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Join coatom element graph of a lattice

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Abstract. Let $\mathcal{L}$ be a bounded distributive lattice. In this paper, we introduce and investigate the join coatom element graph of $\mathcal{L}$, denoted by $\mathbb{CG}(\mathcal{L})$. It is the (undirected) graph with all nontrivial elements of $\mathcal{L}$ as vertices, and for distinct nontrivial elements $a, b \in \mathcal{L}$, the vertices a and b are adjacent if and only if $a \vee b$ is a coatom element of $\mathcal{L}$. The basic properties and possible structures of the graph $\mathbb{CG}(\mathcal{L})$ are investigated. The connectedness, clique number, domination number and independence number of $\mathbb{CG}(\mathcal{L})$ and their relations to algebraic properties of $\mathcal{L}$ are explored.

Keywords: Lattice; coatom, join coatom element graph .

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1. Introduction

One of the meeting points of graph theory and classical algebra is the study of graphs defined in terms of elements and subobjects of the algebraic structure. There are many studies on various graphs associated to modules, rings, lattices and other algebraic structures (see for example [1-8, 11, 13-19, 21]). The aim of the paper is to investigate the interplay

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between lattice properties of a lattice $\mathcal{L}$ and properties of its join coatom element graphs. This will result in classification of lattices in terms of some specific properties of those graphs. All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded.

Beck [4], Anderson and Naseer [2], and Anderson and Livingston [1] et. al. have studied graphs on commutative rings. One of the most important graphs which have been studied is the intersection graph. Bosak [5] defined the intersection graph of semigroups. *Csàkàny* and *Pollàk* [8] studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [7]. The intersection minimal ideal graph of a ring, i.e. a simple graph whose vertices are nontrivial ideals of a ring R and two vertices I, J are adjacent if the intersection of corresponding ideals is a minimal ideal, was investigated by Barman and Rajkhowa in [6]. The intersection graph of ideals of rings, submodules of modules and lattices has been investigated by several authors (see for example [3, 17-19, 21]).

Let $\mathcal{L}$ be a bounded distributive lattice. The purpose of this paper is to investigate a graph associated to a lattice $\mathcal{L}$ called the join coatom element graph of $\mathcal{L}$. The join coatom element graph of $\mathcal{L}$ is a simple graph $\mathbb{CG}(\mathcal{L})$ whose vertices are all nontrivial elements and two distinct vertices are adjacent if and only if the join of the corresponding elements is a coatom element of $\mathcal{L}$. Here is a brief outline of the article. Among many results in this paper, the first, preliminaries section contains elementary observations needed later on. In Section 3, Section 4 and Section 5, We characterize the lattices for which the join coatom element graphs are connected, complete bipartite, star. The concepts of planarity, clique number, domination number and split character are also investigated.

2. Preliminaries

Let G be a simple graph with vertex set $\mathcal{V}$ (G) and edge set $\mathcal{E}$ (G). The degree of a vertex v of the graph G, denoted by $\deg_G(v)$, is the number of edges incident to v. The (open) neighborhood $N(v)$ of a vertex v of $\mathcal{V}(G)$ is the set of vertices which are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices. If the vertices of G can be partitioned into two disjoint sets V_1 and V_2 with every vertex of V_1 is adjacent to any vertex of V_2 and no two vertices belonging to same set are adjacent, then G is called a complete bipartite graph. If

$|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. If one of the partite sets contains exactly one element, then the graph becomes a star graph. If graph G does not have K_5 or $K_{3,3}$ as its subgraph, then G is planar. Let u and v be elements of $\mathcal{V}$ (G). We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is $\mathrm{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\mathrm{gr}(G) = \infty$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. A subset S of $\mathcal{V}$ (G) is said to be an independent set if no two vertices of S are adjacent. If $\mathcal{V}$ (G) can be partitioned in an independent set and a clique then G is said to be split. A set $D \subseteq \mathcal{V}$ (G) is said to be a dominating set if every vertex not in D is adjacent to at least one of the members of D. The cardinality of smallest dominating set is the domination number of the graph G and is denoted by $\gamma(G)$. Note that a graph whose vertices set is empty is a null graph and a graph whose edge set is empty is an empty graph. For a connected graph G, x is a cut vertex of G if $G \setminus \{x\}$ is not connected [20].

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in \mathcal{L}$). We say that an element x in a lattice $\mathcal{L}$ is an atom (resp. coatom) if there is no $y \in \mathcal{L}$ such that $0 < y < x$ (resp. $x < y < 1$). The set of all coatom (resp. atom) elements of $\mathcal{L}$ is denoted by $\mathcal{CA}(\mathcal{L})$ (resp. $\mathcal{A}(\mathcal{L})$). If $\mathcal{L}$ is a complete lattice, then the Jacobson radical of $\mathcal{L}$ is the meet of all coatom elements of $\mathcal{L}$, and is denoted as $\mathrm{Rad}(\mathcal{L})$ (i.e. $\mathrm{Rad}(\mathcal{L}) = \bigwedge_{c \in \mathcal{CA}(\mathcal{L})} c$). In a lattice $\mathcal{L}$ with 1 an element $a \in \mathcal{L}$ is called small, denoted by $a \ll \mathcal{L}$, if $a \vee b \neq 1$ holds for every $b \neq 1$. An element x of a lattice $\mathcal{L}$ is nontrivial (resp. proper) if $x \neq 0, 1$ (resp. $x \neq 1$). A nonzero element x of a lattice $\mathcal{L}$ is called semisimple, if for each element y of $\mathcal{L}$ with $y < x$, there exists an element z of $\mathcal{L}$ such that $x = y \vee z$ and $y \wedge z = 0$. In this case, we say that y is a direct join of x, and we write $x = y \oplus z$ [15]. A lattice $\mathcal{L}$ is Artinian (satisfies DCC) if there is no infinite strictly descending chain $a_0 > a_1 > \cdots$ in $\mathcal{L}$. A lattice

$\mathcal{L}$ is called local if it has exactly one coatom element c that $x \leq c$ for each proper element x. The undefined terms related to lattice theory are taken from [9, 10] and terms related to graph theory are taken from [20].

