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# The Two-Parameter Exponentiated-Sine Modified Lindley Distribution: A Flexible Model for Reliability Analysis

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Abstract. This paper introduces a two-parameter generalization of the sine-modified Lindley distribution by incorporating a shape parameter $\alpha$. This extended model retains the flexibility of the sine-modified Lindley distribution while enhancing its ability to model a wider range of lifetime and reliability data. The distribution's properties, including the probability density function, cumulative distribution function, survival function, moments, and hazard rate function, are derived. Maximum likelihood estimation is used for parameter estimation, and its performance is evaluated through

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simulations. The model is applied to real-world datasets and compared with existing lifetime distributions using goodness-of-fit criteria. Results demonstrate its superiority in modeling skewed and heavy-tailed data.

Keywords: Sine modified Lindley distribution; Lifetime modelling; Reliability analysis; Two-parameter extension; Maximum likelihood estimation; Hazard function; Survival analysis.

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## 1. Introduction

In recent years, the increasing complexity of real-world data has driven the need for more sophisticated statistical tools. Traditional probability distributions form the cornerstone of classical statistical modeling; however, their often rigid assumptions can hinder accurate representation of data exhibiting skewness, heavy tails, or multimodal characteristics. To address these challenges, researchers have developed several generalized families of distributions. These extended models preserve desirable properties of classical distributions while offering greater flexibility, thereby improving both theoretical depth and practical applicability across a wide range of disciplines.

As a result, generalized probability distributions have been modified to better accommodate complex data structures. By introducing additional parameters to existing distributions, these extensions have significantly improved the modeling of natural phenomena. Various researchers have contributed to this effort, including the development of the Lomax-power Rayleigh ($t$-$x$) distribution by [3], the odd Lindley power Rayleigh distribution by [5], an extension of the exponentiated exponential distribution using the alpha power transformation by [2], and the SMP Kumaraswamy distribution introduced by [16].

Among the many generalized families, the sine-G family has gained prominence for its flexibility and effectiveness in modeling a wide range of real-world data. Prominent instances include the sine exponential distribution by [18], sine Kumaraswamy-G family by [6], sine-modified Lindley distribution by [24], sine power Lomax distribution by [23], and the sine power Rayleigh distribution by [21].

The sine-modified Lindley (SML) distribution extends the modified Lindley (ML) distribution introduced by [24] through the addition of a sine transformation, significantly improving its ability to model diverse data patterns. Although the ML distribution effectively represents increasing, reverse bathtub (unimodal), and constant hazard rate functions, the range of shape variability remains somewhat limited. Many extensions have been developed to enhance the flexibility and practical applicability of the ML distribution. For instance, [7] introduced the inverted modified Lindley (IML) distribution, [13] proposed the Marshall–Olkin ML distribution, [9] developed the wrapped ML distribution, [8] presented the weighted ML distribution, and the power ML distribution was independently proposed by [10]. Additionally, a three-parameter ML (TPML) distribution, obtained by mixing the modified Weibull and generalized gamma distributions, was presented in [19], the power inverted ML distribution was introduced by [17], the extended slash ML distribution by [12], and the two-parameter sine modified Lindley (TPSML) distribution was developed by [11].

The cumulative distribution function (cdf) and probability density function (pdf) of the SML distribution are given, respectively, by

$$F(x;\theta) = \begin{cases} \cos\left[\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right], & x > 0, \\ 0, & x \leq 0, \end{cases} \tag{1}$$

and

$$f(x;\theta) = \begin{cases} \frac{\pi}{2}\ \frac{\alpha\theta}{1+\theta}e^{-2\theta x}\left[(1+\theta)e^{\theta x}+2\theta x-1\right]\sin\left[\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right], & x > 0, \\ 0, & x \leq 0, \end{cases} \tag{2}$$

with $\theta > 0$.

This paper introduces a new flexible distribution, the two-parameter exponentiated sine-modified Lindley (ESML) distribution, which extends the SML model through an additional shape parameter $\alpha$. The proposed distribution significantly enhances the flexibility of the baseline model.The inclusion of this parameter gives the ESML distribution greater flexibility and enhances its ability to model a wide range of data patterns, making it especially suitable for survival analysis, reliability engineering, and risk assessment. The motivation for proposing the ESML distribution arises from its key advantages. It provides high flexibility, accommodating a wide range of density shapes from sharply peaked to heavy-tailed. Its hazard function can display increasing, decreasing, constant, or bathtub patterns, making it suitable for diverse reliability scenarios. Furthermore, it effectively handles skewed data and demonstrates superior performance compared to well-known competing models on two real-life data sets.

