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Counting Actions and Non-Isomorphic Actions of a Finitely Generated Abelian Group on a Finite Set

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Abstract. The main goal of the paper is to compute the number of group actions and actions up to isomorphism of a finitely generated Abelian group $G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t}$, on a set of m elements, and several illustrative examples are presented to clarify the main results.

Keywords: Orbit-stabilizer, Finitely generated Abelian group, Group actions.

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1. Introduction

A group action of a group G on a set X describes how elements of G permute elements of X, offering insights into the symmetries of X. Such actions arise in symmetry classification (e.g., molecular chirality) [3], permutation representations (e.g., character theory of Abelian groups) [1], and algebraic combinatorics (e.g., orbit counting in posets or graphs) [7]. For a finitely generated Abelian group $G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t}$,

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where $\mathbb{Z}^k$ is the free Abelian group of rank k, and $\mathbb{Z}_{d_i}$ are cyclic groups of order d_i with $d_i \mid d_{i+1}$, we aim to compute:

(1) The total number of group actions on a set X with m elements, corresponding to the number of homomorphisms $\mathrm{Hom}(G, S_m)$, where S_m is the symmetric group on m elements.
(2) The number of group actions up to isomorphism, where two actions $\phi, \psi : G \to S_m$ are isomorphic if there exists $\sigma \in S_m$ such that $\psi(g) = \sigma\phi(g)\sigma^{-1}$ for all $g \in G$.

Since G is Abelian, $\phi(G) \subseteq S_m$ is Abelian, hence contained in a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^b$ corresponding to partitions of m into singletons and pairs. We use the orbit-stabilizer theorem and Burnside's lemma. Our results *generalize and extend* prior work on free Abelian groups [4, 6]., which only considered $k \geq 1$ and $t = 0$, by including torsion and mixed cases.

The flow is: Section 2 gives preliminaries, including Lemma 2.6 counting homomorphisms into finite Abelian groups. Section 3 derives the main counting formulas in Theorems 3.1 and 3.2. Section 4 provides examples and concludes.

2. Preliminaries

Definition 2.1. A *group action* of a group G on a set X is a map $\phi : G \times X \to X$ satisfying:

(1) (Identity) For the identity $e \in G$, $\phi(e, x) = x$ for all $x \in X$.
(2) (Compatibility) For $g, h \in G$, $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $x \in X$.

This corresponds to a group homomorphism $\rho : G \to S_X$, where S_X is the symmetric group on X [2].

Definition 2.2. Two group actions $\phi, \psi : G \to S_m$ on a set $X = \{1, 2, \ldots, m\}$ are *isomorphic* if there exists $\sigma \in S_m$ such that:

$$\psi(g) = \sigma\phi(g)\sigma^{-1} \text{ for all } g \in G.$$

This means ϕ and ψ are conjugate under the action of S_m on $\mathrm{Hom}(G, S_m)$.

Definition 2.3. For a group action of G on X, and $x \in X$:

- The *orbit* of x is $\mathrm{Orb}(x) = \{g \cdot x \mid g \in G\}$.
- The *stabilizer* of x is $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$.

Theorem 2.4 (Orbit-Stabilizer Theorem)**.** *For a group G acting on a set X, and any $x \in X$, the size of the orbit is:*

$$|Orb(x)| = [G : Stab(x)] = \frac{|G|}{|Stab(x)|},$$

if G is finite, where $[G : Stab(x)]$ is the index of the stabilizer [2].

Theorem 2.5 (Burnside's Lemma)**.** *For a group G acting on a set X, the number of orbits is:*

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|,$$

where $Fix(g) = \{x \in X \mid g \cdot x = x\}$ is the set of elements fixed by g [5].

