta_{333}+16\beta_{342}+5\beta_{351}),$
$\beta_{522}=\frac{9}{20}(4\beta_{324}+9\beta_{333}+4\beta_{342}),\ \beta_{540}=\beta_{342}+\frac{5}{4}\beta_{351},$
$\beta_{504}=\frac{5}{4}\beta_{315}+\beta_{324},\ \beta_{612}=\frac{1}{2}\beta_{315}+\frac{12}{5}\beta_{324}+\frac{27}{10}\beta_{333}+\frac{4}{5}\beta_{342},$
$\beta_{621}=\frac{1}{10}(8\beta_{324}+27\beta_{333}+24\beta_{342}+5\beta_{351}),\ \beta_{630}=\frac{1}{2}\beta_{333}$
$+\frac{4}{3}\beta_{342}+\frac{5}{6}\beta_{351},\ \beta_{603}=\frac{5}{6}\beta_{315}+\frac{4}{3}\beta_{324}+\frac{1}{2}\beta_{333},\ \beta_{711}=\frac{1}{35}(5\beta_{315}$
$+32\beta_{324}+54\beta_{333}+32\beta_{342}+5\beta_{351}),\beta_{702}=\frac{1}{14}(5\beta_{315}+12\beta_{324}$
$+9\beta_{333}+\beta_{342}),\ \beta_{720}=\frac{1}{14}(2\beta_{324}+9\beta_{333}+12\beta_{342}+5\beta_{351}),$
$\beta_{810}=\frac{1}{56}(\beta_{315}+8\beta_{324}+18\beta_{333}+16\beta_{342}+5\beta_{351}),$
$\beta_{801}=\frac{1}{56}(5\beta_{315}+16\beta_{324}+18\beta_{333}+8\beta_{342}+\beta_{351}),$
$\beta_{900}=\frac{5}{504}(\beta_{315}+2\beta_{324}+6\beta_{333}+2\beta_{342}+\beta_{351}).$

(5) The case when k =9
The conditions of 2-cocycle shows that only one 2-cocycle spans the cohomology group, where $\alpha_{ijkl}=-\alpha_{jikl}$, given by

$$\alpha_{4331}=-\frac{1}{\nu}(2(3+2\lambda)\alpha_{4340}+5\alpha_{5330}),$$
$$\alpha_{3413}=\frac{1}{\lambda}((6+4\nu)\alpha_{4304}+5\alpha_{5303}),$$
$$\alpha_{5321}=\frac{1}{5\nu}(\nu(1+2\nu)\alpha_{3422}+3(1+\lambda)\left((6+4\lambda)\alpha_{4340}+5\alpha_{5330}\right)),$$
$$\alpha_{5312}=\frac{1}{5\lambda}(\lambda(1+2\lambda)\alpha_{3422}+3(1+\nu)\left((6+4\nu)\alpha_{4304}+5\alpha_{5303}\right)),$$
$$\alpha_{6410}=-\alpha_{6410}=\frac{1}{9}\left(\nu\alpha_{4511}+(1+2\lambda)\alpha_{4520}\right),$$
$$\alpha_{4601}=-\frac{1}{9}(1+2\nu)\alpha_{4502}-\frac{1}{9}\lambda\alpha_{4511},$$
$$\alpha_{3620}=\frac{1}{45}\left(\nu(1+2\nu)\alpha_{3422}+6(3+5\lambda+2\lambda^2\right)\alpha_{4340}-10\alpha_{4520}+30(1+\lambda)\alpha_{5330}),$$
$$\alpha_{3602}=\frac{1}{45}(\lambda(1+2\lambda)\alpha_{3422}+6\left(3+5\nu+2\nu^2\right)\alpha_{4304}-10\alpha_{4502}+30(1+\nu)\alpha_{5303}),$$

