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GROUPOID ASSOCIATED TO A SMOOTH MANIFOLD

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Abstract. In this paper, we introduce the structure of a groupoid associated to a vector field on a smooth manifold. We show that in the case of the 1-dimensional manifolds, our groupoid has a smooth structure such that makes it into a Lie groupoid. Using this approach, we associated to every vector field an equivalence relation on the Lie algebra of all vector fields on the smooth manifolds.

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1. Introduction

The concept of a groupoid is a generalization of the concept of a group, the main difference being that not any two elements of a groupoid are composable. Note that groupoids generalize not only the notion of a group but also the notion of a group action. A groupoid can be endowed with the algebraic, geometric or topological structures and in this case we can study the compatibility among these structures and groupoid.

Note that the theory of groupoids has developed in different fields of mathematics. The algebraic, topological and differentiable groupoids play an important role in algebra, measure theory, harmonic analysis,

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differential geometry and symplectic geometry. This can also be seen from a view at the list of references (see [2, 4, 5, 6, 7, 8, 9]).

A set $\mathcal{G}^{(1)}$ has the structure of a *groupoid* with the set of units $\mathcal{G}^{(0)}$, if there are defined maps $\Delta : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$ (the *unit map*), an involution $\iota : \mathcal{G}^{(1)} \to \mathcal{G}^{(1)}$ called the *inverse map* and denoted by $\iota(\alpha) = \alpha^{-1}$, a *target map* $r : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$, a *source map* $s : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ and an associative multiplication $m : (\alpha, \beta) \to \alpha\beta$ defined on the set

$$\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G}^{(1)} \times \mathcal{G}^{(1)} \mid s(\alpha) = r(\beta)\},$$

satisfying the conditions

(i) $s(\alpha) = r(\alpha^{-1}), \quad \alpha\alpha^{-1} = \Delta(r(\alpha))$,

(ii) $r(\Delta(t)) = t = s(\Delta(t)), \ \alpha\Delta(s(\alpha)) = \alpha, \ \Delta(r(\alpha))\alpha = \alpha$,

for all $\alpha \in \mathcal{G}^{(1)}$ and $t \in \mathcal{G}^{(0)}$.

Here, we have only considered groupoids where $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(0)}$ are sets. In some interesting cases, however, they have more structure. For example, they could be topological spaces, in which case $(\mathcal{G}^{(1)}, \mathcal{G}^{(0)})$ is a topological groupoid. In this paper, we will be concerned mainly with the case when $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(0)}$ are smooth manifolds. A *Lie groupoid* or a *differentiable groupoid* is a groupoid $(\mathcal{G}^{(1)}, \mathcal{G}^{(0)})$ such that $\mathcal{G}^{(0)}, \mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ are smooth manifolds, $s, r : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ are smooth submersions, $\Delta : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$ is an immersion, and all other maps are smooth.

Any Lie group is obviously a Lie groupoid, taking as the set of units a set with a single element. Also, if M is a smooth manifold, then $(M \times M, M)$ is a Lie groupoid. It is known that for an arbitrary groupoid $(\mathcal{G}^{(1)}, \mathcal{G}^{(0)})$ there is an equivalence relation on the unit space $\mathcal{G}^{(0)}$. For two elements $t, l \in \mathcal{G}^{(0)}$ the relation $t \sim l$ if and only if $s^{-1}(t) \cap r^{-1}(l) \neq \emptyset$ is an equivalence relation on $\mathcal{G}^{(0)}$.

Note that Lie groupoids are used in the study of manifolds. They are geometric objects that interpolate between differentiable manifolds and Lie groups. To a compact smooth manifold M one associates the commutative algebra $C^\infty(M, \mathbb{R})$ of its differentiable functions, whereas to a Lie group $\mathcal{G}$ one associates the convolution algebra $C_c(\mathcal{G})$ of compactly supported smooth functions on the group. The algebras $C^\infty(M, \mathbb{R})$ and $C_c(\mathcal{G})$ are particular cases of the convolution algebra of a differential groupoid and in this way, differentiable groupoids provide a link between geometry and harmonic analysis (see [3]).

We now give a brief summary of how the paper is organized.