3. Connectedness of $\mathbb{CG}(\mathcal{L})$

Throughout this paper we shall assume, unless otherwise stated, that $\mathcal{L}$ is a bounded distributive lattice. In this section, we collect some basic properties concerning the join coatom element graph $\mathbb{CG}(\mathcal{L})$ of $\mathcal{L}$. We remind the reader with the following definition.

Definition 3.1. The join coatom element graph $\mathbb{CG}(\mathcal{L})$ of $\mathcal{L}$ is simple undirected graph whose vertices are all nontrivial elements of $\mathcal{L}$ and any two distinct vertices a and b are adjacent if and only if $a \vee b$ is a coatom element of $\mathcal{L}$.

Lemma 3.2. *Every non-coatom vertex of the graph $\mathbb{CG}(\mathcal{L})$ is adjacent to at least one of the coatom elements of $\mathcal{L}$.*

Proof. Let x be a non-atom vertex of the graph $\mathbb{CG}(\mathcal{L})$. Then there exists a coatom element c of $\mathcal{L}$ such that $x \lneqq c$ by [14, Lemma 2.1]. Hence, $x \vee c = c$ is a coatom element, as needed. $\square$

Proposition 3.3. *If $\mathcal{L}$ is an Artinian lattice and $\mathbb{CG}(\mathcal{L})$ has a non-coatom universal vertex a, then a is an atom element.*

Proof. On the contrary, assume that a is not an atom element. By assumption, there exists an atom element a' of $\mathcal{L}$ such that $a' \lneqq a$. Then a is an universal vertex gives $a = a \vee a'$ is a coatom element which is impossible. Thus, a is an atom element. $\square$

Proposition 3.4. *If $\mathcal{L}$ is complete and $\mathrm{Rad}(\mathcal{L}) \neq 0$, then every element of $\mathcal{CA}(\mathcal{L})$ is adjacent to $\mathrm{Rad}(\mathcal{L})$.*

Proof. Since $\mathrm{Rad}(\mathcal{L}) \neq 1$, we conclde that $\mathrm{Rad}(\mathcal{L})$ is a vertex of the graph $\mathbb{CG}(\mathcal{L})$. Let $c \in \mathcal{CA}(\mathcal{L})$. Then $\mathrm{Rad}(\mathcal{L}) \leq c$ gives $c \vee \mathrm{Rad}(\mathcal{L}) = c \in \mathcal{CA}(\mathcal{L})$ and so c adjacent to $\mathrm{Rad}(\mathcal{L})$. $\square$

Proposition 3.5. *The subgraph induced by the coatom elements of $\mathcal{L}$ is empty.*

Proof. Let $c, c' \in \mathcal{CA}(\mathcal{L})$ with $c \neq c'$. Then $c, c' \leq c \vee c'$ gives $c \vee c' = 1$ which implies that c is not adjacent to c', as required. $\square$

Corollary 3.6. *If $\mathbb{CG}(\mathcal{L})$ has a coatom universal vertex c, then $\mathcal{L}$ is a local lattice with unique coatom element c.*

Proof. Apply Proposition 3.5. $\square$

Proposition 3.7. *If $\mathcal{L}$ is a local lattice with unique coatom element c, then c is a universal vertex.*

Proof. If x is a nontrivial element of $\mathcal{L}$, then $x \leq c$ gives $x \vee c = c$ and so c is a universal vertex. $\square$

Theorem 3.8. *If $\mathbb{CG}(\mathcal{L})$ is a complete graph, then $\mathcal{L}$ is a local lattice.*

Proof. If $c, c' \in \mathcal{CA}(\mathcal{L})$ with $c \neq c'$, then c and c' are not adjacent in $\mathbb{CG}(\mathcal{L})$ by Proposition 3.5 which is impossible, as $\mathbb{CG}(\mathcal{L})$ is complete. Hence, $\mathcal{L}$ is a local lattice. $\square$

The following example shows that, in general, the converse of Theorem 3.8 is not true.

Example 3.9. Assume that $\mathcal{L} = \{0, a, b, c, d, e, 1\}$ is a lattice with the relations $0 \leq e \leq a \leq b \leq c \leq 1$, $0 \leq e \leq a \leq d \leq c \leq 1$, $d \vee b = c$ and $d \wedge b = a$. Clearly, $\mathcal{CA}(\mathcal{L}) = \{c\}$. Then $b \vee a = b$ is not a coatom element gives $\mathbb{CG}(\mathcal{L})$ is not a complete graph.

In the following theorem, we give a condition under which the graph $\mathbb{CG}(\mathcal{L})$ is complete.