$$\begin{aligned}
\alpha_{7310} = & \frac{1}{630\lambda}((3\lambda(1+2\lambda)\nu(1+2\nu))\alpha_{3422} + (6\nu(3+2\nu)\left(1+3\nu+2\nu^2\right))\alpha_{4304} \\
& + (12\lambda\left(3+11\lambda+12\lambda^2+4\lambda^3\right))\alpha_{4340} - (20\lambda\nu\alpha_{4511} - 20\lambda(1+2\lambda))\alpha_{4520} \\
& + (15\nu\left(1+3\nu+2\nu^2\right))\alpha_{5303} + (45\lambda\left(1+3\lambda+2\lambda^2\right))\alpha_{5330}), \\
\alpha_{7301} = & \frac{1}{630\nu}((3\lambda(1+2\lambda)\nu(1+2\nu))\alpha_{3422} + (12\nu\left(3+11\nu+12\nu^2+4\nu^3\right))\alpha_{4304} \\
& + (6\lambda(3+2\lambda)\left(1+3\lambda+2\lambda^2\right))\alpha_{4340} - 20\nu(1+2\nu)\alpha_{4502} - 20\lambda\nu\alpha_{4511} \\
& + 45\nu\left(1+3\nu+2\nu^2\right)\alpha_{5303} + 15\lambda\left(1+3\lambda+2\lambda^2\right)\alpha_{5330}), \\
\alpha_{6500} = & \frac{1}{4725}(3\lambda(1+2\lambda)\nu(1+2\nu)\alpha_{3422} + 9\nu\left(53+11\nu+12\nu^2+4\nu^3\right)\alpha_{4304} \\
& - 9\lambda\left(53+11\lambda+12\lambda^2+4\lambda^3\right)\alpha_{4340} + 5\nu(103+26\nu)\alpha_{4502} + 130\lambda\nu\alpha_{4511} \\
& + 5\lambda(103+26\lambda)\alpha_{4520} - 30\nu\left(-14+3\nu+2\nu^2\right)\alpha_{5303} - 30\lambda\left(-14+3\lambda+2\lambda^2\right)\alpha_{5330}), \\
\alpha_{3800} = & \frac{1}{13230}(6\lambda(1+2\lambda)\nu(1+2\nu)\alpha_{3422} + 9\nu\left(1+22\nu+24\nu^2+8\nu^3\right)\alpha_{4304} \\
& + 9\lambda\left(1+22\lambda+24\lambda^2+8\lambda^3\right)\alpha_{4340} - 10\nu(-2+5\nu)\alpha_{4502} - 50\lambda\nu\alpha_{4511} \\
& - 10\lambda(-2+5\lambda)\alpha_{4520} + 15\nu\left(7+12\nu+8\nu^2\right)\alpha_{5303} + 15\lambda\left(7+12\lambda+8\lambda^2\right)\alpha_{5330}), \\
\alpha_{6311} = & \frac{1}{45}(-2(1+2\lambda)(1+2\nu)\alpha_{3422} - \frac{1}{\lambda}(6(3+2\nu)\left(1+3\nu+2\nu^2\right))\alpha_{4304} \\
& - \frac{1}{\nu}(6(3+2\lambda)\left(1+3\lambda+2\lambda^2\right))\alpha_{4340} + 10\alpha_{4511} - \frac{1}{\lambda}(15\left(1+3\nu+2\nu^2\right))\alpha_{5303} \\
& - \frac{1}{\nu}(15\left(1+3\lambda+2\lambda^2\right))\alpha_{5330}).
\end{aligned}$$

Let us study the triviality of this 2-cocycle. A direct computation proves that

- $\nu=0, \lambda=0$ the constants $\beta_{433}, \beta_{541}, \beta_{424}, \beta_{352}, \beta_{442}, \beta_{343}, \beta_{514}$, β_{505}, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is zero-dimensional.