The remainder of this paper is structured as follows. Section 2 introduces the proposed ESML distribution. Sections 3–5 examine its reliability properties, moments, and order statistics. Section 6 discusses parameter estimation based on maximum likelihood. Section 7 presents a simulation study. Section 8 provides applications to real data sets, and concluding remarks are given in Section 9.

## 2. Exponentiated Sine-Modified Lindley Distribution

The exponentiated technique proposed by [22] enhances flexibility by introducing a shape parameter to the baseline distribution. Let G(x) and g(x) denote the cdf and pdf of the baseline distribution, respectively. The corresponding cdf and pdf of the exponentiated G-family of distributions are then defined, respectively, as:

$$F(x) = [G(x)]^{\alpha}, \qquad \alpha > 0, \tag{3}$$

and

$$f(x) = \alpha\, g(x)\, [G(x)]^{\alpha-1}, \qquad \alpha > 0. \tag{4}$$

Equation (3) is also well-known in the literature as the proportional reversed hazard rate model, which has been extensively studied by several authors [14], [15], and [4].

By taking the SML distribution as the baseline, with cdf and pdf given in equations (1) and (2), the ESML distribution has the following cdf and pdf, respectively:

$$F(x;\alpha,\theta) = \begin{cases} \left[\cos\left(\frac{\pi}{2}\left(1+\dfrac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right]^{\alpha}, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{5}$$

and

$$f(x;\alpha,\theta) = \frac{\pi}{2}\frac{\alpha\theta}{1+\theta}e^{-2\theta x}\left[(1+\theta)e^{\theta x} + 2\theta x - 1\right]\sin\left[\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right] \times \left[\cos\left(\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right]^{\alpha-1}, \qquad x > 0. \tag{6}$$

Clearly, for $\alpha = 1$, the ESML distribution reduces to the original SML distribution. Plots of the pdf of the ESML distribution for selected values of $\theta$ and $\alpha$ , are shown in Figure 1. The plots demonstrate the distribution's flexibility, showing a variety of shapes including sharply decreasing, nearly uniform, and unimodal forms. Low $\alpha$ with small $\theta$ produces a sharp peak near zero, while higher values of $\alpha$ flatten the density. Moderate parameter values generate a unimodal shape, and larger $\theta$ values spread the density over a wider range.

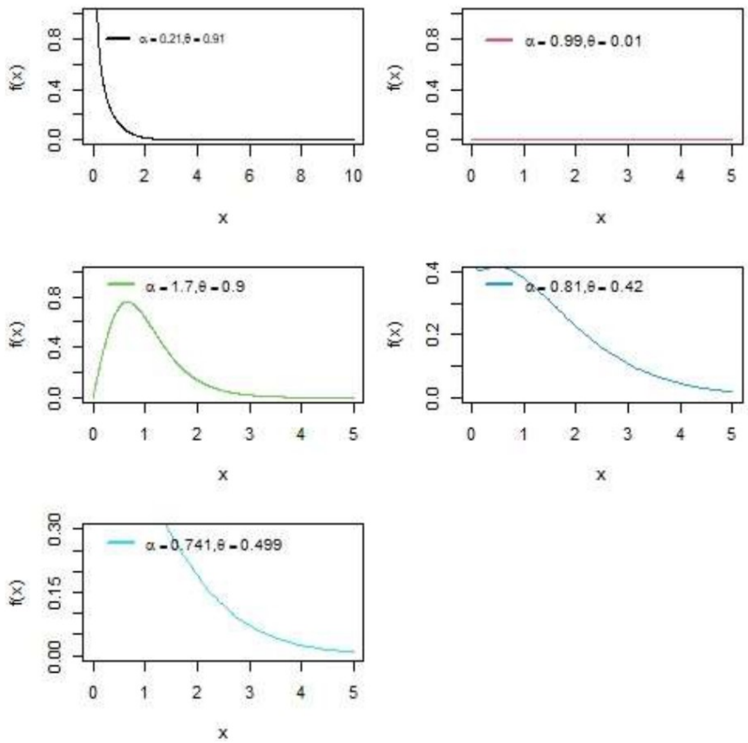


FIGURE 1. Plots of the pdf of the ESML distribution for some selected values of parameters $\alpha$ and $\theta$.