Theorem 3.10. *Suppose that $\mathcal{L}$ is an Artinian lattice and let c be a vertex of $\mathbb{CG}(\mathcal{L})$ with degree 1. If c is a coatom in $\mathcal{L}$ which is not an atom element, then $\mathbb{CG}(\mathcal{L}) \cong K_2$.*

Proof. By the hypothesis, $a \lneqq c$ for some atom element a of $\mathcal{L}$; so $a \backsim c$. If a' is an element of $\mathcal{L}$ such that $a \lneqq a' \lneqq c$, then $a' \backsim c$ gives $\deg_{\mathbb{CG}(\mathcal{L})}(c) \geq 2$ which is a contradiction. Thus, $\mathcal{V}\ (\mathbb{CG}(\mathcal{L})) = \{a, c\}$, as required. $\square$

The following theorems, we give a condition under which $\mathbb{CG}(\mathcal{L})$ is a empty graph.

Theorem 3.11. *Every nontrivial element of a lattice of $\mathcal{L}$ is a coatom if and only if $\mathbb{CG}(\mathcal{L})$ is an empty graph.*

Proof. If every nontrivial element of $\mathcal{L}$ is coatom, then $\mathbb{CG}(\mathcal{L})$ is an empty graph by Proposition 3.5. Conversely, assume that $\mathbb{CG}(\mathcal{L}$ is empty and let x be any vertex of the graph of $\mathbb{CG}(\mathcal{L}$ such that $x \notin \mathcal{CA}(\mathcal{L})$. Then by Lemma 3.2, there exists a coatom element y of $\mathcal{L}$ such that y adjacent to x which is impossible, as needed. $\square$

Theorem 3.12. *If $c_1 \oplus c_2 = 1$ for some coatom elements c_1 and c_2 of $\mathcal{L}$, then $\mathbb{CG}(\mathcal{L})$ is an empty graph.*

Proof. Let a be a nontrivial element of $\mathcal{L}$. Then $a = a \vee 0 = (a \vee c_1) \wedge (a \vee c_2)$. If $a \vee c_1 = 1 = a \vee c_2$, then $a = 1$ which is a contradiction. If

$a \vee c_1 \neq 1$ (so $a \leq c_1$, as c_1 is a coatom) and $a \vee c_2 \neq 1$ (so $a \leq c_2$), then $a = c_1 \wedge c_2 = 0$ which is impossible. Without loss of generality, let $a \vee c_1 \neq 1$ (so $a \leq c_1$) and $a \vee c_2 = 1$ which implies that $a = c_1$. Therefore, every nontrivial element of $\mathcal{L}$ is a coatom element. Therefore, $\mathbb{CG}(\mathcal{L})$ is an empty graph by Theorem 3.5. □

Theorem 3.13. *If $\mathcal{L}$ is complete and* $\mathrm{Rad}(\mathcal{L}) \neq 0$*, then the subgraph induced by the non-coatom elements of $\mathcal{L}$ is connected graph of diameter not bigger than* 4*.*

Proof. Suppose that x and y are distinct non-coatom vertices of the graph $\mathbb{CG}(\mathcal{L})$. If x adjacent to y, then $x \backsim y$ is a path. So we may assume that $x \vee y$ is not a coatom. By assumption, there exist $a, b \in \mathcal{CA}(\mathcal{L})$ such that $x \lneqq a$ and $y \lneqq b$. If $a = b$, then $x \backsim a \backsim y$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. If $a \neq b$, then $a \wedge b \neq 1$ and $0 \neq \mathrm{Rad}(\mathcal{L}) \leq a \wedge b$ gives $a \wedge b$ is a vertex of the graph $\mathbb{CG}(\mathcal{L})$ and so $x \backsim a \backsim a \wedge b \backsim b \backsim y$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 4$, i.e. the result holds. □

The next theorem gives a more explicit description of the diameter of $\mathbb{CG}(\mathcal{L})$.

Theorem 3.14. *The graph $\mathbb{CG}(\mathcal{L})$ is connected with* $\mathrm{diam}(\mathbb{CG}(\mathcal{L})) \leq 4$ *if and only if the meet of any two distinct coatom elements of $\mathcal{L}$ is not* 0 *or $\mathcal{L}$ is a local lattice.*

Proof. Suppose that $\mathcal{L}$ is a local lattice with unique coatom element c and let x and y be distinct non-coatom vertices of the graph $\mathbb{CG}(\mathcal{L})$. Then $x \backsim c \backsim y$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. So suppose that $|\mathcal{CA}(\mathcal{L})| \neq 1$ and the meet of any two distinct coatom elements of $\mathcal{L}$ is not 0. Consider two distinct vertices b and c of $\mathbb{CG}(\mathcal{L})$. If b adjacent to c, then $b \backsim c$ is a path. So we may assume that $b \vee c$ is not a coatom element of $\mathcal{L}$. Then either $b \vee c = 1$ or $b \vee c \lneqq a$ for some $a \in \mathcal{CA}(\mathcal{L})$. If $b \vee c \lneqq a$ (so $b \lneqq a$ and $c \lneqq a$), then $b \backsim a \backsim c$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. If $b \vee c = 1$, we split the proof into three cases:

Case 1. $b, c \in \mathcal{A}(\mathcal{L})$. Then $b \backsim b \wedge c \backsim c$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 2$.

Case 2. If exactly one of b and c is coatom, then without loss of generality, assume that $b \in \mathcal{CA}(\mathcal{L})$ and $c \notin \mathcal{CA}(\mathcal{L})$. By [14, Lemma 2.1], there is a coatom element a of $\mathcal{L}$ such that $c \lneqq a$ which implies that $b \backsim a \wedge b \backsim a \backsim c$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 4$.