In Section 2 we begin with our basic construction. We construct a groupoid associated to a vector field X on a smooth manifold M. If M is a smooth 1-dimensional manifold, we show that our groupoid admits a smooth structure such that makes it into a Lie groupoid.

Section 3 contains some conclusions. In this section an equivalence relation on the Lie algebra of all vector fields $\chi(M)$ on M is introduced and we will give the conditions that the equivalence classes are abelian Lie subalgebras of the Lie algebra $\chi(M)$.

Our basic reference for groupoids is [1], and for an extensive use of them one can refer to [3].

Throughout this paper, all smooth manifolds are assumed to be real, Hausdorff, and finite-dimensional. All vector fields on manifolds are assumed to be smooth.

2. The structure of a groupoid

Assume that M is a smooth manifold and TM its tangent bundle. Let $\Gamma_M = C^\infty(M, \mathbb{R}^*) \times C^\infty(M, \mathbb{R})$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. For each $p \in M$, let $T_p(M)$ be the space of tangent vectors to M at p.

Fix a vector field X on M, and let Y_p be a non-zero tangent vector in $T_p(M)$. Let $\mathcal{G}_X(Y_p)$ be the set of all $(f, g) \in \Gamma_M$ such that:

$$Y_p(f) = X_p(g) + g(p), \qquad f(p)Y_p \neq g(p)X_p.$$

We introduce the set $\mathcal{G}_X(M)$ as follows:

$$\mathcal{G}_X(M) = \{(p, Y_p, f, g) \bigm| p \in M, Y_p \in T_p(M)\backslash\{0_p\},\ (f, g) \in \mathcal{G}_X(Y_p)\}.$$

Define an equivalence relation over $\mathcal{G}_X(M)$.

We say that two elements (p, Y_p, f, g) and (q, W_q, h, k) of $\mathcal{G}_X(M)$ are equivalent if the following conditions are hold:

$$p = q,\ \ Y_p = W_q, \qquad f(p)Y_p - g(p)X_p = h(q)W_q - k(q)X_q.$$

Let $[a]$ be the equivalence class of any $a \in \mathcal{G}_X(M)$ and the set of all equivalence classes of $\mathcal{G}_X(M)$, is denoted by $\mathcal{G}_X^{(1)}(M)$.

Let $\mathcal{G}^{(0)}(M) = \{(p, Y_p) \in TM \mid Y_p \neq 0_p\}$. We have to show that the pair $(\mathcal{G}_X^{(1)}(M), \mathcal{G}^{(0)}(M))$ has the structure of a groupoid.

Lemma 2.1. *Let M be a smooth manifold and X an arbitrary vector field on M. Then the pair $(\mathcal{G}_X^{(1)}(M), \mathcal{G}^{(0)}(M))$ has the structure of a groupoid, that is, the maps*

$$\Delta(p, Y_p) = [(p, Y_p, 1, 0)], \qquad \textit{the unit map}$$

$$r([(p, Y_p, f, g)]) = (p, Y_p), \qquad \textit{the target map}$$

$$s([(p, Y_p, f, g)]) = (p, f(p)Y_p - g(p)X_p), \qquad \textit{the source map}$$

and the inverse given by

$$\iota([(p, Y_p, f, g)]) = [(p, f(p)Y_p - g(p)X_p, \frac{1}{f}, -\frac{g}{f})],$$

and the multiplication m *given by*

$$m([(p,Y_p,f,g)],[(p,f(p)Y_p-g(p)X_p,h,k)])=[(p,Y_p,fh,gh+k)],$$

satisfy the axioms of a groupoid.

Proof. It is clear that $\Delta:\mathcal{G}^{(0)}(M)\to\mathcal{G}_X^{(1)}(M)$ is well-defined. Assume that $[(p,Y_p,f,g)]\in\mathcal{G}_X^{(1)}(M)$. Since

$$Y_p(f)=X_p(g)+g(p),$$

and $f(p)Y_p(\frac{1}{f})+\frac{Y_p(f)}{f(p)}=Y_p(f\frac{1}{f})=Y_p(1)=0$, so we have

$$\begin{aligned}(f(p)Y_p-g(p)X_p)(\frac{1}{f})+X_p(\frac{g}{f})&=f(p)Y_p(\frac{1}{f})+\frac{Y_p(f)}{f(p)}-\frac{g(p)}{f(p)}\\&=-\frac{g(p)}{f(p)}.\end{aligned}$$

