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On a Class of Infinite Semipositone problems via Sub and Supersolutions Method

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Abstract. Using the method of sub-super solutions, we study the existence of positive solutions for a class of infinite semipositone problems involving nonlocal operator.

Keywords: Kirchhoff-type problems, Infinite semipositone, Sub and Supersolutions method.

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1. Introduction

In this paper, we are interested in the existence of positive solutions for the Kirchhoff-type problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = au - bu^\gamma - f(u) - \frac{c}{u^\alpha}, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1.1}$$

where Δ denotes the Laplacian operator defined by $\Delta z = \operatorname{div}(\nabla z)$, $\gamma > 1$, $\alpha \in (0,1)$, a, b and c are positive constants, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $f : [0,\infty) \to \mathbb{R}$ is a continuous function. This model describes the steady states of a logistic growth

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model with grazing and constant yield harvesting. It also describes the dynamics of the fish population with natural predation and constant yield harvesting.

We make the following assumptions:

(H_1) $f : [0, \infty) \to \mathbb{R}$ is a continuous function such that $\lim_{s\to\infty} f(s) = \infty$.

(H_2) There exist $A > 0$ and $\beta > 1$ such that $f(s) \leq As^{\beta}$, for all $s \geq 0$.

(H_3) $M : [0, \infty) \to \mathbb{R}$ is a continuous and increasing function such that $0 < M_0 \leq M(t) \leq M_\infty$ for all t.

In [25], the authors have studied the equation $-\Delta u = g(u) - (c/u^{\alpha})$ with Dirichlet boundary conditions, where g is nonnegative and nondecreasing and $\lim_{u\to\infty} g(u) = \infty$. The case $g(u) := au - f(u)$ has been studied in [16], where $f(u) \geq au - N$ and $f(u) \leq Au^{\beta}$ on $[0, \infty)$ for some N, $A > 0$, $\beta > 1$ and this g may have a falling zero. Here u is the population density and $au - bu^{\gamma} - f(u)$ represents logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [17]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{c}{u^{\alpha}}$. At high levels of vegetation density this term saturates to c as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations (see [17, 26]). The diffusive logistic equation with constant yield harvesting, in the absence of grazing was studied in [20]. More recently, reaction-diffusion models have been used to describe spatiotemporal phenomena in disciplines other than ecology, such as physics,chemistry (see [9, 27]). For more information, we refer the reader to [12, 13, 19]. In this paper, we study the equation

$$-M\left(\int_{\Omega} |\nabla u|^2\, dx\right) \Delta u = au - bu^{\gamma} - f(u) - (c/u^{\alpha}) \qquad (1.2)$$

with Dirichlet boundary conditions. Let $F(u) := au - bu^{\gamma} - f(u) - (c/u^{\alpha})$, then

$\lim_{u\to 0^+} F(u) = -\infty$ and hence we refer to (1.1) as an infinite semipositone problem. We refer to [21, 22] for additional results on semipositone problems. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [4, 5] in which the authors have used variational method and topological method to get the existence of solutions for (1.1). In this paper, motivated by the ideas introduced in [8] and the properties of Kirchhoff type operators in [1]-[3],[11, 15, 18], we study problem (1.1) in the semipositone case ($F(0) < 0$ but finite) (see [6, 7, 14]). The main tool used in this study is the method of sub- and super solutions. Our result in this note improves the previous one [8] in which $M(t) \equiv 1$. To our best knowledge, this is a new research topic for nonlocal problems, see [4, 15, 23].

2. Main result

A function ψ is said to be a subsolution of problem (1.1) if it is in $W^{1,2}(\Omega)$ such that $\psi = 0$ on $\partial\Omega$ and satisfies

$$M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx \le \int_\Omega \left(a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^\alpha}\right) w\,dx, \tag{2.1}$$
$$\forall w \in W,$$

where $W := \{w \in C_0^\infty(\Omega) :\ w \ge 0 \text{ in } \Omega\}$. A function $z \in W^{1,2}(\Omega)$ with $z = 0$ on $\partial\Omega$ in the trace sense is said to be a supersolution if satisfies

$$M\left(\int_\Omega |\nabla z|^2\,dx\right)\int_\Omega \nabla z\cdot\nabla w\,dx \ge \int_\Omega \left(az - bz^\gamma - f(z) - \frac{c}{z^\alpha}\right) w\,dx, \tag{2.2}$$
$$\forall w \in W.$$

Our main result is given by the following theorem.