Lemma 2.6. *For an Abelian group $G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t}$, with generators $\{e_1, \ldots, e_k, f_1, \ldots, f_t\}$, where e_i generate the free part and f_i generate $\mathbb{Z}_{d_i}$ with $d_i f_i = 0$, the number of homomorphisms to a finite Abelian group $H \cong (\mathbb{Z}/d\mathbb{Z})^r$ is:*

$$|Hom(G,H)| = d^{k\cdot r} \cdot \prod_{i=1}^{t} \gcd(d_i, d)^r.$$

Proof. A homomorphism $\phi : G \to H$ is determined by the images of the generators. For $H \cong (\mathbb{Z}/d\mathbb{Z})^r$, each of the k free generators e_i can map to any of the $|H| = d^r$ elements. Each torsion generator f_i satisfies $d_i f_i = 0$, so $\phi(f_i) \in H$ must have order dividing d_i. In $(\mathbb{Z}/d\mathbb{Z})^r$, the number of elements of order dividing d_i is $\gcd(d_i, d)^r$. Thus:

$$|\mathrm{Hom}(G,H)| = (d^r)^k \cdot \prod_{i=1}^{t} \gcd(d_i, d)^r.$$

□

3. Main Results

The goal is to determine explicit formulas for both the total and non-isomorphic actions for $G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t}$ on a set of m elements. The double factorial is defined as $(2b-1)!! = 1 \cdot 3 \cdot \cdots \cdot (2b-1)$, with $(-1)!! = 1$ for $b = 0$.

Theorem 3.1. *The number of group actions of $G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t}$ on a set of m elements is:*

$$\sum_{b=0}^{\lfloor m/2 \rfloor} \binom{m}{2b} (2b-1)!! \cdot 2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b,$$

where $(2b-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2b-1)$, with $(-1)!! = 1$ for $b = 0$.

Proof. A group action of G on $X = \{1, 2, \ldots, m\}$ corresponds to a homomorphism $\phi : G \to S_m$. Since G is Abelian, $\phi(G) \subseteq S_m$ is an Abelian subgroup. The maximal Abelian subgroups of S_m arise from partitions of $\{1, 2, \ldots, m\}$:

$$S_{m_1} \times \cdots \times S_{m_r} \leq S_m,$$

where $m_1 + \cdots + m_r = m$. Since S_m is non-Abelian for $m \geq 3$, each $m_i \leq 2$ [2].

Consider a partition with a parts of size 1 and b parts of size 2, so $a + 2b = m$, or $a = m - 2b$. The corresponding subgroup is:

$$H \cong S_1^{m-2b} \times S_2^b \cong (\mathbb{Z}/2\mathbb{Z})^b,$$

since S_1 is trivial and $S_2 \cong \mathbb{Z}/2\mathbb{Z}$, with $|H| = 2^b$.

For a homomorphism $\phi : G \to H$, the action on X has orbits corresponding to the partition:

- Orbits of size 1: Fixed points, where $\mathrm{Orb}(i) = \{i\}$, and $\mathrm{Stab}(i) = \phi(G)$.
- Orbits of size 2: Pairs $\{i, j\}$, where the image in $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ gives $\mathrm{Orb}(i) = \{i, j\}$. The stabilizer $\mathrm{Stab}(i)$ has index 2 in the image restricted to the pair.

If $\phi(G) \subseteq (\mathbb{Z}/2\mathbb{Z})^b$, the orbit-stabilizer theorem confirms orbit sizes of 1 or 2, as $|\mathrm{Orb}(i)| = [\phi(G) : \mathrm{Stab}(i)]$, and $\phi(G)$ is a quotient of G, possibly infinite if $k \geq 1$.

By Lemma 2.6, for $H \cong (\mathbb{Z}/2\mathbb{Z})^b$, with $d = 2$, the number of homomorphisms is:

$$|\mathrm{Hom}(G, (\mathbb{Z}/2\mathbb{Z})^b)| = 2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b,$$

where $\gcd(d_i, 2) = 2$ if d_i is even, and 1 if odd .

The number of partitions with b pairs and $m - 2b$ singletons is:

- Choose $2b$ elements for pairs: $\binom{m}{2b}$.
- Form b unordered pairs: $(2b - 1)!! = \frac{(2b)!}{2^b b!}$.

