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Hesitant T-spherical Fuzzy Hamacher Aggregation Operators and Their Applications in Multiple-Criteria Group Decision Making

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Abstract. Building upon the foundation of Hesitant Fuzzy (HF) sets and T-Spherical Fuzzy (T-SFS) sets, this paper introduces the Hesitant T-Spherical Fuzzy (HT-SFS) set and presents its fundamental set-theoretical and Hamacher operations. To provide a flexible and general framework for aggregating HT-SF information, this paper leverages the generalized Hamacher t-norm and t-conorm, which subsume several other operational laws as special cases. We introduce a family of novel aggregation operators based on these Hamacher operations, including the hesitant T-spherical fuzzy Hamacher weighted arithmetic averaging (HTSFHWAA) operator, hesitant T-spherical fuzzy Hamacher weighted geometric averaging (HTSFHWGA) operator, hesitant T-spherical fuzzy Hamacher ordered weighted arithmetic averaging (HTSFHOWAA) operator,

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and hesitant T-spherical fuzzy Hamacher ordered weighted geometric averaging (HTSFHOWGA) operator. The fundamental properties of these operators are investigated. Additionally, we provide a multi-criteria group decision-making (MCGDM) method and algorithm of the suggested method under the hesitant T-spherical fuzzy environment. To demonstrate the proposed method, we present an example related to filling a project manager position in child protection at an international organization. In addition, we provide a comparative analysis with existing operators to reveal the advantages and validity of our technique. The results confirm the validity and flexibility of the proposed approach, demonstrating its effectiveness in handling complex group decision-making problems.

Keywords: Hesitant fuzzy set, T-spherical fuzzy set, HT-spherical fuzzy set, Hamacher operator, decision-making.
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1. Introduction

In order to model specific problems involving uncertainty, Zadeh [1] introduced the concept of fuzzy set (FS) in 1965. The FS has applications in a wide range of disciplines, including data mining, clustering, robotics, and computer science. An FS is defined by Zadeh [1] as a membership function whose codomain lies within the interval $[0, 1]$. If an element in an FS has a membership degree (MsD) of p, then it has a non-membership degree (NMsD) of $1 - p$. This viewpoint has certain limitations. Therefore, Atanassov [2] suggested the idea of intuitionistic FS (IFS) as a generalization of FSs to get over these restrictions.

An IFS is defined by assigning two functions in the range ($[0, 1]$), referred as membership function (p) and non-membership function (r), under the condition that $0 \leq p + r \leq 1$ for all constituents of the functional universe. However, if $p + r > 1$, this set is useless. Therefore, Yager [3, 4] presented the Pythagorean FS (PyFS) as an expansion of IFS under the constraint $0 \leq p^2 + r^2 \leq 1$. The Picture FS (PFS), proposed by Cuong [5, 6], is another expansion of IFS. It is an effective method for simulating human perception, since a PFS can replicate evaluations of a thing or a suggestion utilizing variations in acceptance, rejection, abstention, and yes. With the criterion $0 \leq p + q + r \leq 1$, belongingness of an element in a PFS is expressed by three values, referred to as membership degree (MsD) p, abstinence degree (AD) q and nonmemership degree (NMsD) r. Despite the fact that PFS has numerous applications in areas like decision-making (DM), [7, 9, 10, 11, 12, 13, 14], similarity measure [15, 16, 18, 17, 19], correlation coefficient [20, 21], and clustering [22, 23], if $p + q + r > 1$, it is insufficient for modeling various issues. Consequently, G ungogdu and Kahraman [24, 25] introduced the Spherical Fuzzy Set (SFS) as an advancement of the PFS, satisfying the condition $0 \leq p^2 + q^2 + r^2 \leq 1$. Additionally, they researched SFS operations and how to use this set to solve DM issues.