Case 3. $b, c \notin \mathcal{CA}(\mathcal{L})$. Then there exist $a, a' \in \mathcal{CA}(\mathcal{L})$ such that $b \lneqq a$ and $c \lneqq a'$ by [14, Lemma 2.1]. If $a = a'$, then $b \backsim a \backsim c$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 2$. If $a \neq a'$, then $b \backsim a \backsim a \wedge a' \backsim a' \backsim c$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(b, c) = 4$. Hence, we infer that $\mathbb{CG}(\mathcal{L})$ is connected with $\mathrm{diam}(\mathbb{CG}(\mathcal{L})) \leq 4$.

Conversely, assume that $\mathbb{CG}(\mathcal{L})$ is connected. If $\mathcal{L}$ is a local lattice, then we are done. So we may assume that $|\mathcal{CA}(\mathcal{L})| \neq 1$. On the contrary, assume that there are two coatom elements c_1 and c_2 of $\mathcal{L}$ such that $c_1 \wedge c_2 = 0$. We claim that c_2 is an atom element of $\mathcal{L}$. Assume to the contrary, $0 \lneqq x \lneqq c_2$ for some $x \in \mathcal{L}$. Then $c_1 \wedge x \leq c_2 \wedge c_1 = 0$ gives $x \wedge c_1 = 0$. If $x \vee c_1 \neq 1$, then $x \leq c_1$ implies that $0 = x \wedge c_1 = x$ which is impossible. Thus, $x \vee c_1 = 1$. Now, we have $c_2 = c_2 \wedge 1 = c_2 \wedge (x \vee c_1) = (c_2 \wedge x) \vee (c_2 \wedge c_1) = x$, a contradiction. So c_2 is an atom element. Similarly, c_1 is an atom element. By the hypothesis, $c_1 \backsim v$ for some vertex v of $\mathbb{CG}(\mathcal{L})$. Then $c_1 \vee v = c_1$ since $c_1 \vee v$ is a coatom (so $v \lneqq c_1$), a contradiction Since c_1 is an atom element, as required. $\square$

Theorem 3.15. *Assume that $\mathcal{L}$ is a complete lattice with* $\mathrm{Rad}(\mathcal{L}) \neq 0$ *and let* $\mathbb{CG}(\mathcal{L})$ *be a graph which contains a cycle. Then* $\mathrm{gr}(\mathbb{CG}(\mathcal{L})) = 3, 4$.

Proof. Assume that $\mathrm{Rad}(\mathcal{L}) \neq 0$ (so $\mathrm{Rad}(\mathcal{L})$ is a vertex of $\mathbb{CG}(\mathcal{L})$ since $\mathrm{Rad}(\mathcal{L}) \neq 1$) and let $x \backsim y$. By Proposition 3.5, either $x \notin \mathcal{CA}(\mathcal{L})$ or $y \notin \mathcal{CA}(\mathcal{L})$. If $x, y \notin \mathcal{CA}(\mathcal{L})$, then $x \backsim x \vee y \backsim y \backsim x$ is a cycle (so $\mathrm{gr}(\mathbb{CG}(\mathcal{L})) = 3$). Suppose that one of x or y is a coatom element. We can assume that $x \in \mathcal{CA}(\mathcal{L})$ and $y \notin \mathcal{CA}(\mathcal{L})$. By Lemma 3.2, there is a coatom element a of $\mathcal{CA}(\mathcal{L})$ such that $y \lneqq a$. Hence we obtain the cycle $x \backsim y \backsim a \backsim \mathrm{Rad}(\mathcal{L}) \backsim x$ which implies that $\mathrm{gr}(\mathbb{AG}(\mathcal{L})) = 4$, as needed. $\square$

Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2), \cdots, (\mathcal{L}_n, \leq_n)$ are lattices ($n \geq 2$) and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice.*

Lemma 3.16. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice. If c_i is a coatom element of $\mathcal{L}_i$ for $i \in \{1, \cdots, n\}$, then $c'_1 = (c_1, 1, \cdots, 1)$, $c'_2 = (1, c_2, 1, \cdots, 1)$, $\cdots$, and $c'_n = (1, 1, \cdots, c_n)$ are coatom elements of $\mathcal{L}$.*

Proof. On the contrary, assume that

$$c'_1 = (c_1, 1, \cdots, 1) \lneqq x = (x_1, x_2, \cdots, x_n) \lneqq (1, 1, \cdots, 1)$$

for some element x of $\mathcal{L}$ (so $x_i = 1$ for $i \in \{2, 3, \cdots, n\}$). It follows that $c_1 \lneqq x_1 \lneqq 1$, a contradiction, as c_1 is a coatom element. Thus c'_1 is a coatom element of $\mathcal{L}$. Similarly, $c'_2, \cdots, c'_n$ are coatom elements of $\mathcal{L}$. $\square$

Theorem 3.17. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice such that $\mathcal{CA}(\mathcal{L}_i) = \{c_i\}$ for $i \in \{1, 2, \cdots, n\}$. Then* $\mathrm{diam}(\mathbb{CG}(\mathcal{L})) \leq 2$.