Also, $f(p)Y_p-g(p)X_p\neq 0$ and $\frac{1}{f(p)}(f(p)Y_p-g(p)X_p)+\frac{g(p)}{f(p)}X_p=Y_p\neq 0$. Hence, the map $\iota:\mathcal{G}_X^{(1)}(M)\to\mathcal{G}_X^{(1)}(M)$ is well-defined. Let

$$\mathcal{G}_X^{(2)}(M)=\{(\alpha,\beta)\in\mathcal{G}_X^{(1)}(M)\times\mathcal{G}_X^{(1)}(M)\,\big|\quad s(\alpha)=r(\beta)\}.$$

To show $m(\mathcal{G}_X^{(2)}(M))\subseteq\mathcal{G}_X^{(1)}(M)$, assume that

$$([(p,Y_p,f,g)],[(p,f(p)Y_p-g(p)X_p,h,k)])\in\mathcal{G}_X^{(2)}(M).$$

Since $Y_p(f)=X_p(g)+g(p)$ and $f(p)Y_p(h)=g(p)X_p(h)+X_p(k)+k(p)$, so one can get

$$\begin{aligned}Y_p(fh)-X_p(gh+k)&=Y_p(f)h(p)+k(p)-X_p(g)h(p)\\&=(gh+k)(p).\end{aligned}$$

Also, $Y_p\neq 0$ and $h(p)f(p)Y_p-g(p)h(p)X_p-k(p)X_p\neq 0$. Therefore, the map $m:\mathcal{G}_X^{(2)}(M)\to\mathcal{G}_X^{(1)}(M)$ is well-defined. It is easy to check that m is associative. It is straightforward to check that the following axioms are true

$$\begin{aligned}&s(\alpha)=r(\alpha^{-1}),\qquad \alpha\alpha^{-1}=\Delta(r(\alpha)),\\&r(\Delta(t))=t=s(\Delta(t)),\\&\alpha\Delta(s(\alpha))=\alpha,\qquad \Delta(r(\alpha))\alpha=\alpha,\end{aligned}$$

for all $\alpha\in\mathcal{G}_X^{(1)}(M)$ and $t\in\mathcal{G}^{(0)}(M)$. $\square$

We illustrate our construction in the case $M=\mathbb{R}$ and $X=\frac{\partial}{\partial x}$.

Example 2.2. Apply the above lemma to $M = \mathbb{R}$ and $X = \frac{\partial}{\partial x}$. One can identify $\mathcal{G}^{(0)}(\mathbb{R})$ with $\mathbb{R} \times \mathbb{R}^*$ and $\mathcal{G}_X^{(1)}(\mathbb{R})$ with $\mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^*$ and their maps are as follows:

$$\Delta(x^1, x^2) = (x^1, x^2, x^2), \qquad r(x^1, x^2, x^3) = (x^1, x^2),$$
$$s(x^1, x^2, x^3) = (x^1, x^3),$$

and the inverse is given by

$$\iota(x^1, x^2, x^3) = (x^1, x^3, x^2),$$

and the multiplication is given by

$$(x^1, x^2, x^3)(x^1, x^3, y) = (x^1, x^2, y).$$

Remark 2.3. *As a consequence of Lemma 2.1, one gets an equivalence relation on $\mathcal{G}^{(0)}(M)$ (see Section 1 above). Fix a vector field X. We say that two elements (p, Y_p) and (q, W_q) of $\mathcal{G}^{(0)}(M)$ are equivalent if and only if there exists a pair $(f, g) \in \Gamma_M$ such that:*

$$p = q, \qquad f(p)W_p = g(p)X_p + Y_p, \qquad W_p(f) = X_p(g) + g(p).$$

The next theorem indicate situations in which $\mathcal{G}_X^{(1)}(M)$ has a smooth structure as a smooth manifold and $(\mathcal{G}_X^{(1)}(M), \mathcal{G}^{(0)}(M))$ is a Lie groupoid.