- $\nu=0, \lambda=-\frac{1}{2}$ the constants $\beta_{343}, \beta_{505}, \beta_{541}, \beta_{442}, \beta_{424}, \beta_{433}, \beta_{352}$, β_{514} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is one-dimensional

- $\nu=0, \lambda=-1$ the constants $\beta_{343}, \beta_{433}, \beta_{505}, \beta_{541}, \beta_{442}, \beta_{424}, \beta_{352}$, β_{514} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is one-dimensional

- $\nu=0,\lambda=-\frac{3}{2}$ the constants $\beta_{343},\beta_{541},\beta_{505},\beta_{442},\beta_{433},\beta_{424},\beta_{514},$ β_{352} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is one-dimensional

- $\nu=0,\lambda=-2$ the constants $\beta_{505},\beta_{442},\beta_{433},\beta_{343},\beta_{424},\beta_{514},\beta_{352},$ β_{451} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional

- $\nu=0,\lambda=-\frac{5}{2}$ the constants $\beta_{361},\beta_{505},\beta_{433},\beta_{442},\beta_{343},\beta_{424},\beta_{541},$ β_{514} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is one-dimensional

- $\nu=0,\lambda=-3$ the constants $\beta_{541},\beta_{352},\beta_{505},\beta_{442},\beta_{433},\beta_{343},\beta_{424},$ β_{514} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2},\lambda=0$ the constants $\beta_{352},\beta_{550},\beta_{514},\beta_{424},\beta_{442},\beta_{433},\beta_{343},$ β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2},\lambda=-\frac{1}{2}$ the constants $\beta_{343},\beta_{550},\beta_{352},\beta_{433},\beta_{442},\beta_{424},$ β_{514},β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2},\lambda=-1$ the constants $\beta_{343},\beta_{442},\beta_{433},\beta_{352},\beta_{550},\beta_{541},\beta_{514}$ $,\beta_{424}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle . Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2},\lambda=-\frac{3}{2}$ the constants $\beta_{352},\beta_{433},\beta_{550},\beta_{514},\beta_{343},\beta_{442},\beta_{424},$ β_{451} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2},\lambda=-2$ the constants $\beta_{352},\beta_{433},\beta_{550},\beta_{514},\beta_{343},\beta_{442},\beta_{424},$ β_{451} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2}, \lambda=-\frac{5}{2}$ the constants $\beta_{352}, \beta_{361}, \beta_{424}, \beta_{343}, \beta_{514}, \beta_{442}, \beta_{433}$, β_{505} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{1}{2}, \lambda=-3$ the constants $\beta_{352}, \beta_{442}, \beta_{343}, \beta_{1000}, \beta_{514}, \beta_{433}$, β_{505} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional.