Proof. By Lemma 3.16, $\mathcal{CA}(\mathcal{L}) = \{c'_1, \cdots, c'_n\}$, where we have that

$$c'_k = (1, 1, \cdots, 1, c_k, 1, \cdots, 1)$$

for $k \in \{1, 2, \cdots, n\}$. Let x and y be two vertices of $\mathbb{CG}(\mathcal{L})$ such that they are not adjacent. If $x \leq c'_i$ and $y \leq c'_i$ for some c'_i, then $x \backsim c'_i \backsim y$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. Otherwise, there are $c'_i, c'_j \in \mathcal{CA}(\mathcal{L})$ such that $x \leq c'_i$, $y \nleq c'_i$, $y \leq c'_j$ and $x \nleq c'_j$. We may assume that $i < j$. Consider the element $z = (1, 1, \cdots, 1, c_i, 1, \cdots, 1, c_j, 1, \cdots, 1)$. Since $x \nleq c'_j$ and $y \nleq c'_i$, we conclude that $z \vee x = c'_i$ and $z \vee y = c'_j$. This shows that $x \backsim z \backsim y$ is a path in $\mathbb{CG}(\mathcal{L})$ with $\mathrm{d}(x, y) = 2$. Thus, $\mathrm{diam}(\mathbb{CG}(\mathcal{L})) \leq 2$. □

Theorem 3.18. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice such that $\mathcal{CA}(\mathcal{L}_i) = \{c_i\}$ for $i \in \{1, 2, \cdots, n\}$. Then* $\mathrm{gr}(\mathbb{CG}(\mathcal{L})) = 3$.

Proof. By Lemma 3.16, $\mathcal{CA}(\mathcal{L}) = \{c'_1, \cdots, c'_n\}$, where we have that $c'_k = (1, 1, \cdots, 1, c_k, 1, \cdots, 1)$ for $k \in \{1, 2, \cdots, n\}$. Now we consider the elements $x = (c_1, c_2, 1, \cdots, 1)$, $y = (c_1, 1, c_3, 1, \cdots, 1)$ and $z = (1, c_2, c_3, 1, \cdots, 1)$. Since $x \vee y = c'_1$, $x \vee z = c'_2$ and $y \vee z = c'_3$, we get the cycle $x \backsim y \backsim z \backsim x$. This shows that $\mathrm{gr}(\mathbb{CG}(\mathcal{L})) = 3$. □

4. Further results in related lattices

The following theorem provides some condition under which the graph $\mathbb{CG}(\mathcal{L})$ is star.

Proposition 4.1. *If a chain is formed by the nontrivial elements of $\mathcal{L}$, then $\mathbb{CG}(\mathcal{L})$ is a star graph.*

Proof. By assumption, there is an element c of $\mathcal{CA}(\mathcal{L})$ such that $c \backsim x$ for every $x \in \mathcal{V}$ $(\mathbb{CG}(\mathcal{L}))$. If $x \neq c$ and $y \neq c$ are two distinct vertices of $\mathbb{CG}(\mathcal{L})$, then either $x \leq y$ or $y \leq x$. For both cases, x and y are not adjacent vertices. Thus $\mathbb{CG}(\mathcal{L})$ is a star graph with center c. □

In the following theorem, we describe the relation between cut vertex element and coatom elements.

Theorem 4.2. *Suppose that the meet of any two distinct coatom elements of $\mathcal{L}$ is not 0 and let a be a cut vertex of $\mathbb{CG}(\mathcal{L})$. Then there exist coatom elements b, c of $\mathcal{L}$ such that $a = b \wedge c$.*

Proof. If a is a coatom element, then $a = a \wedge a$. So we may assume that a is not a coatom element. Let b and c be two vertices of $\mathbb{CG}(\mathcal{L})$ such that $b \in V_1$ and $c \in V_2$, where V_1 and V_2 are the distinct components $\mathbb{CG}(\mathcal{L}) \setminus \{a\}$. Now we consider the various possibilities for b and c.

Case 1. $b, c \in \mathbb{CA}(\mathcal{L})$. Since $b \backsim b \wedge c \backsim c$ is a path in $\mathbb{CG}(\mathcal{L})$ and a is a cut vertex, we infer that $a = b \wedge c$.

Case 2. $b \in \mathbb{CA}(\mathcal{L})$ and $c \notin \mathbb{CA}(\mathcal{L})$. By assumption, $c \leq u$ for some coatom element u of $\mathcal{L}$ by Lemma 3.2; hence $u \in V_2$. Since $b \backsim b \wedge u \backsim u$ is a path in $\mathbb{CG}(\mathcal{L})$, $b \in V_1$ and a is a cut vertex, we conclude that $a = b \wedge u$.

Case 3. $b \notin \mathbb{CA}(\mathcal{L})$ and $c \in \mathbb{CA}(\mathcal{L})$. By an argument like that in Case 2, $a = c \wedge v$ for some coatom element v of $\mathcal{L}$ with $b \leq v$.

Case 4. $b, c \notin \mathbb{CA}(\mathcal{L})$. By assumption, $b \lneqq a$ and $c \lneqq a'$ for some coatom elements a and a' of $\mathcal{L}$ by Lemma 3.2; so $a \in V_1$ and $a' \in V_2$. Since $a \backsim a \wedge a' \backsim a'$ is a path in $\mathbb{CG}(\mathcal{L})$ and a is a cut vertex, we infer that $a = a \wedge a'$. □

We next give one other characterization of the Jacobson radical of $\mathcal{L}$.

Theorem 4.3. *If $\mathcal{L}$ is a complete lattice, then* $\mathrm{Rad}(\mathcal{L}) = \bigvee_{e \ll \mathcal{L}} e$.