## 3. Reliability Analysis

In this section, we discuss several reliability functions for the ESML distribution, including the survival function, hazard rate function, reverse hazard rate, second rate of failure, and cumulative hazard rate function.

3.1. **Survival Function.** The survival function measures the probability that a component survives beyond time $x$. For the ESML distribution, it is

$$S(x;\alpha,\theta) = \begin{cases} 1 - \left[\cos\left(\frac{\pi}{2}\left(1 + \frac{\theta x}{1+\theta} e^{-\theta x}\right) e^{-\theta x}\right)\right]^{\alpha}, & x > 0, \\ 1, & x \leq 0. \end{cases} \tag{7}$$

Further, Figure 2 shows the behavior of the survival function for some selected values of parameters $\alpha$ and $\theta$. The results show that parameter choices strongly affect survival behavior: some configurations produce

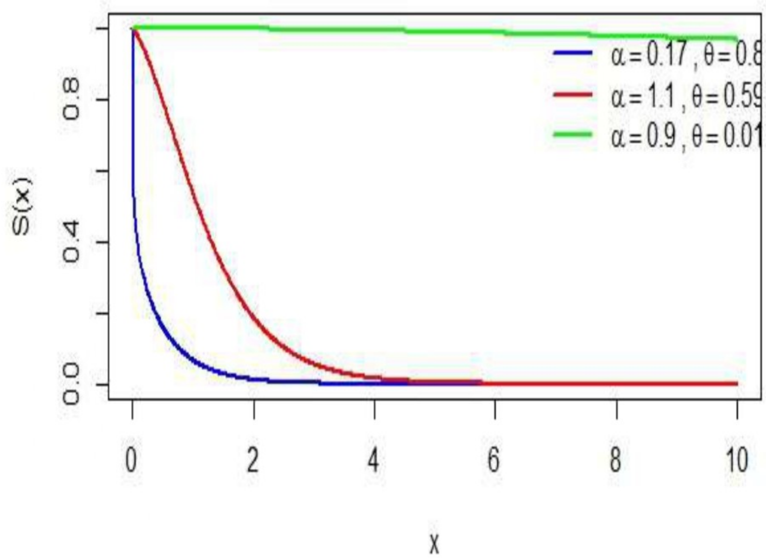


FIGURE 2. Survival function plots for some selected values of parameters $\alpha$ and $\theta$.

rapid early decline, while others yield slower decay or near-constant survival, reflecting notable differences in risk behavior across the models.

3.2. **Hazard Rate Function.** The hazard rate function is defined as

$$h(x;\alpha,\theta) = \frac{f(x;\alpha,\theta)}{S(x;\alpha,\theta)},$$

which for the ESML distribution becomes

$$h(x;\alpha,\theta) = \begin{cases} \dfrac{\frac{\pi}{2}\frac{\alpha\theta}{1+\theta}e^{-2\theta x}\left[(1+\theta)e^{\theta x}+2\theta x-1\right]\sin\left[\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right]\left[\cos\left(\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right]^{\alpha-1}}{1-\left[\cos\left(\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right]^{\alpha}}, & x>0, \\ 0, & x\le 0. \end{cases} \tag{8}$$

Based on the hazard rate expression, Figure 3 demonstrates the flexibility of the model in producing various hazard behaviors depending on the parameter values, including sharply decreasing, unimodal (increasing then decreasing), approximately constant near-zero, and gradually decreasing patterns.