Thus:

$$\binom{m}{2b} \cdot (2b - 1)!! = \frac{m!}{2^b b!(m - 2b)!}.$$

The total number of homomorphisms is:

$$\sum_{b=0}^{\lfloor m/2 \rfloor} \frac{m!}{2^b b!(m - 2b)!} \cdot 2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b = \sum_{b=0}^{\lfloor m/2 \rfloor} \binom{m}{2b} (2b-1)!! \cdot 2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b.$$

□

Theorem 3.2. *The number of group actions, up to isomorphism, of $G \cong \mathbb{Z}^k \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_t}$ on a set of m elements is:*

$$\sum_{b=0}^{\lfloor m/2 \rfloor} \left[2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b \geq 1 \right],$$

where $[P] = 1$ *if predicate* P *is true, and 0 otherwise. Equivalently, it is the number of partitions of* m *into parts of size 1 or 2 for which there exists at least one homomorphism from* G *to* $(\mathbb{Z}/2\mathbb{Z})^b$*.*

Proof. Two homomorphisms $\phi, \psi : G \to S_m$ define isomorphic actions if there exists $\sigma \in S_m$ such that $\psi(g) = \sigma\phi(g)\sigma^{-1}$ for all $g \in G$. This corresponds to orbits of the conjugation action of S_m on $\mathrm{Hom}(G, S_m)$. Since $\phi(G)$ is Abelian, it is contained in a conjugate of $S_1^{m-2b} \times S_2^b \cong (\mathbb{Z}/2\mathbb{Z})^b$.

By Burnside's lemma, the number of orbits is:

$$\frac{1}{|S_m|} \sum_{\sigma \in S_m} |\mathrm{Fix}(\sigma)|,$$

where $\mathrm{Fix}(\sigma) = \{\phi : G \to S_m \mid \sigma\phi(g)\sigma^{-1} = \phi(g) \text{ for all } g \in G\}$. For ϕ to be fixed, $\phi(G) \subseteq C_{S_m}(\sigma)$, the centralizer of σ. Since $\phi(G)$ is Abelian, $C_{S_m}(\sigma)$ must be Abelian, which restricts σ to permutations with cycle types having parts of size 1 or 2 [2].

Conjugate homomorphisms have the same orbit structure. Each partition $(m - 2b) \cdot 1 + b \cdot 2$ corresponds to a unique isomorphism class, provided $|\mathrm{Hom}(G, (\mathbb{Z}/2\mathbb{Z})^b)| \geq 1$. From Lemma 2.6:

$$|\mathrm{Hom}(G, (\mathbb{Z}/2\mathbb{Z})^b)| = 2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b.$$

A homomorphism exists if this is at least 1, i.e., $k \geq 1$ or (if $k = 0$) $\prod_{i=1}^{t} \gcd(d_i, 2)^b \geq 1$, requiring at least one d_i even for $b \geq 1$.

Thus, the number of non-isomorphic actions is the number of b from 0 to $\lfloor m/2 \rfloor$ for which a homomorphism exists:

$$\sum_{b=0}^{\lfloor m/2 \rfloor} \left[2^{bk} \cdot \prod_{i=1}^{t} \gcd(d_i, 2)^b \geq 1 \right].$$

If $k \geq 1$, then $2^{bk} \geq 1$, so all partitions contribute. If $k = 0$, non-trivial actions require some d_i even. □

Logical connection: Lemma 2.6 provides a general formula for the number of homomorphisms from G to any finite Abelian group $H \cong (\mathbb{Z}/d\mathbb{Z})^r$. In Theorem 3.1, we specialize this to $d = 2$ and $r = b$ to count all homomorphisms into each maximal Abelian subgroup $(\mathbb{Z}/2\mathbb{Z})^b$ of S_m, which are indexed by the possible values of b (number of pairs in the partition of m). Each such b corresponds to a family of actions with the same orbit type (orbits of size 1 or 2). In Theorem 3.2, we use the same homomorphism count to determine *which* of these b values admit at least one homomorphism, thereby counting the distinct isomorphism classes via the conjugation action of S_m on $\mathrm{Hom}(G, S_m)$. This progression from general homomorphism counting, to total actions, to non-isomorphic actions forms the core argument of the paper.

4. Examples

We provide examples where $n = k + t \neq m$, including cases with $k = 0$, to illustrate both theorems.