By combining the SFS and TOPSIS methods, Kahraman et al. [24] suggested a DM technique and provided an example of how it was used to choose the location of a hospital. Mahmood et al. [26], first introduced the T-spherical FS (T-SFS), generalizing the SFS with the constraint $0 \leq p^t + q^t + r^t \leq 1$, and demonstrated its applications in medical diagnosis and decision-making. The similarity measurements for T-SFSs were first introduced by Ullah et al. [27], the individual also has shown proficiency in the development of a pattern recognition program. Garg et al. [28] proposed a set of improved interactive aggregation operators for T-SFSs and investigated their operational principles. In their study, Ullah et al. [30] provided a comprehensive overview of several hybrid geometric (HG) and ordered weighted geometric (OWG) operators. Additionally, they demonstrated the use of these operators using a numerical example that included a multi-attribute DM issue. Ullah et al. [33] established the idea of interval-valued T-SFS (IVT-SFS) and provided an overview of its fundamental operations. In addition, the authors introduced two aggregation operators on IVT-SFS, namely weighted geometric and weighted averaging operators, and proposed a multi-criteria decision making (MCDM) technique. The SFS and T-SFS operating rules have a number of flaws that Liu et al. [34] highlighted, and they suggested some new operational laws for these systems. The Power Muirhead Mean for T-SFS was formulated by the integration of the Power Average and the Muirhead Mean Operators. Additionally, the researchers introduced an MAGDM approach that relies on the aforementioned operators. The topic of T-SFS has garnered significant interest among scholars who are studying various aspects of multi-criteria decision-making (MCDM) methods, multi-criteria group decision-making (MCGDM) methods, and aggregation operators. For instance, the T-SFSs' divergence metric [31], instantaneous probabilistic T-SFS operators for interactive averaging aggregation [32]. For instance, Quek et al. (2019) explored the use of generalized T-SF weighted aggregation operators on neutrosophic sets. Guleria and Bajaj [35] introduced T-SF soft sets and related aggregation operators. Other studies have examined correlation coefficients for T-SFSs [39], T-SF Hamacher aggregation operators [40], T-SF Einstein hybrid aggregation operators [37], and complex T-SF aggregation operators [41, 38]. Another extension of the FS used to model issues when decision-makers have conflicting views on an alternative or component of the universe under consideration is the hesitant FS (HFS). Torra and Narukawa defined the HFS in [42, 43].

We use the following scenario to illustrate the fundamental principle behind the concept of the hesitant fuzzy set: two decision-makers are debating whether to give an element in a set a membership grade of 0.7 or 0.3. It can be challenging to reach a consensus in certain situations. The HFS is a helpful tool in this situation. Numerous researchers have created various decision-making methods and their applications in HF environments as a result of the benefits of HFS [44, 45, 46, 47, 48, 49]. In their research, Xia et al. [50] provided a comprehensive description of many hesitant fuzzy (HF) aggregation operators and presented the development of a system designed for facilitating group decision-making. The concept of interval-valued hesitant fuzzy sets (IvHFSs), which is an extension of hesitant fuzzy sets (HFS), was proposed by Chen *et al.* [51]. In their study, Peng *et al.* [52] used the continuous OWA operator to investigate the aggregation operators in continuous HF. They introduced the C-HFOWG operator and C-HFOWA operator, highlighting their distinctive characteristics. These operators were also expanded to include interval-valued HFS. For hesitant fuzzy elements, Mu *et al.* [53] introduced an innovative aggregation principle. Amin *et al.* [54] proposed many aggregation methods for trapezoidal linguistically hesitant fuzzy (TrCHF) numbers and triangular cubic linguistically hesitant fuzzy sets. Fahmi *et al.* [55] introduced novel operation rules and aggregation operators specifically designed for trapezoidal cubic hesitant fuzzy (TrCHF) numbers. Using the Hamacher t-conorm and t-norm, Jiang *et al.* [56] established the notion of interval-valued Dual HFS and provided a comprehensive discussion on aggregation processes within the context of interval-valued Dual HF settings. References to scholarly research on the topic of the growth of HFS, the aggregation operator of HFS, and decision-making may be found in the publication of [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 72, 73]. Liu *et al.* [57] introduced the Dombi aggregation operations for interval-valued hesitant fuzzy collections by using the Dombi t-conorm and t-norm. In our previous study, [73], we introduced the concept of (HT-SFS) and we presented some aggregation operators based on Dombi operators.