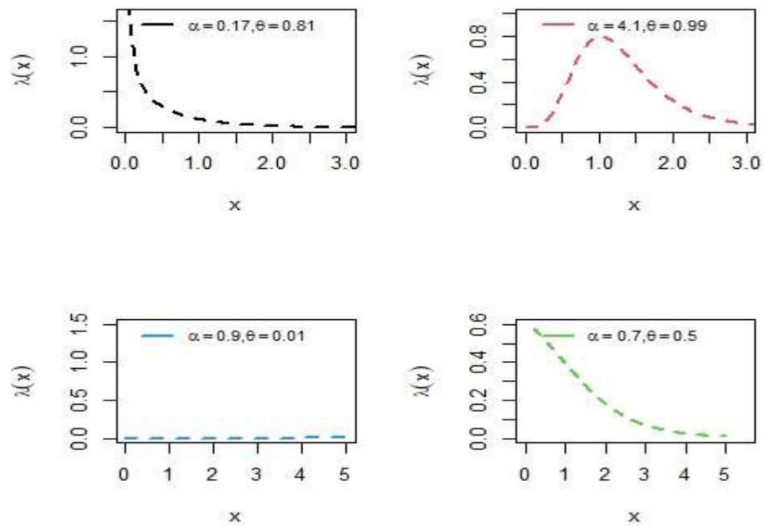


FIGURE 3. Hazard rate function plots for some selected values of parameters $\alpha$ and $\theta$.

3.3. **Reverse Hazard Rate Function.** The reverse hazard rate function is

$$h_r(x;\alpha,\theta) = \frac{f(x;\alpha,\theta)}{F(x;\alpha,\theta)} = \begin{cases} \dfrac{f(x;\alpha,\theta)}{\left[\cos\left(\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right]^{\alpha}}, & x > 0, \\ 0, & x \le 0. \end{cases} \tag{9}$$

3.4. **Second Rate of Failure.** The second rate of failure is

$$h^*(x;\alpha,\theta) = \begin{cases} \ln \frac{S(x;\alpha,\theta)}{S(x+1;\alpha,\theta)}, & x > 0, \\ 0, & x \le 0. \end{cases} \tag{10}$$

3.5. **Cumulative Hazard Rate Function.** The cumulative hazard function is

$$\Lambda(x;\alpha,\theta) = \begin{cases} -\ln S(x;\alpha,\theta) = -\ln\left[1-\left(\cos\left(\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right)^{\alpha}\right], & x > 0, \\ 0, & x \le 0. \end{cases} \tag{11}$$

## 4. Moment Analysis

The moments of the ESML distribution provide measures of central tendency, dispersion, skewness, and kurtosis.

**Theorem 4.1.** *The rth moment of the ESML distribution is*

$$E(X^r) = r\int_0^\infty x^{r-1}\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^k\binom{n}{k}\binom{k}{j}\frac{1}{2^k}\cos\left[(k-2j)\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right]dx. \tag{12}$$

*Proof.* Using

$$E(X^r) = \int_0^\infty x^r f(x;\alpha,\theta)\,dx = r\int_0^\infty x^{r-1}S(x;\alpha,\theta)\,dx,$$

and applying the binomial expansion along with the trigonometric identity $\cos^k(x) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\cos\big((k-2j)x\big)$, we obtain equation (12). □

From the above moment formulas, we can easily derive the mean, variance, moment skewness coefficient, and moment kurtosis coefficient. Table 1 presents selected values corresponding to different $\alpha$ and $\theta$ parameters. From Table 1, we can observe that as $\alpha$ increases (with $\theta$ fixed at 0.1), both the mean and variance of the ESML distribution increase, indicating a shift toward higher values and greater dispersion. In contrast, skewness and kurtosis decrease, suggesting that the distribution becomes less asymmetric and exhibits lighter tails for higher$\alpha$ values. These observations highlight the significant influence of $\alpha$ on the shape and spread of the ESML distribution.

Table 1. Mean, variance, skewness, and kurtosis of the ESML distribution for selected $\alpha$ and $\theta$.

| $\alpha$ | $\theta$ | Mean | Variance | Skewness | Kurtosis |
|---|---|---|---|---|---|
| 0.1 | 0.1 | 1.4616 | 9.8898 | 3.7078 | 21.9433 |
| 0.4 | 0.1 | 4.5044 | 22.6848 | 1.8071 | 7.6018 |
| 0.7 | 0.1 | 6.5136 | 27.4196 | 1.4128 | 5.8819 |
| 1.0 | 0.1 | 7.8200 | 30.7064 | 1.5224 | 6.7904 |

## 5. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the ESML distribution, and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the corresponding order statistics. The pdf of the $r$th order statistic is given by

$$f_{r:n}(x;\alpha,\theta) = \frac{n!}{(r-1)!(n-r)!}[F(x;\alpha,\theta)]^{r-1}[1-F(x;\alpha,\theta)]^{n-r}f(x;\alpha,\theta), \quad x>0. \tag{14}$$