Theorem 2.1. *Let (H1), (H2) and (H3) hold. If $a > (\frac{2}{1+\alpha})\lambda_1$, then there exists a positive constant $c^* := c^*(a, A, \alpha, \gamma, \Omega)$ such that for $c \le c^*$, problem (1.1) has a positive solution, where λ_1 is the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions.*

Proof. We shall establish our result by constructing positive sub-supersolutions to equation (1.1). From an anti-maximum principle (see [10, pages 155-156]), there exists a $\sigma(\Omega) > 0$ such that the solution z_λ of

$$\begin{cases} -\Delta z - \lambda z = -1, & x \in \Omega,\\ z = 0 & x \in \partial\Omega \end{cases}$$

for $\lambda \in (\lambda_1, \lambda_1 + \sigma)$ is positive in Ω and is such that $\frac{\partial z}{\partial v} < 0$ on $\partial\Omega$, where v is outward normal vector on $\partial\Omega$. Fix $\lambda^* \in (\lambda_1, \min\{\lambda_1 + \sigma, (\frac{1+\alpha}{2})a\})$ and let

$$K := \min\left\{ \left(\frac{M_\infty(2+\alpha)}{2b\|z_{\lambda^*}\|_\infty^{\frac{\gamma(\alpha-1)}{1+\alpha}}}\right)^{\frac{1}{\gamma-1}}, \left(\frac{a - (\frac{2}{1+\alpha})\lambda^*}{3M_\infty b\|z_{\lambda^*}\|_\infty^{\frac{2(\gamma-1)}{1+\alpha}}}\right)^{\frac{1}{\gamma-1}},\right.$$
$$\left.\left(\frac{M_\infty(2+\alpha)}{2A\|z_{\lambda^*}\|_\infty^{\frac{\beta(\alpha-1)}{1+\alpha}}}\right)^{\frac{1}{\beta-1}}, \left(\frac{a - (\frac{2}{1+\alpha})\lambda^*}{3M_\infty A\|z_{\lambda^*}\|_\infty^{\frac{2(\beta-1)}{1+\alpha}}}\right)^{\frac{1}{\beta-1}} \right\}.$$

Define $\psi = K z_{\lambda^*}^{\frac{2}{1+\alpha}}$. Then

$$\nabla\psi = K(\frac{2}{1+\alpha}) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \nabla z_{\lambda^*}$$

and

$$\begin{aligned}
& M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx \\
= \; & M\left(\int_\Omega |\nabla\psi|^2\,dx\right) K(\frac{2}{1+\alpha})\int_\Omega \left[\left\{(\frac{1-\alpha}{1+\alpha}) z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}} |\nabla z_{\lambda^*}|^2 + z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \Delta z_{\lambda^*}\right\}\right] w\,dx \\
= \; & M\left(\int_\Omega |\nabla\psi|^2\,dx\right) K(\frac{2}{1+\alpha})\int_\Omega \left[\left\{(\frac{1-\alpha}{1+\alpha}) z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}} |\nabla z_{\lambda^*}|^2 + z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} (1-\lambda^* z_{\lambda^*})\right\}\right] w\,dx \\
\le \; & M_\infty K(\frac{2}{1+\alpha})\int_\Omega \left[\left\{\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right\}\right] w\,dx.
\end{aligned}$$

Let $\delta > 0$, $\mu > 0$ and $m > 0$ be such that $|\nabla z_{\lambda^*}|^2 \ge m$ in $\overline{\Omega}_\delta$ and $z_{\lambda^*} \ge \mu$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$. Let

$$c^* := K^{1+\alpha}\min\left\{M_\infty(\frac{2}{1+\alpha})\left(\frac{1-\alpha}{1+\alpha}\right)m^2, \frac{1}{3}\mu^2\left(a - M_\infty(\frac{2}{1+\alpha})\lambda^*\right)\right\}.$$

Let $x \in \overline{\Omega}_\delta$ and $c \le c^*$. Since $(\frac{2}{1+\alpha})\lambda^* < a$, we have

$$M_\infty K(\frac{2}{1+\alpha})\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} < a\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right). \tag{2.3}$$

From the choice of K, we have

$$\frac{M_\infty}{2}(\frac{2}{1+\alpha}) \ge bK^{\gamma-1}\|z_{\lambda^*}\|_\infty^{\frac{2\gamma(\alpha-1)}{1+\alpha}}, \tag{2.4}$$

$$\frac{M_\infty}{2}(\frac{2}{1+\alpha}) \ge AK^{\beta-1}\|z_{\lambda^*}\|_\infty^{\frac{2\beta(\alpha-1)}{1+\alpha}}. \tag{2.5}$$

By (2.4), (2.5) and (H2), we know that

$$-\frac{M_\infty}{2}K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \le -b\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\gamma, \tag{2.6}$$

$$-\frac{M_\infty}{2}K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \le -A\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\beta \le -f\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right). \tag{2.7}$$