In this article, we also present some aggregation operators based on Hamacher operators, such as HTSFHWAA, HTSFHWGA, HTSFHOWAA, and HTSFHOWGA. We also provide an algorithm for the suggested method using (MCGDM) method under the HT-SF environment. We use the selection of the best candidate for a project manager position in child protection at an international organization as an example to demonstrate the steps of the suggested method. Additionally, we provided a comparison table that compares the suggested clusters with other expansions of the FS as well as a comparison of the proposed approaches with one another.

2. Hesitant T-spherical Fuzzy Sets

In this section, some basic notions of TSFS and terms related to this work are presented.

Definition 2.1. Let Ψ be a nonempty fixed set.

- A hesitant fuzzy set (HFS) on Ψ is an object having the form
$$\vartheta = \{< \mathfrak{x}, \mathfrak{h}_\vartheta(\mathfrak{x}) > | \mathfrak{x} \in \Psi\},$$
where $\mathfrak{h}_\vartheta(\mathfrak{x})$ is a set that contains values belongs to $[0,1]$, which is called the membership degree of $\mathfrak{x}$ in ϑ. In this case, the set $\mathfrak{h} = \mathfrak{h}_\vartheta(\mathfrak{x})$ is called a hesitant fuzzy element (HFE) [42, 43].

- A T-spherical fuzzy set (TSFS) on Ψ has a mathematical representation
$$\vartheta = \{(\mathfrak{x}, p(\mathfrak{x}), q(\mathfrak{x}), r(\mathfrak{x})) | \mathfrak{x} \in \Psi\},$$
where $p, q, r : \Psi \longrightarrow [0,1]$ are three functions which are called the membership degree, neutral degree and non-membership degree of $\mathfrak{x}$, respectively. Also, for some fixed $t \in \mathbb{Z}^+$ the following condition holds:
$$0 \leq p^t(\mathfrak{x}) + q^t(\mathfrak{x}) + r^t(\mathfrak{x}) \leq 1.$$
In this case, $\kappa(\mathfrak{x}) = \sqrt[t]{1 - \big(p^t(\mathfrak{x}) + q^t(\mathfrak{x}) + r^t(\mathfrak{x})\big)}$ is called the refusal degree of $\mathfrak{x}$ in ϑ and the triplet (p, q, r) is known as T-spherical fuzzy number (T-SFN) [26].
- A hesitant T-spherical fuzzy set (HTSFS) over Ψ is an object
$$\digamma = \Big\{(\mathfrak{x}, h(\mathfrak{x})) : \mathfrak{x} \in \Psi\Big\},$$
such that $h(\mathfrak{x})$ is a set that contains finite number of T-SFNs for all $\mathfrak{x} \in \Psi$. The set $h = h(\mathfrak{x})$ is called a hesitant T-spherical fuzzy element (HTSFE). In a special case, the following HTSFSs are called the empty and universal HTSFS, respectively [73]
$$\emptyset = \Big\{(\mathfrak{x}, \{(0,0,1)\}) : \mathfrak{x} \in \Psi\Big\}\ ,\ \mathcal{U} = \Big\{(\mathfrak{x}, \{(1,0,0)\}) : \mathfrak{x} \in \Psi\Big\}.$$

The length of an HTSFE $h(\mathfrak{x})$ will be denoted by the symbol ℓ_h. It is the total number of elements in the $h(\mathfrak{x})$.

In the rest of this section, we define the basic set-theoretic operations (union $\Cup$, intersection $\Cap$, and subset $\Subset$) for HTSFSs based on score value comparisons. These operations are distinct from the Hamacher operations ($\oplus$ and $\otimes$) that will be introduced in Section 3 for aggregating

HT-SFEs. The set-theoretic operations provide a foundation for comparing and combining HTSFSs, while the Hamacher operations enable flexible aggregation based on t-norm and t-conorm theory.