For the ESML distribution, the pdf of the rth order statistic can be derived and, by applying a trigonometric expansion, simplified to

$$f_{r:n}(x;\alpha,\theta)=\frac{n!}{(r-1)!(n-r)!}f(x;\alpha,\theta)\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j}\left[\cos\left(\frac{\pi}{2}\left(1+\frac{\theta x}{1+\theta}e^{-\theta x}\right)e^{-\theta x}\right)\right]^{\alpha(j+r-1)}. \tag{15}$$

Using equation (15), the $s$th order moment of of $X_{(r:n)}$ can be expressed as

$$E(X_{(r:n)}^{s})=\int_{0}^{\infty}x^{s}f_{r:n}(x;\alpha,\theta)\,dx.$$

While a closed-form expression is not available, numerical techniques can be used to obtain accurate approximations.

## 6. Parameter Estimation

Let $x_1,x_2,\ldots,x_n$ be a $n$ independent observations from the ESML distribution. The likelihood function is defined as

$$L(\alpha,\theta)=\prod_{i=1}^{n}f(x_i;\alpha,\theta),$$

with the corresponding log-likelihood given by

$$\ell(\alpha,\theta)=\sum_{i=1}^{n}\ln f(x_i;\alpha,\theta).$$

The maximum likelihood estimates (MLEs) of $\alpha$ and $\theta$ can be obtained by solving the following non-linear equations:

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\cos\left(\frac{\pi}{2}\frac{\theta x_i}{1+\theta}e^{-\theta x_i}\right)=0,$$

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta(1+\theta)}-2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\frac{(1+\theta)e^{\theta x_i}+2\theta x_i}{(1+\theta)e^{\theta x_i}+2\theta x_i-1}+\cdots=0,$$

where $U_i=\frac{\pi}{2}\left[x_ie^{-2\theta x_i}\left(\frac{1-2\theta x_i(1+\theta)}{(1+\theta)^2}\right)-x_ie^{-\theta x_i}\right]$.

Since these equations do not have closed-form solutions, the maximum likelihood estimates of $\alpha$ and $\theta$ must be obtained numerically, using iterative methods such as the Newton-Raphson algorithm.

TABLE 2. Simulation results for $\alpha = 1.8$, $\theta = 2.2$

| 2*$n$ | Estimate | | MSE | | Bias | | Variance | |
|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $\theta$ | $\alpha$ | $\theta$ | $\alpha$ | $\theta$ | $\alpha$ | $\theta$ |
| 25 | 2.137617 | 2.376995 | 0.707236 | 0.232265 | 0.337617 | 0.176995 | 0.593251 | 0.200938 |
| 150 | 1.857472 | 2.236056 | 0.073757 | 0.039345 | 0.057472 | 0.036055 | 0.070454 | 0.038045 |
| 600 | 1.805591 | 2.206224 | 0.013201 | 0.007703 | 0.005590 | 0.006223 | 0.013170 | 0.007665 |

TABLE 3. Simulation results for $\alpha = 2.5$, $\theta = 1.5$

| 2*$n$ | Estimate | | MSE | | Bias | | Variance | |
|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $\theta$ | $\alpha$ | $\theta$ | $\alpha$ | $\theta$ | $\alpha$ | $\theta$ |
| 25 | 3.319804 | 1.714418 | 2.268891 | 0.084430 | 0.819803 | 0.214418 | 1.596811 | 0.038455 |
| 150 | 2.840104 | 1.617157 | 0.192596 | 0.015426 | 0.340103 | 0.117157 | 0.076925 | 0.001701 |
| 600 | 2.782966 | 1.600803 | 0.094715 | 0.010187 | 0.282966 | 0.100802 | 0.014644 | 0.000026 |

## 7. Numerical Illustration

7.1. **Simulation.** In this section, a simulation study is carried out using R software to explore the effectiveness and performance of the maximum likelihood estimates of the ESML distribution. Samples of sizes $n =$ 25, 150, and 600 were generated, with 500 repetitions for each sample size from the ESML distribution under two parameter settings: ($\alpha =$ 1.8, $\theta = 2.2$) and ($\alpha = 2.5$, $\theta = 1.5$).