Theorem 2.4. *Let M be a smooth 1-dimensional manifold and X a vector field on M. Then $\mathcal{G}_X^{(1)}(M)$ has a topology and a smooth structure such that make it into a smooth 3-dimensional manifold. Moreover, $(\mathcal{G}_X^{(1)}(M), \mathcal{G}^{(0)}(M))$ is a Lie groupoid.*

Proof. We begin by defining the maps that will become our smooth charts. Let (x, U) be a smooth coordinate chart on M. For every $p \in U$, write $X_p = X_p^1(\frac{\partial}{\partial x})_p$ in terms of the coordinate basis. Let

$$\widetilde{U} = \{[(p, Y_p, f, g)] \in \mathcal{G}_X^{(1)}(M) \mid p \in U\}.$$

Let $D_1 = \mathbb{R}^* \times \mathbb{R}^*$. If we define $\widetilde{x} : \widetilde{U} \to x(U) \times D_1$ as

$$\widetilde{x}([(p, a(\frac{\partial}{\partial x})_p, f, g)]) = (x(p), a, f(p)a - g(p)X_p^1),$$

then $\widetilde{x}$ is a bijection, with the inverse $\widetilde{x}^{-1} : x(U) \times D_1 \to \widetilde{U}$ by setting

$$\widetilde{x}^{-1}(x(p), a, b) = [(p, a(\frac{\partial}{\partial x})_p, f, g)],$$

where $(f, g) \in \Gamma_M$ is a pair with the following properties:

$$f(p)a = g(p)X_p^1 + b, \qquad a(\frac{\partial f}{\partial x})_p = X_p(g) + g(p).$$

It is easy to check that the pair (f, g) exists, and $\widetilde{x}^{-1}$ is well-defined.

We can therefore use $\widetilde{x}$ to transfer the topology of $x(U) \times D_1$ to $\widetilde{U}$: a set A in $\widetilde{U}$ is open if and only if $\widetilde{x}(A)$ is open in $x(U) \times D_1$. Let $\mathcal{B}$ be the collection of all open subsets of $\widetilde{U_i}$ as U_i runs over all coordinate open sets in M. Then $\mathcal{B}$ satisfies the conditions for a collection of subsets to be a basis for some topology on $\mathcal{G}_X^{(1)}(M)$.

Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis of M consisting of coordinate open sets. Since

$$\widetilde{U_i} = \{[(p, Y_p, f, g)] \in \mathcal{G}_X^{(1)}(M) \mid p \in U_i\} \simeq U_i \times D_1,$$

it is diffeomorphic to an open subset of $\mathbb{R}^3$ and is therefore second countable. For each i, choose a countable basis $\{B_{i,j}\}_{j=1}^{\infty}$ for $\widetilde{U_i}$. Then $\{B_{i,j}\}_{i,j=1}^{\infty}$ is a countable basis for $\mathcal{G}_X^{(1)}(M)$. Now, suppose we are given two charts (x, U) and (y, V) for M, and let $(\widetilde{x}, \widetilde{U}), (\widetilde{y}, \widetilde{V})$ be the corresponding charts on $\mathcal{G}_X^{(1)}(M)$. Then the sets

$$\widetilde{x}(\widetilde{U} \cap \widetilde{V}) = x(U \cap V) \times D_1,$$

and

$$\widetilde{y}(\widetilde{U} \cap \widetilde{V}) = y(U \cap V) \times D_1,$$

are both open subsets of $\mathbb{R}^3$. If $\widetilde{U} \cap \widetilde{V} \neq \emptyset$, then the transition maps $\widetilde{y} \circ \widetilde{x}^{-1}$ and $\widetilde{x} \circ \widetilde{y}^{-1}$ are clearly smooth. Therefore, if $\{(x_i, U_i)\}$ is a smooth atlas for M, then $\{(\widetilde{x}_i, \widetilde{U}_i)\}$ is a smooth atlas for $\mathcal{G}_X^{(1)}(M)$ and $\mathcal{G}_X^{(1)}(M)$ is a smooth 3-dimensional manifold.