- $\nu=-1, \lambda=0$ the constants $\beta_{550}, \beta_{514}, \beta_{424}, \beta_{442}, \beta_{352}, \beta_{433}, \beta_{541}$, β_{343} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-1, \lambda=-\frac{1}{2}$ the constants $\beta_{343}, \beta_{433}, \beta_{424}, \beta_{514}, \beta_{541}, \beta_{352}, \beta_{442}$, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-1, \lambda=-1$ the constants $\beta_{433}, \beta_{343}, \beta_{550}, \beta_{514}, \beta_{541}, \beta_{352}, \beta_{442}$, β_{424} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-1, \lambda=-\frac{3}{2}$ the constants $\beta_{343}, \beta_{541}, \beta_{550}, \beta_{514}, \beta_{352}, \beta_{433}, \beta_{424}$, β_{442} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-1, \lambda=-2$ the constants $\beta_{550}, \beta_{352}, \beta_{514}, \beta_{343}, \beta_{433}, \beta_{424}$, $\beta_{1000}, \beta_{451}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-1, \lambda=-\frac{5}{2}$ the constants $\beta_{361}, \beta_{343}, \beta_{424}, \beta_{442}, \beta_{514}, \beta_{433}$, $\beta_{1000}, \beta_{505}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-1, \lambda=-3$ the constants $\beta_{343}, \beta_{1000}, \beta_{424}, \beta_{433}, \beta_{514}, \beta_{505}, \beta_{442}$, β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=0$ the constants $\beta_{514},\beta_{550},\beta_{424},\beta_{334},\beta_{433},\beta_{352},\beta_{442}$, β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=-\frac{1}{2}$ the constants $\beta_{334},\beta_{424},\beta_{514},\beta_{352},\beta_{541},\beta_{433},\beta_{442}$, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=-1$ the constants $\beta_{334},\beta_{514},\beta_{541},\beta_{352},\beta_{433},\beta_{442},\beta_{424}$, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=-\frac{3}{2}$ the constants $\beta_{343},\beta_{550},\beta_{514},\beta_{541},\beta_{352},\beta_{334},\beta_{442}$, β_{424} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=-2$ the constants $\beta_{433},\beta_{550},\beta_{352},\beta_{514},\beta_{334},\beta_{424},\beta_{451}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=-\frac{5}{2}$ the constants $\beta_{424},\beta_{361},\beta_{334},\beta_{442},\beta_{433},\beta_{514},\beta_{505}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{3}{2},\lambda=-3$ the constants $\beta_{334},\beta_{1000},\beta_{424},\beta_{505},\beta_{514},\beta_{433},\beta_{442}$, β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2,\lambda=0$ the constants $\beta_{550},\beta_{352},\beta_{433},\beta_{442},\beta_{343},\beta_{541},\beta_{325}$, β_{415} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2,\lambda=-\frac{1}{2}$ the constants $\beta_{325},\beta_{352},\beta_{541},\beta_{442},\beta_{343},\beta_{433},\beta_{550}$, β_{415} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2, \lambda=-1$ the constants $\beta_{343}, \beta_{325}, \beta_{352}, \beta_{541}, \beta_{442}, \beta_{433}$, β_{550}, β_{415} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2, \lambda=-\frac{3}{2}$ the constants $\beta_{343}, \beta_{325}, \beta_{352}, \beta_{541}, \beta_{442}, \beta_{550}, \beta_{415}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2, \lambda=-2$ the constants $\beta_{343}, \beta_{550}, \beta_{325}, \beta_{352}, \beta_{433}, \beta_{415}, \beta_{451}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2, \lambda=-\frac{5}{2}$ the constants $\beta_{415}, \beta_{361}, \beta_{325}, \beta_{433}, \beta_{442}, \beta_{343}, \beta_{505}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-2, \lambda=-3$ the constants $\beta_{325}, \beta_{1000}, \beta_{343}, \beta_{415}, \beta_{505}, \beta_{442}, \beta_{541}$, β_{433} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2}, \lambda=0$ the constants $\beta_{316}, \beta_{550}, \beta_{433}, \beta_{424}, \beta_{343}, \beta_{352}, \beta_{442}$, β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2}, \lambda=-\frac{1}{2}$ the constants $\beta_{343}, \beta_{316}, \beta_{442}, \beta_{352}, \beta_{541}, \beta_{424}$, β_{433}, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2}, \lambda=-1$ engendrer par $\beta_{343}, \beta_{316}, \beta_{442}, \beta_{352}, \beta_{541}, \beta_{433}, \beta_{550}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2}, \lambda=-\frac{3}{2}$ the constants $\beta_{343}, \beta_{433}, \beta_{316}, \beta_{352}, \beta_{442}, \beta_{541}, \beta_{550}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2},\lambda=-2$ the constants $\beta_{451},\beta_{316},\beta_{352},\beta_{433},\beta_{442},\beta_{343},\beta_{550}$, β_{1000} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2},\lambda=-\frac{5}{2}$ the constants $\beta_{316},\beta_{1000},\beta_{460},\beta_{442},\beta_{433},\beta_{343},\beta_{424}$, β_{361} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-\frac{5}{2},\lambda=-3$ the constants $\beta_{316},\beta_{1000},\beta_{406},\beta_{541},\beta_{442},\beta_{433},\beta_{424}$, β_{343}, can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-3,\lambda=0$ the constants $\beta_{514},\beta_{424},\beta_{343},\beta_{550},\beta_{433},\beta_{352},\beta_{442}$, β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-3,\lambda=-\frac{1}{2}$ the constants $\beta_{343},\beta_{352},\beta_{1000},\beta_{442},\beta_{541},\beta_{433}$ $,\beta_{550}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is tow-dimensional.