For each parameter combination, the average MLEs along with their corresponding bias, variance, and mean squared errors (MSEs) were computed. The numerical results are displayed in Tables 2 and 3.

From the simulation tables, the following conclusions can be drawn as the sample size $n$ increases:

- The estimates stabilize around the true parameter values.
- The bias decreases, demonstrating increasing accuracy.
- The MSE decreases, confirming the consistency of the estimators.

7.2. **Applications.** This section illustrates the practical application of the proposed ESML distribution. To evaluate its effectiveness, we conduct a comparative analysis against several well-known competing models using two real-life datasets. The models considered for comparison include the SML distribution proposed by [24], the sine power Rayleigh (SPR) distribution introduced by [21], and the MTI inverted exponential (MTIIE) distribution developed by [1].

Model performance is then assessed using standard information criteria, including the Akaike Information Criterion (AIC), its corrected version (AICc), the Bayesian Information Criterion (BIC), the Hannan–Quinn Information Criterion (HQIC), and the corrected Bayesian Information Criterion (BICc). All model parameters are estimated using the

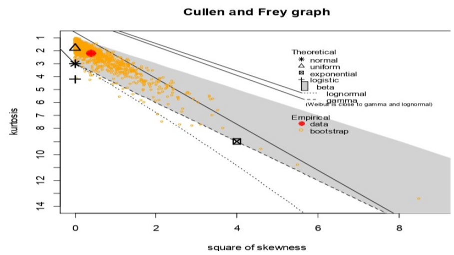


FIGURE 4. Cullen and Frey plot for data set 1.

maximum likelihood method. These measures are computed based on the log-likelihood, the number of estimated parameters, and the sample size, with lower values indicating a better fit. The criteria are defined as follows:

$$\text{AIC} = -2\hat{\ell} + 2k, \quad \text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}, \quad \text{BIC} = -2\hat{\ell} + k\ln(n),$$

$$\text{HQIC} = -2\hat{\ell} + 2k\ln(\ln(n)), \quad \text{BICc} = -2\hat{\ell} + k\ln(n) + \frac{k(k+1)}{n-k-1},$$

where $\hat{\ell}$ is the maximized log-log-likelihood, $k$ is the number of estimated parameters, and $n$ is the sample size.

The analyses are conducted on the following datasets:

*Data set 1:* The first data set, from [18] represents the failure times (minutes) of 15 electronic components in an accelerated life test and is given as: {1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23.0, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2}. The descriptive statistical measures of this dataset are presented in Table 4. Figures 4 and 5 show the Cullen and Frey plot and the histogram for data set 1, respectively.

TABLE 4. Descriptive statistics for data set 1.

| Count | Min | Max | Mean | Median | SD | IQR |
|---|---|---|---|---|---|---|
| 15 | 1.4 | 66.2 | 27.6 | 22.2 | 20.7 | 31.9 |

*Data Set 2.* The second data set, from [18] corresponds to the remission times (in weeks) for 30 leukemia patients receiving similar treatment and is given as:{1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31, 42, 45, 50, 57, 60, 71, 85, 91}. The descriptive statistical measures of this dataset are presented in Table 5. Figures 6 and 7 show the Cullen and Frey plot and the histogram for data set 2, respectively.

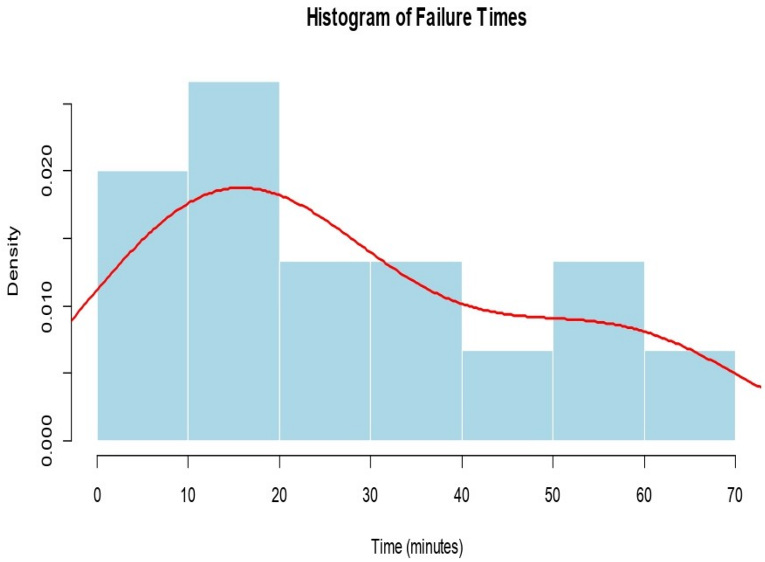


Figure 5. Histogram plot for data set 1.