Also, $\mathcal{G}^{(0)}(M)$ is a smooth 2-dimensional manifold and its smooth atlas is as follows. Given any coordinate chart (x, U) for M and set

$$\widetilde{\widetilde{U}} = \{(p, Y_p) \in \mathcal{G}^{(0)}(M) \mid p \in U\}.$$

If we define $\widetilde{\widetilde{x}} : \widetilde{\widetilde{U}} \to \mathbb{R}^2$ by

$$\widetilde{\widetilde{x}}(p, a(\frac{\partial}{\partial x})_p) = (x(p), a),$$

then its image set is $x(U) \times \mathbb{R}^*$, which is an open subset of $\mathbb{R}^2$. It is a bijection onto its image, because its inverse can be written explicitly as

$$(x(p), a) \mapsto (p, a(\frac{\partial}{\partial x})_p).$$

One can check that if $\{(x_i, U_i)\}$ is a smooth atlas for M, then $\{(\widetilde{\widetilde{x}}_i, \widetilde{\widetilde{U}}_i)\}$ is a smooth atlas for $\mathcal{G}^{(0)}(M)$. Also, $\mathcal{G}_X^{(2)}(M)$ is a smooth 4-dimensional submanifold of $\mathcal{G}_X^{(1)}(M) \times \mathcal{G}_X^{(1)}(M)$ and $s, r : \mathcal{G}_X^{(1)}(M) \to \mathcal{G}^{(0)}(M)$ are smooth submersions, $\Delta : \mathcal{G}^{(0)}(M) \to \mathcal{G}_X^{(1)}(M)$ is an immersion, and the maps $\iota : \mathcal{G}_X^{(1)}(M) \to \mathcal{G}_X^{(1)}(M)$ and $m : \mathcal{G}_X^{(2)}(M) \to \mathcal{G}_X^{(1)}(M)$ are smooth. Hence, $(\mathcal{G}_X^{(1)}(M), \mathcal{G}^{(0)}(M))$ is a Lie groupoid. $\square$

3. An equivalence relation on the Lie algebra of all vector fields on a smooth manifold

Let M be a smooth manifold and $\chi(M)$ be the Lie algebra of all vector fields on M. In this section, we introduce an equivalence relation on $\chi(M)$ associated to a vector field X. The idea comes from Remark 2.3. We say that two elements Y and W of $\chi(M)$ are equivalent iff there exists a pair $(f,g) \in \Gamma_M$ such that:

$$fW = gX + Y, \qquad W(f) = X(g) + g,$$

and write $(f,g) : Y \to W$, where Γ_M is defined as before by $\Gamma_M = C^\infty(M, \mathbb{R}^*) \times C^\infty(M, \mathbb{R})$.

We have $(1,0) : Y \to Y$, for every $Y \in \chi(M)$. If $(f,g) : Y \to W$, then $(\frac{1}{f}, \frac{-g}{f}) : W \to Y$. Also, if $(f,g) : Y \to W$ and $(h,k) : W \to Z$, then $(fh, fk+g) : Y \to Z$. Therefore, we obtain an equivalence relation on $\chi(M)$.

Let $[Y]_X$ be the equivalence class of any $Y \in \chi(M)$. Let us give an example.

Example 3.1. The vector field X on $\mathbb{R}^2$ defined in terms of the identity chart x by

$$X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$$

has integral curves $\gamma = (z_1 \exp t, z_2 \exp t)$. Let ζ be the zero vector field, defined by $\zeta(f) = 0$ for each $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$. We have

$$[\zeta]_X = \{\frac{g}{f} X \mid (f,g) \in \Gamma_{\mathbb{R}^2},\ gX(f) - fX(g) = fg\}.$$

Assume that $W = \frac{g}{f} X \in [\zeta]_X$ such that g is a non-zero function. Since $\frac{d}{dt}(\frac{g}{f} \circ \gamma) = X(\frac{g}{f}) \circ \gamma$, it follows that

$$f(z_1, z_2) g(z_1 \exp t, z_2 \exp t) \exp t = f(z_1 \exp t, z_2 \exp t) g(z_1, z_2),$$

on $\mathbb{R}$, for all $(z_1, z_2) \in \mathbb{R}^2$. Hence, we have

$$f(z_1, z_2) g(z_1 s, z_2 s) s = f(z_1 s, z_2 s) g(z_1, z_2),$$

for all $(z_1, z_2) \in \mathbb{R}^2$ and all $s > 0$. Since $\frac{g}{f} \neq 0$ we can choose $z \in \mathbb{R}^2$ such that $\frac{g(z)}{f(z)} \neq 0$. Then $\lim_{s \to 0} |\frac{g(sz)}{f(sz)}|$ would be infinite and this would imply that $\frac{g}{f}$ is not continuous at 0 and this is a contradiction. Hence, we have $[\zeta]_X = \{\zeta\}$.