Definition 2.2. [73] Let $a = (p, q, r)$ be a TSFE. Then, the score value and accuracy value of a are $SC(a) = p^t - r^t$ and $AC(a) = p^t + q^t + r^t$, respectively. If h is an HT-SFE then the score value (respectively, accuracy value) of h is defined to be the average of score values (respectively, accuracy values) of fuzzy numbers in h, that is

$$SC(h) = \frac{1}{\ell_h} \sum_{a \in h} SC(a) \ , \ AC(h) = \frac{1}{\ell_h} \sum_{a \in h} AC(a). \tag{2.1}$$

Also, the upper and lower bounds of h are defined as

$$h^+ = \max\{SC(a) : a \in h\} \ , \ h^- = \min\{SC(a) : a \in h\},$$

respectively.

It is evident that $SC(h)$ lies between -1 and 1 and it increases monotonically when p increases and decreases monotonically when r decreases.

To compare HT-SFEs, we propose a comparison rule based on score and accuracy values. Note that if the score and accuracy value of HT-SFEs are the same, they represent the same information.

Definition 2.3. [73] Let h_1 and h_2 be two HT-SFEs. Then,

(1) If $SC(h_1) < SC(h_2)$ then $h_1 < h_2$;
(2) If $SC(h_1) = SC(h_2)$, there are three cases:
  (a) If $AC(h_1) < AC(h_2)$, then $h_1 < h_2$;
  (b) If $AC(h_1) > AC(h_2)$, then $h_1 > h_2$;
  (c) If $AC(h_1) = AC(h_2)$, then $h_1 \approx h_2$.

Definition 2.4. Let $\digamma_1 = \{(\mathfrak{x}, h_1(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$ and $\digamma_2 = \{(\mathfrak{x}, h_2(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$ be two HT-SFSs over the set Ψ. Then,

- If for any $\mathfrak{x} \in \Psi$, we have $h_1(\mathfrak{x}) < h_2(\mathfrak{x})$, then we say that $\digamma_1$ is an HT-SF subset of $\digamma_2$ and we denote it by $\digamma_1 \Subset \digamma_2$.
- If for any $\mathfrak{x} \in \Psi$, we have $h_1(\mathfrak{x}) \approx h_2(\mathfrak{x})$, then we say that $\digamma_1$ and $\digamma_2$ are equivalent and we denote it by $\digamma_1 \equiv \digamma_2$.

Example 2.5. Let $\digamma_1$ and $\digamma_2$ be two HT-SFSs over $\Psi = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3\}$, for $n = 3$ given as follows:

$$\begin{aligned}
\digamma_1 &= \{(\mathfrak{x}_1, \{(0.7, 0.2, 0.6), (0.6, 0.5, 0.4), (0.5, 0.4, 0.3)\}), \\
&\quad (\mathfrak{x}_2, \{(0.7, 0.1, 0.6), (0.6, 0.2, 0.4), (0.5, 0.5, 0.2)\}), \\
&\quad (\mathfrak{x}_3, \{(0.9, 0.5, 0.1), (0.5, 0.5, 0.5), (0.6, 0.3, 0.5)\})\} \\
\digamma_2 &= \{(\mathfrak{x}_1, \{(0.8, 0.3, 0.6), (0.7, 0.5, 0.5), (0.7, 0.6, 0.5)\}), \\
&\quad (\mathfrak{x}_2, \{(0.6, 0.2, 0.2), (0.8, 0.3, 0.3), (0.5, 0.7, 0.1)\}), \\
&\quad (\mathfrak{x}_3, \{(0.9, 0.2, 0.2), (0.5, 0.1, 0.3), (0.7, 0.3, 0.3)\})\}.
\end{aligned}$$

Then, for $\mathfrak{x}_i \in \Psi (i = 1, 2, 3)$, the score values of HT-SFEs are obtained by using Eq. (2.1):

	$\mathfrak{x}_1$	$\mathfrak{x}_2$	$\mathfrak{x}_3$
$SC(h_1)$	0.126	0.132	0.272
$SC(h_2)$	0.244	0.272	0.378

From this table, it is clear that $\digamma_1 \Subset \digamma_2$.