Since $|\nabla z_{\lambda^*}|^2 \ge m$ in $\overline{\Omega}_\delta$, from the choice of c^* we have

$$\begin{aligned}
& -M_\infty K(\tfrac{2}{1+\alpha})\left(\tfrac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\
& \le M_\infty K(\tfrac{2}{1+\alpha})\left(\tfrac{1-\alpha}{1+\alpha}\right)\frac{m^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\
& \le \frac{c}{\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\alpha}.
\end{aligned} \tag{2.8}$$

Hence for c^*, combining (2.3), (2.6), (2.7), (2.8) and (H3) distributing the terms, we obtain

$$\begin{aligned}
& M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx \\
= \quad & M\left(\int_\Omega |\nabla\psi|^2\,dx\right)K(\frac{2}{1+\alpha}) \\
\times \quad & \int_\Omega \left(\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right) w\,dx, \\
& \leq M_\infty K(\frac{2}{1+\alpha})\int_\Omega \Bigg\{\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - \frac{1}{2}K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \\
& -\frac{1}{2}K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \\
& K(\frac{2}{1+\alpha})\left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\Bigg\} w\,dx, \\
& \leq \int_\Omega \left(a\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - b\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\gamma - f\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - \frac{c}{\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\alpha}\right) w\,dx, \\
& \leq \int_\Omega \left(a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^\alpha}\right) w\,dx, \quad \forall w\in W,\ x\in\overline{\Omega}_\delta.
\end{aligned}$$

Next in $\Omega\setminus\overline{\Omega}_\delta$, for $c\leq c^*$ from the choice of c^* and K, we know that

$$\frac{c}{K^\alpha} \leq \frac{1}{3}Kz_{\lambda^*}^2\left(a - (\frac{2}{1+\alpha})\lambda^*\right), \tag{2.9}$$

$$bK^{\gamma-1}z_{\lambda^*}^{\frac{p(\gamma-1)}{1+\alpha}} \leq \frac{1}{3}\left(a - (\frac{2}{1+\alpha})\lambda^*\right), \tag{2.10}$$

and

$$AK^{\beta-1}z_{\lambda^*}^{\frac{2(\beta-1)}{1+\alpha}} \leq \frac{1}{3}\left(a - (\frac{2}{1+\alpha})\lambda^*\right) \tag{2.11}$$

By combining (2.9), (2.10) and (2.11), we obtain

$$
\begin{aligned}
& M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx \\
& = M\left(\int_\Omega |\nabla\psi|^2\,dx\right)K(\frac{2}{1+\alpha}) \\
\times\quad & \int_\Omega \left(\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - z_{\lambda^*}^{\frac{(1-\alpha)}{1+\alpha}} - \left(\frac{(1-\alpha)}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right) w\,dx, \\
& \leq \int_\Omega \left(M_\infty k(\frac{2}{1+\alpha})\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}}\right) w\,dx, \\
& = \int_\Omega \frac{M_\infty}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\sum_{i=1}^{3}\left(\frac{1}{3}k(\frac{2}{1+\alpha})\lambda^* z_{\lambda^*}^2\right) w\,dx, \\
& \leq \int_\Omega \left\{\frac{1}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\left(\frac{1}{3}kz_{\lambda^*}^2 a - \frac{c}{k^\alpha}\right) + kz_{\lambda^*}^2\left(\frac{1}{3}a - bk^{\gamma-1}z_{\lambda^*}^{\frac{2(\gamma-1)}{1+\alpha}}\right)\right. \\
& \left. + kz_{\lambda^*}^2\left(\frac{1}{3}a - Ak^{\beta-1}z_{\lambda^*}^{\frac{2(\beta-1)}{1+\alpha}}\right)\right\} w\,dx, \\
& \leq \int_\Omega \left(a\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - b\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\gamma - f\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - \frac{c}{\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\alpha}\right) w\,dx, \\
& \leq \int_\Omega \left(a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^\alpha}\right) w\,dx, \quad \forall w \in W,\ x \in \Omega\setminus\overline{\Omega}_\delta.
\end{aligned}
$$

Thus ψ is a positive subsolution of (1.1). From (H_1) and $\gamma > 1$, it is straightforward to verify that a sufficiently large constant z = M is a supersolution of (1.1) with $z \geq \psi$. Thus, the proof is complete. □

3. CONCLUSION

This article concerns the existence of positive solutions for class of infinite semipositone problems involving nonlocal operator. we establish our abstract existence result via the method of sub- and super-solutions.Our result in this note improves the previous one [8] in which $M(t) \equiv 1$.

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