Proof. Let $r = \bigvee_{e \ll \mathcal{L}} e$. By [14, Proposition 2.2], $\mathrm{Rad}(\mathcal{L}) \ll \mathcal{L}$ and so $\mathrm{Rad}(\mathcal{L}) \leq r$. Let $a \ll \mathcal{L}$. If c is a coatom element of $\mathcal{L}$, and if $a \nleq c$, then $a \vee c = 1$; but then since $a \ll \mathcal{L}$, we have $a = 1$, a contradiction. We infer that every small element a of $\mathcal{L}$ we have that $a \leq \mathrm{Rad}(\mathcal{L})$ and so $r \leq \mathrm{Rad}(\mathcal{L}$, as needed. □

Proposition 4.4. *The following hold in a complete lattice $\mathcal{L}$:*

(1) If $x \in \mathcal{L}$, then $x \ll \mathcal{L}$ if and only if $x \leq \mathrm{Rad}(\mathcal{L})$;

(2) If x is a nontrivial element of $\mathcal{L}$ and $\mathrm{Rad}(\mathcal{L}) \lneqq x$, then x is not small in $\mathcal{L}$.

Proof. (1) One side is clear by Theorem 4.3. To prove the other side, assume to the contrary, that x is not small in $\mathcal{L}$. Then there exists a nontrivial element b of $\mathcal{L}$ such that $x \vee b = 1$. By [14, Lemma 2.1], $b \leq c$ for some coatom element c of $\mathcal{L}$. Since $x \leq \mathrm{Rad}(\mathcal{L}) \leq c$, we conclude that $1 = x \vee b \leq c \vee b = c$ which implies that $c = 1$ which is a contradiction. Therefore, $x \ll \mathcal{L}$.

(2) It is a direct consequence of (1). □

Proposition 4.5. *The following hold in $\mathbb{CG}(\mathcal{L})$:*

(1) If $\mathcal{L}$ is complete, $c \in \mathcal{CA}(\mathcal{L})$ and $x \lneqq \mathrm{Rad}(\mathcal{L})$, then $c \backsim x$;

(2) If $x \notin \mathcal{CA}(\mathcal{L})$ and $y \lneqq x$, then x is not adjacent to y.

Proof. (1) Since $x \lneqq \mathrm{Rad}(\mathcal{L}) \leq c$, we conclude that $x \vee c = c$ and so $c \backsim x$.

(2) If $y \lneqq x$, then $x \vee y = x$. Since $x \notin \mathcal{CA}(\mathcal{L})$, we infer that x is not adjacent to y. □

The following theorem shows that when the join coatom element graph is a complete bipartite graph.

Theorem 4.6. *Let $\mathcal{L}$ be a complete lattice with* $\mathrm{Rad}(\mathcal{L}) \neq 0$*. Then every vertex of* $\mathbb{CG}(\mathcal{L})$ *is either coatom or small if and only if* $\mathbb{CG}(\mathcal{L})$ *is a complete bipartite graph.*

Proof. Let V_1 and V_2 be the set of coatom elements and small elements of $\mathcal{L}$, respectively. If $c, c' \in V_1$, then c and c' are not adjacent by Proposition 3.5. If $x, y \in V_2$, then $x \vee y \ll \mathcal{L}$ by [14, Lemma 2.3 (2)] and so $x \vee y \lneqq \mathrm{Rad}(\mathcal{L})$ by Proposition 4.4 which implies that any two vertices of V_2 are not adjacent. Moreover, every vertex in V_1 is adjacent to each vertex V_2 by Proposition 4.5. Therefore, $\mathbb{CG}(\mathcal{L})$ is a complete bipartite graph.

Conversely, suppose that V_1 and V_2 are parts of $\mathbb{CG}(\mathcal{L})$ and let c be an coatom element of $\mathcal{L}$. Without loss of generality, let $c \in V_1$. If $c \neq c' \in \mathcal{CA}(\mathcal{L})$ with $c' \notin V_1$, then $c' \in V_2$ implies that $c \vee c'$ is coatom which is impossible. Thus $\mathcal{CA}(\mathcal{L}) \subseteq V_1$. If $b \in V_1$ with $b \notin \mathcal{CA}(\mathcal{L})$, then there exists a coatom element a of $\mathcal{L}$ (so $a \in V_1$) such that $b \backsim a$ by Lemma 3.2, a contradiction. Therefore, $V_1 = \mathcal{CA}(\mathcal{L})$. Suppose that $c \leq \mathrm{Rad}(\mathcal{L})$ for some vertex c. We show that c is small in $\mathcal{L}$. Let d be any element of $\mathcal{L}$ such that $c \vee d = 1$. If d is coatom, then $c \leq \mathrm{Rad}(\mathcal{L}) \leq d$ and so $d = 1$. If d is not coatom, then $d = 1$. Otherwise, there exists a coatom element u of $\mathcal{L}$ such that $d \leq u$ by [14, Lemma 2.1]. Since $c \leq \mathrm{Rad}(\mathcal{L}) \leq u$, we conclude that $1 = c \vee d \leq u$, a contradiction. Therefore, $\mathcal{S} = \{s \in \mathcal{V}\ (\mathbb{CG}(\mathcal{L}) : s \leq \mathrm{Rad}(\mathcal{L})\}$ is the set of all small elements of $\mathcal{L}$. An easy inspection will show that $V_2 = \mathcal{S}$, as required. □

5. Clique number, domination number and planarity of $\mathbb{CG}(\mathcal{L})$

We continue this section with the investigation of the stability of join coatom element graphs in various lattice-theoretic constructions. In the following results we show that domination numbers are really of interest in indecomposable lattices.