Example 4.1 ($G = \mathbb{Z}, m = 2$)**.** Here, $k = 1$, $t = 0$, $n = 1 \neq 2$.

For Theorem 3.1:

$$\sum_{b=0}^{1} \binom{2}{2b} (2b-1)!! \cdot 2^{b \cdot 1}.$$

- $b = 0$: $\binom{2}{0} \cdot 1 \cdot 2^0 = 1 \cdot 1 \cdot 1 = 1$.
- $b = 1$: $\binom{2}{2} \cdot 1 \cdot 2^1 = 1 \cdot 1 \cdot 2 = 2$.

Total: $1 + 2 = 3$.

Actions: Trivial (all fixed), and two where $\phi(1) = (1\ 2)$ or $\phi(1) = e$, with orbit $\{1, 2\}$, stabilizer trivial.

For Theorem 3.2:

$$\sum_{b=0}^{1} [2^{b \cdot 1} \geq 1].$$

- $b = 0$: $[2^0 \geq 1] = [1 \geq 1] = 1$.
- $b = 1$: $[2^1 \geq 1] = [2 \geq 1] = 1$.

Total: $1 + 1 = 2$.

Isomorphism classes: Trivial action ($b = 0$: two singletons), and non-trivial action ($b = 1$: one 2-orbit). Burnside's lemma distinguishes these by counting orbits under conjugation: the trivial action is fixed by all $\sigma \in S_2$, while the non-trivial action (generated by $(1\,2)$) is conjugate only to itself under even permutations, forming two distinct orbits in $\mathrm{Hom}(G, S_2)$.

Example 4.2 ($G = \mathbb{Z}_3 \oplus \mathbb{Z}_3, m = 4$)**.** Here, $k = 0$, $t = 2$, $d_1 = d_2 = 3$, $n = 2 \neq 4$.

For Theorem 3.1:

$$\sum_{b=0}^{2} \binom{4}{2b} (2b-1)!! \cdot (\gcd(3, 2)^b)^2.$$

Since $\gcd(3, 2) = 1$:

- $b = 0$: $\binom{4}{0} \cdot 1 \cdot 1 = 1$.
- $b = 1$: $\binom{4}{2} \cdot 1 \cdot 1 = 6 \cdot 1 = 6$.
- $b = 2$: $\binom{4}{4} \cdot 3 \cdot 1 = 1 \cdot 3 = 3$.

Total: $1+6+3=10$.

Since $\gcd(3,2)=1$, only trivial homomorphisms contribute for $b \geq 1$, reflecting the number of partitions.

For Theorem 3.2:

$$\sum_{b=0}^{2}[1^b \cdot 1^b \geq 1].$$

- $b=0$: $[1 \geq 1]=1$.
- $b=1$: $[1 \geq 1]=1$.
- $b=2$: $[1 \geq 1]=1$.

Total: $1+1+1=3$.

Classes: 4 singletons, 1 pair + 2 singletons, 2 pairs.

Means: Burnside's lemma shows that all three partition types ($b = 0, 1, 2$) yield distinct conjugacy classes: $b=0$ (trivial), $b=1$ (one pair, three singletons), $b=2$ (two disjoint pairs). Each class is distinguished by the cycle type of the image in S_4, and only the trivial action has non-trivial homomorphisms (since $\gcd(3,2)=1$).

Example 4.3 ($G=\mathbb{Z}_2 \oplus \mathbb{Z}_4, m=3$)**.** Here, $k=0$, $t=2$, $d_1=2$, $d_2=4$, $n=2 \neq 3$.

For Theorem 3.1:

$$\sum_{b=0}^{1}\binom{3}{2b}(2b-1)!! \cdot \gcd(2,2)^b \cdot \gcd(4,2)^b.$$

Since $\gcd(2,2)=2$, $\gcd(4,2)=2$:

- $b=0$: $\binom{3}{0} \cdot 1 \cdot 1 \cdot 1 = 1$.
- $b=1$: $\binom{3}{2} \cdot 1 \cdot 2^1 \cdot 2^1 = 3 \cdot 1 \cdot 2 \cdot 2 = 12$.

Total: $1+12=13$.