In general, one can check that the equivalence classes are not subspaces of $\chi(M)$. Now, we will give the conditions that the equivalence classes are Lie subalgebras of the Lie algebra $\chi(M)$.

Theorem 3.2. *Assume that $X, Y \in \chi(M)$. Then the following properties are equivalent:*

(i) $[Y]_X$ is an abelian Lie subalgebra of the Lie algebra $\chi(M)$,

(ii) $[Y]_X = \{fX \mid f \in C^\infty(M,\mathbb{R}),\ X(f) = -f\}$,

(iii) there is a smooth function $e \in C^\infty(M,\mathbb{R})$ such that $Y = -eX$ and $X(e) = -e$.

Proof. It is easy to check that $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$. So, it suffices to show only $(iii) \Longrightarrow (i)$. Let $e \in C^\infty(M,\mathbb{R})$ such that $Y = -eX$ and $X(e) = -e$. Assume that $W, Z \in [Y]_X$. Therefore, there are the pairs $(f,g), (h,k) \in \Gamma_M$ such that $(f,g) : Y \to W, (h,k) : Y \to Z$. We have to show that $(s,t) = (fh, fk + hg + e(1-h-f)) : Y \to W + Z$. It follows that

$$s(W+Z) - tX = Y,$$

and

$$\begin{aligned}(W+Z)(s) - X(t) &= X(g)h + hg + (g-e)X(h)\\ &\quad + (k-e)X(f) + fX(k) + fk\\ &\quad - X(f)k - fX(k) - X(g)h - gX(h)\\ &\quad - X(e) + X(e)h + eX(h) + X(e)f + eX(f)\\ &= t.\end{aligned}$$

Hence, $W + Z \in [Y]_X$. On the other hand, it is simple to see that $\lambda W \in [Y]_X$, for all $\lambda \in \mathbb{R}$. Therefore, $[Y]_X$ is a subspace of the space $\chi(M)$. Also,

$$\begin{aligned}fh[W,Z] &= (g-e)X(k-e)X - (g-e)\underbrace{X(h)\frac{1}{h}(k-e)}_{X(k)+k}X\\ &\quad - (k-e)X(g-e)X + (k-e)\underbrace{X(f)\frac{1}{f}(g-e)}_{X(g)+g}X\\ &= 0.\end{aligned}$$

Since $fh \in C^\infty(M,\mathbb{R}^*)$, it follows that $[W,Z] = 0$. □

A non-zero function $h \in C^\infty(M,\mathbb{R})$ such that $X(h) = \lambda h$, for some real number λ, is said to be an *eigenfunction* of the vector field X and λ is called the corresponding *eigenvalue*. Note that a non-zero function $h \in C^\infty(M,\mathbb{R})$ is an eigenfunction of a vector field X corresponding to a zero eigenvalue if and only if it is constant on the range of every integral curve.

Example 3.3. (i) Any vector field X on a compact manifold M has all its eigenvalues zero. Let $\zeta : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ be the zero vector field. Using Theorem 3.2, one gets $[\zeta]_X = \{\zeta\}$.

(ii) The vector field X on $\mathbb{R}^2$ defined in terms of the identity chart x by

$$X = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$$

has every eigenvalue zero, since its integral curves

$$\gamma = (a \sin(t + b), a \cos(t + b))$$

are all periodic, that is, there exists $r > 0$ such that $\gamma(t_1) = \gamma(t_2)$ if and only if $t_1 - t_2 = kr$ for some $k \in \mathbb{Z}$. Let $\zeta : C^\infty(\mathbb{R}^2, \mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R})$ be the zero vector field. Hence, from Theorem 3.2, we have $[\zeta]_X = \{\zeta\}$.

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