Table 5. Descriptive statistics for data set 2.

| Count | Min | Max | Mean | Median | SD | IQR |
|---|---|---|---|---|---|---|
| 30 | 1 | 91 | 26.4 | 18.5 | 23.5 | 28.5 |

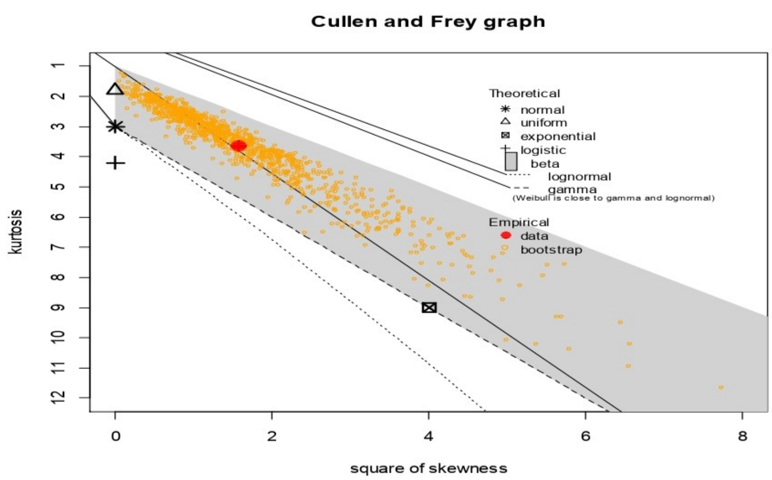


Figure 6. Cullen and Frey plot for data set 2.

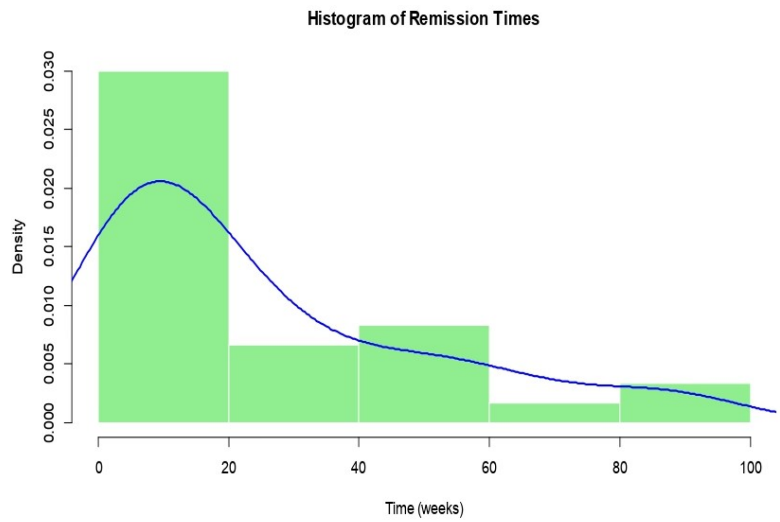


FIGURE 7. Histogram plot for data set 2.

We can observe from Tables 6 and 7, the ESML distribution consistently achieves the lowest information criteria values, confirming its superior fit compared to the competing models for both datasets. This superiority is further supported by graphical tools, including histograms, Cullen and Frey plots, and Kaplan–Meier survival estimates.

The Cullen and Frey plots in Figures 4 and 6 indicate that the empirical data exhibit skewness and kurtosis patterns similar to gamma, lognormal, or Weibull distributions, indicating that the flexible ESML distribution is appropriate for modeling these data.

Kaplan–Meier survival curves in Figure 8 highlight distinct patterns between the two datasets. For electronic component failure times, survival probability declines rapidly, reaching zero near 66 minutes, whereas for leukemia remission times, survival decreases more gradually, with some patients remaining in remission beyond 80 weeks.