For $b=1$, each generator maps to $\{e, (1\ 2)\}$, giving $2 \cdot 2 = 4$ homomorphisms per pair ($\binom{3}{2}=3$).

For Theorem 3.2:

$$\sum_{b=0}^{1}[2^b \cdot 2^b \geq 1].$$

- $b=0$: $[1 \cdot 1 \geq 1]=1$.
- $b=1$: $[2^1 \cdot 2^1 \geq 1]=[4 \geq 1]=1$.

Total: $1+1=2$.

Classes: 3 singletons, 1 pair + 1 singleton.

Means: Burnside's lemma distinguishes: (i) the trivial action ($b=0$), fixed by all $\sigma \in S_3$; (ii) the non-trivial actions ($b=1$), where the image is in a copy of $\mathbb{Z}/2\mathbb{Z}$ acting on a pair. All 12 non-trivial homomorphisms are conjugate under S_3, forming a single orbit. Thus, two isomorphism classes.

Example 4.4 ($G = \mathbb{Z}^2 \oplus \mathbb{Z}_6, m = 4$)**.** Here, $k = 2$, $t = 1$, $d_1 = 6$, $n = 3 \neq 4$.

For Theorem 3.1:

$$\sum_{b=0}^{2} \binom{4}{2b} (2b-1)!! \cdot 2^{b \cdot 2} \cdot \gcd(6,2)^b.$$

Since $\gcd(6, 2) = 2$:

- $b = 0$: $\binom{4}{0} \cdot 1 \cdot 2^0 \cdot 1 = 1$.
- $b = 1$: $\binom{4}{2} \cdot 1 \cdot 2^2 \cdot 2^1 = 6 \cdot 1 \cdot 4 \cdot 2 = 48$.
- $b = 2$: $\binom{4}{4} \cdot 3 \cdot 2^4 \cdot 2^2 = 1 \cdot 3 \cdot 16 \cdot 4 = 192$.

Total: $1 + 48 + 192 = 241$.

For Theorem 3.2:

$$\sum_{b=0}^{2} [2^{b \cdot 2} \cdot 2^b \geq 1].$$

- $b = 0$: $[2^0 \cdot 1 \geq 1] = 1$.
- $b = 1$: $[2^2 \cdot 2^1 \geq 1] = [8 \geq 1] = 1$.
- $b = 2$: $[2^4 \cdot 2^2 \geq 1] = [64 \geq 1] = 1$.

Total: $1 + 1 + 1 = 3$.

Classes: 4 singletons, 1 pair + 2 singletons, 2 pairs.

Example 4.5 ($G = \mathbb{Z} \oplus \mathbb{Z}_2, m = 4$)**.** Here, $k = 1$, $t = 1$, $d_1 = 2$ (even), $n = 2 \neq 4$.

For Theorem 3.1:

$$\sum_{b=0}^{2} \binom{4}{2b} (2b-1)!! \cdot 2^{b \cdot 1} \cdot \gcd(2,2)^b = \sum_{b=0}^{2} \binom{4}{2b} (2b-1)!! \cdot 2^b \cdot 2^b.$$

- $b = 0$: 1.
- $b = 1$: $\binom{4}{2} \cdot 1 \cdot 4 = 24$.
- $b = 2$: $\binom{4}{4} \cdot 3 \cdot 16 = 48$.

Total: $1 + 24 + 48 = 73$.

Non-isomorphic: 3 by Theorem 3.2. Since $k \geq 1$, all $b = 0, 1, 2$ are possible. Burnside's lemma confirms three distinct conjugacy classes in $\mathrm{Hom}(G, S_4)$: one for each b, distinguished by the number of 2-cycles in the support of the action.

4.1. **Conclusion.** We have derived formulas for both the total number of group actions and the number of actions up to isomorphism for a finitely generated Abelian group on an m-element set. The orbit-stabilizer theorem ensures orbits are of size 1 or 2, and Burnside's lemma simplifies the count of non-isomorphic actions to the number of compatible partitions. The examples demonstrate the formulas' application across various cases, including purely torsion groups. Future work could explore actions of non-Abelian groups or actions on structured sets.

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