Theorem 5.1. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice such that* $\mathbb{CA}(\mathcal{L}_i) = \{c_i\}$ *for* $i \in \{1, 2\}$*. Then* $\gamma(\mathbb{CG}(\mathcal{L})) = 2$*;*

Proof. By Lemma 3.16, $\mathcal{CA}(\mathcal{L}) = \{c'_1, c'_2\}$, where we have that $c'_1 = (c_1, 1)$ and $c'_2 = (1, c_2)$. Let $a = (a_1, a_2)$ be a nontrivial element of $\mathcal{L}$. Since $a \vee_c c'_1 = (a_1 \vee c_1, 1)$ and $a \vee_c c'_2 = (1, c_2 \vee a_2)$, we infer that any vertex of the graph $\mathbb{CG}(\mathcal{L})$ is adjacent to at least one of the elements of the set $\{c'_1, c'_2\}$. This shows that $\gamma(\mathbb{CG}(\mathcal{L})) = 2$. □

Theorem 5.2. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ $(n \geq 3)$ be a decomposable lattice such that* $\mathbb{CA}(\mathcal{L}_i) = \{c_i\}$ *for* $i \in \{1, 2, \cdots, n\}$*. Then* $\gamma(\mathbb{CG}(\mathcal{L})) \leq n$*.*

Proof. By Lemma 3.16, $\mathcal{CA}(\mathcal{L}) = \{c'_1, \cdots, c'_n\}$, where we have that $c'_k = (1, 1, \cdots, 1, c_k, 1, \cdots, 1)$ for $k \in \{1, 2, \cdots, n\}$. Then the set $\mathcal{CA}(\mathcal{L})$ dominates all the vertices of the graph $\mathbb{CG}(\mathcal{L})$; hence $\gamma(\mathbb{CG}(\mathcal{L})) \leq n$. $\square$

The following example shows that in general Theorem 5.2 is not true in the case $\gamma(\mathbb{CG}(\mathcal{L})) = n$.

Example 5.3. Suppose that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \{0, 1\}$ and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$ be a decomposable lattice. Then we have

$$\mathcal{V}(\mathbb{CG}(\mathcal{L})) = \{(1, 0, 0), (1, 1, 0), (0, 1, 0), (0, 1, 1), (1, 0, 1), (0, 0, 1)\}.$$

If $D = \{(0, 0, 1), (1, 0, 0)\}$, then the set D dominates all the vertices of the graph $\mathbb{CG}(\mathcal{L})$; hence $\gamma(\mathbb{CG}(\mathcal{L})) = 2 \neq 3$.

Proposition 5.4. *Let $a, b \notin \mathcal{CA}(\mathcal{L})$ such that they are adjacent. Then there is a unique coatom element c of $\mathcal{L}$ such that $c \in N(a) \cap N(b)$.*

Proof. Since $a \vee b$ is a coatom element, we infer that $a \vee b$ is adjacent to both a and b which gives $a \vee b \in N(a)$ and $a \vee b \in N(b)$. Let $x \in \mathcal{CA}(\mathcal{L})$ such that $x \in N(a) \cap N(b)$. It suffices to show that $x = a \vee b$. On the contrary, assume that $x \neq a \vee b$. First, we prove that $x \in N(a) \cap N(b)$ if and only if $x \in N(a \vee b)$. If $x \backsim a \vee b$, then $x \vee (a \vee b)$ is a coatom element of $\mathcal{L}$ which gives $x \vee a \neq 1$ and $x \vee b \neq 1$. Since $x \leq a \vee x$ and $x \leq x \vee b$, we conclude that $a \vee x = x = x \vee b$ and so $x \backsim a$ and $x \backsim b$. Conversely, assume that $a \backsim x$ and $b \backsim x$. Then $a \vee x$ and $x \vee b$ are coatom elements gives $a \vee x = x = x \vee b$ which implies that $(a \vee b) \vee x = x$; so $x \backsim a \vee b$. Since $x \vee (a \vee b)$ is a coatom element, we infer that $x \vee (a \vee b) \neq 1$ which implies that $a \vee b \leq x$. Therefore, $x = a \vee b$, as $a \vee b$ is a coatom element, a contradiction. Thus $x = a \vee b$. $\square$

Theorem 5.5. *Let $\mathcal{G}$ be a clique in $\mathbb{CG}(\mathcal{L})$. Then $\mathcal{G}$ is contained in the subgraph induced by $\{x \in \mathcal{V}\ (\mathbb{CG}(\mathcal{L})) : x \leq c\}$ for some coatom element c of $\mathcal{L}$.*

Proof. By Proposition 3.5, clique $\mathcal{G}$ has at most one coatom element. If $\mathcal{G} \cap \mathcal{CA}(\mathcal{L}) = \{c\}$, then $x \leq c$ for every $x \in \mathcal{G}$ and so $\mathcal{G}$ is contained in the subgraph induced by $\{x \in \mathcal{V}\ (\mathbb{CG}(\mathcal{L})) : x \leq c\}$. So we may assume that $\mathcal{G} \cap \mathcal{CA}(\mathcal{L}) = \emptyset$. The adjacency of every two vertices of $\mathcal{G}$ and Proposition 5.4 shows that there exists a unique coatom element c' of $\mathcal{L}$ for which $\mathcal{G}$ is a subgraph of the graph induced by $\{x \in \mathcal{V}\ (\mathbb{CG}(\mathcal{L})) : x \leq c'\}$. $\square$

The following theorem provides some condition under which $\mathbb{CG}(\mathcal{L})$ is a split graph.