These graphical analyses are valuable complements to numerical criteria. Cullen and Frey plots help identify the theoretical distribution that best matches the data by comparing sample skewness and kurtosis to expected regions, supporting parametric model selection. Kaplan–Meier curves provide non-parametric estimates of survival probabilities over time, enabling visual assessment of survival patterns and differences across groups. Together, these tools help detect deviations, outliers, and distributional features not evident from summary statistics, improving model selection and interpretation.

TABLE 6. Model comparison criteria for data set 1.

| Model | Estimates | LogLik | AIC | BIC | AICc | HQIC | BICC |
|---|---|---|---|---|---|---|---|
| ESML | $\alpha = 0.6621,\ \theta = 0.0235$ | -64.1233 | 132.2467 | 133.6628 | 133.2467 | 132.2316 | 134.6628 |
| SML | $\alpha = 2.1000,\ \theta = 0.0289$ | -64.9191 | 133.8382 | 135.2543 | 134.8382 | 133.8231 | 136.2543 |
| SPR | $\beta = 0.001,\ \theta = 0.001$ | -345.3878 | 694.7755 | 696.1916 | 695.7755 | 694.7604 | 697.1916 |
| MTIIE | $\alpha = 0.3220,\ \lambda = 3.8479$ | -67.5562 | 139.1125 | 140.5286 | 140.1125 | 274.2099 | 141.5286 |

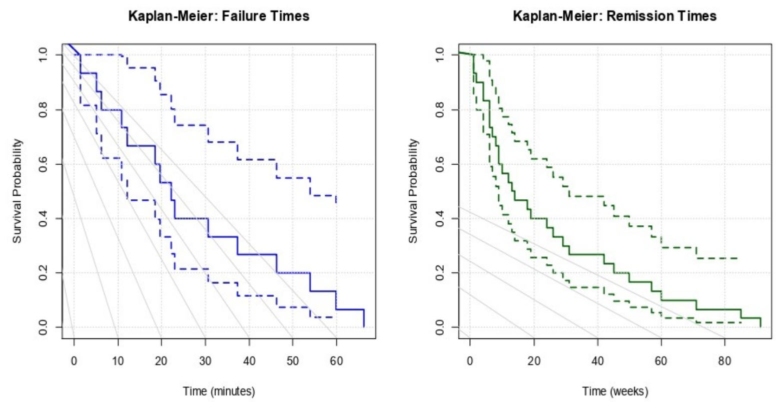


FIGURE 8. Kaplan–Meier survival estimates for data set 1 and data set 2.

These findings demonstrate the flexibility of the ESML distribution in modeling both rapidly failing systems and long-term survival phenomena.

TABLE 7. Model comparison criteria for data set 2.

| **Model** | **Estimates** | **LogLik** | **AIC** | **BIC** | **AICC** | **HQIC** | **BICC** |
|---|---|---|---|---|---|---|---|
| ESML | $\alpha = 0.4497; \theta = 0.0199$ | $-127.5860$ | 259.1720 | 261.9744 | 259.6164 | 260.0685 | 262.4188 |
| SML | $\alpha = 2.1000; \theta = 0.0315$ | $-134.6053$ | 273.2105 | 276.0129 | 273.6550 | 274.1070 | 276.4574 |
| SPR | $\beta = 0.0130; \theta = 3.8591$ | $-303.4614$ | 610.9227 | 613.7251 | 611.3672 | 611.8192 | 614.1696 |
| MTIIE | $\alpha = 0.2904; \theta = 2.4181$ | $-129.2190$ | 262.4379 | 265.2403 | 262.8824 | 521.7724 | 265.6848 |

## 8. CONCLUSION

In this article, a new lifetime distribution, namely the two-parameter exponentiated sine-modified Lindley (ESML) distribution, is introduced and studied. The ESML distribution provides a more flexible probability density function, and its hazard rate function exhibits a variety of shapes. Several structural properties and reliability characteristics of the distribution have been derived.

For parameter estimation, the maximum likelihood estimation method has been employed, and a simulation study has been conducted to evaluate the performance of the estimators. Furthermore, the usefulness of the proposed model is demonstrated by applying it to two real-life datasets and comparing its performance with existing lifetime distributions. The results indicate that the ESML distribution possesses wide applicability and outperforms the competing models in fitting the datasets considered, thereby offering valuable potential in various areas of applied research.

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