- $\nu=-3,\lambda=-2$ the constants $\beta_{343},\beta_{352},\beta_{1000},\beta_{442},\beta_{433},\beta_{541},\beta_{424}$, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-3,\lambda=-\frac{3}{2}$ the constants $\beta_{343},\beta_{352},\beta_{1000},\beta_{442},\beta_{433},\beta_{550},\beta_{424}$, β_{541} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-3,\lambda=-2$ the constants $\beta_{352},\beta_{1000},\beta_{451},\beta_{442},\beta_{343},\beta_{433},\beta_{424}$, β_{550} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu=-3,\lambda=-\frac{5}{2}$ the constants $\beta_{361},\beta_{1000},\beta_{460},\beta_{442},\beta_{541},\beta_{433}$, β_{343},β_{424} can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

- $\nu = -3, \lambda = -3$ the constants $\beta_{343}, \beta_{370}, \beta_{1000}, \beta_{541}, \beta_{514}, \beta_{424}, \beta_{442}, \beta_{433}$ can be chosen in such a way that once adding the trivial 2-cocycle $\delta b_9^{\lambda,\nu}$ to our 2-cocycle. Hence, the cohomology group is one-dimensional.

(6) The case when k =10
The proof here is the same as in the previous section. We just point out that the space of solutions of the 2-cocycle is tow-dimensional for $(\lambda,\nu) = (0,0), (0,-2), (-\frac{3}{2},-3), (\frac{3}{2},-\frac{7}{2}), (-1,-\frac{7}{2}), (-2,-\frac{5}{2}), (-2,-3)$ $, (-2,-\frac{7}{2}), (-\frac{5}{2},-2), (-\frac{5}{2},-3), (-\frac{5}{2},-\frac{7}{2}), (-3,-\frac{3}{2}), (-3,-3),$ $(-3,-\frac{7}{2}), (-\frac{7}{2},-1), (-\frac{7}{2},-\frac{3}{2}), (-\frac{7}{2},-2), (-\frac{7}{2},-\frac{5}{2}), (-\frac{7}{2},-3), (-\frac{7}{2},-\frac{7}{2});$ one-dimensional for $(\lambda,\nu) = (0,-\frac{1}{2}), (0,-1), (0,-\frac{3}{2}), (0,-\frac{5}{2}), (0,-3),$ $(0,-\frac{1}{2}), (0,-1), (0,-\frac{3}{2}), (0,-2), (0,-\frac{3}{2}), (0,-2), (0,-\frac{5}{2}), (-\frac{1}{2},0), (-\frac{3}{2},0),$ $(-\frac{3}{2},-\frac{5}{2}), (-1,0), (-1,-3), (-2,0), (-2,-2), (-\frac{5}{2},0), (-\frac{5}{2},-\frac{3}{2}), (-3,0),$ $(-3,-1), (-3,-\frac{7}{2}), (-3,-2), (-\frac{7}{2},-\frac{1}{2})$ and zero-dimensional for otherwise.

(7) The case when k =11
The proof here is the same as in the previous section.

(8) The case when k =12
The proof here is the same as in the previous section. We will investigate the dimension of the space of operators that satisfy the 2-cocycle condition. We discuss the following cases: the space of solutions of the system is 12-dimensional for $(\lambda,\nu) = (\frac{1}{108}(-389\pm\sqrt{32737}), \frac{1}{108}(-389\pm\sqrt{32737})), (\frac{1}{108}(-389\pm\sqrt{32737}), 0),(\frac{1}{108}(-389\pm\sqrt{32737}), -1),(-1.0),(0,-1),$ $(-1,-1)$ and 14-dimensional for $(\lambda,\nu) = (0,0)$. Second, taking into account these conditions, we will study all trivial 2-cocycles.

(9) The case when $k \geq 13$
For $k \geq 13$, We get $\Upsilon(X,Y)(\phi,\psi) = 0$ car the number of variables generating any 2-cocycle is much smaller than the number of equations coming out from the 2-cocycle condition (for $k = 13$ we have 90 variables and 113 equations). For generic λ and ν, the number of equations will generates a one-dimensional space, which give a unique cohomology class. This cohomology class is indeed trivial because the expression $\delta b(X,Y)(\phi,\psi)$ is also a 2-cocycle.

$$\mathrm{H}^2(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\nu,\mu}) = 0.$$

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