Theorem 5.6. *If $\mathbb{CG}(\mathcal{L})$ is not empty and $\mathcal{V}\ (\mathbb{CG}(\mathcal{L})) = \mathbb{A}(\mathcal{L}) \cup \mathcal{CA}(\mathcal{L})$, then $\mathbb{CG}(\mathcal{L})$ is a split graph.*

Proof. At first, we show that the subgraph induced by the atom elements of $\mathcal{L}$ is a complete graph. It is enough to show if c and d are atom elements of $\mathcal{L}$ with $c \neq d$, then $c \backsim d$. We claim that $c \vee d \neq 1$. On the contrary, let $c \vee d = 1$. Since $0 \leq c \wedge d \lneqq c$, we infer that $c \wedge d = 0$. Let $c \leq e \neq 1$ for some element e of $\mathcal{L}$. As $0 \leq e \wedge d \lneqq d$ and $1 = c \vee d \leq e \vee d$, we conclude that $e \wedge d = 0$ and $e \vee d = 1$ which implies that $c = c \wedge 1 = c \wedge (e \vee d) = (c \wedge e) \vee (c \wedge d) = c \wedge e = e$. This shows that c is coatom. Similarly, d is coatom. It follows that $\mathcal{V}\ (\mathbb{CG}(\mathcal{L})) = \mathbb{CA}(\mathcal{L})$ and so $\mathbb{CG}(\mathcal{L})$ is empty by Proposition 3.5 which is impossible. Therefore, $c \vee d \neq 1$. It is clear that $c \vee d \notin \mathcal{A}(\mathcal{L})$; so $c \vee d \in \mathbb{CA}(\mathcal{L})$, i.e. any two atom elements are adjacent. Consider the subgraph induced by $\mathcal{A}(\mathcal{L})$ of $\mathbb{CG}(\mathcal{L})$. Let $c, d \in \mathcal{A}(\mathcal{L})$ with $c \neq d$. Then $c \vee d \in \mathbb{CA}(\mathcal{L})$ which implies that the subgraph induced by $\mathcal{A}(\mathcal{L})$ is complete. Moreover, by Proposition 3.5, the subgraph induced by $\mathbb{CA}(\mathcal{L})$ is empty. Thus, $\mathbb{CG}(\mathcal{L})$ is a split graph. $\square$

Corollary 5.7. *If* $\mathbb{CG}(\mathcal{L})$ *is not empty and* $\mathcal{V}\ (\mathbb{CG}(\mathcal{L})) = \mathbb{A}(\mathcal{L}) \cup \mathcal{CA}(\mathcal{L})$, *then* $|\mathcal{A}(\mathcal{L})| \leq \omega(\mathbb{CG}(\mathcal{L}))$.

Proof. Since the subgraph of $\mathbb{CG}(\mathcal{L})$ with the vertex set of $\mathcal{A}(\mathcal{L})$ is a complete subgraph of $\mathbb{CG}(\mathcal{L})$, we conclude that $|\mathcal{A}(\mathcal{L})| \leq \omega(\mathbb{CG}(\mathcal{L}))$. $\square$

The following example shows that the equality does not hold necessarily in Corollary 5.7.

Example 5.8. Let $\mathcal{L} = \{0, a, b, c, d, 1\}$ be a lattice with the relations $0 \leq d \leq c \leq a \leq 1$, $0 \leq d \leq c \leq b \leq 1$, $a \vee b = 1$ and $a \wedge b = c$. Clearly, the nontrivial elements of $\mathcal{L}$ are a, b, c and d. An inspection will show that $\mathcal{A}(\mathcal{L}) = \{d\}$, $\mathcal{CA}(\mathcal{L}) = \{a, b\}$ and $\mathcal{G} = \{a, c\}$ is a clique. Hence $|\mathcal{A}(\mathcal{L})| = 1 < \omega(\mathbb{CG}(\mathcal{L})) = 2$.

The following theorem provides some condition under which $\mathbb{CG}(\mathcal{L})$ is a planar graph.

Theorem 5.9. *Suppose that* $\mathbb{CG}(\mathcal{L})$ *is not empty and let* $\mathcal{V}\ (\mathbb{CG}(\mathcal{L})) = \mathbb{A}(\mathcal{L}) \cup \mathcal{CA}(\mathcal{L})$. *If* $|\mathcal{A}(\mathcal{L})| \leq 3$, *then* $\mathbb{CG}(\mathcal{L})$ *is a planar graph.*

Proof. Note that $\mathbb{CG}(\mathcal{L})$ is a split graph by Theorem 5.6. Since $|\mathcal{A}(\mathcal{L})| \leq 3$, we infer that any subgraph induced by five vertices is not complete; so K_5 is not a subgraph of $\mathbb{CG}(\mathcal{L})$. It suffices to show that $K_{3,3}$ is not a subgraph of $\mathbb{CG}(\mathcal{L})$. On the contrary, assume that $K_{3,3}$ is a subgraph of $\mathbb{CG}(\mathcal{L})$ with partite sets $|V_1| = 3$ and $|V_2| = 3$. Therefore, either $V_1 \subseteq \mathbb{CA}(\mathcal{L})$ or $V_2 \subseteq \mathbb{CA}(\mathcal{L})$. If $V_1 \subseteq \mathbb{CA}(\mathcal{L})$, then $V_2 \subseteq \mathcal{A}(\mathcal{L})$, a contradiction since the subgraph induced by the atom elements of $\mathcal{L}$ is a complete graph. Therefore, $\mathbb{CG}(\mathcal{L})$ is a planar graph. $\square$

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