Definition 2.6. [73] Let $\digamma_1 = \{(\mathfrak{x}, h_1(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$ and $\digamma_2 = \{(\mathfrak{x}, h_2(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$ be two HT-SFSs over Ψ. Then, the set theoretical operations between $\digamma_1$ and $\digamma_2$ are defined as follows:

(1) Union: $\digamma_1 \Cup \digamma_2 = \{(\mathfrak{x}, h_{\Cup}(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$, where

$$h_{\Cup}(\mathfrak{x}) = h_1(\mathfrak{x}) \vee h_2(\mathfrak{x}) := \begin{cases} h_1(\mathfrak{x}) & h_1(\mathfrak{x}) > h_2(\mathfrak{x}) \\ h_2(\mathfrak{x}) & h_2(\mathfrak{x}) > h_1(\mathfrak{x}) \\ h_1(\mathfrak{x}) & h_2(\mathfrak{x}) \approx h_1(\mathfrak{x}) \end{cases}$$

(2) Intersection: $\digamma_1 \Cap \digamma_2 = \{(\mathfrak{x}, h_{\Cap}(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$, where

$$h_{\Cap}(\mathfrak{x}) = h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x}) := \begin{cases} h_1(\mathfrak{x}) & h_1(\mathfrak{x}) < h_2(\mathfrak{x}) \\ h_2(\mathfrak{x}) & h_2(\mathfrak{x}) < h_1(\mathfrak{x}) \\ h_1(\mathfrak{x}) & h_2(\mathfrak{x}) \approx h_1(\mathfrak{x}) \end{cases}$$

(3) Complement:

$$\digamma_1^c = \{(\mathfrak{x}, h_1^c) : \mathfrak{x} \in \Psi\},$$

where

$$h_1^c = \bigcup_{(p_1, q_1, r_1) \in h_1} \{(r_1, q_1, p_1)\}.$$

Example 2.7. Let $\digamma_1$ and $\digamma_2$ be two HT-SFSs over $\Psi = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3\}$, for $n = 3$, given as follows:

$$\begin{aligned}
\digamma_1 &= \{(\mathfrak{x}_1, \{(0.8, 0.3, 0.6), (0.7, 0.5, 0.5), (0.7, 0.6, 0.5)\}),\\
&\quad (\mathfrak{x}_2, \{(0.7, 0.1, 0.6), (0.6, 0.2, 0.4), (0.5, 0.5, 0.2)\}),\\
&\quad (\mathfrak{x}_3, \{(0.9, 0.2, 0.2), (0.5, 0.1, 0.3), (0.7, 0.3, 0.3)\})\}\\
\digamma_2 &= \{(\mathfrak{x}_1, \{(0.7, 0.2, 0.6), (0.6, 0.5, 0.4), (0.5, 0.4, 0.3)\}),\\
&\quad (\mathfrak{x}_2, \{(0.6, 0.2, 0.2), (0.8, 0.3, 0.3), (0.5, 0.7, 0.1)\}),\\
&\quad (\mathfrak{x}_3, \{(0.9, 0.5, 0.1), (0.5, 0.5, 0.5), (0.6, 0.3, 0.5)\})\}.
\end{aligned}$$

Then, for $\mathfrak{x}_i \in \Psi(i = 1, 2, 3)$, the score values of HT-SFEs are obtained by using Eq. (2.1):

	$\mathfrak{x}_1$	$\mathfrak{x}_2$	$\mathfrak{x}_3$
$SC(h_1)$	0.244	0.132	0.378
$SC(h_2)$	0.126	0.272	0.272

From this table and by using Definition 2.6, we can find

$$\begin{aligned}
\digamma_1 \Cup \digamma_2 &= \{(\mathfrak{x}_1, \{(0.8, 0.3, 0.6), (0.7, 0.5, 0.5), (0.7, 0.6, 0.5)\}),\\
&\quad (\mathfrak{x}_2, \{(0.6, 0.2, 0.2), (0.8, 0.3, 0.3), (0.5, 0.7, 0.1)\}),\\
&\quad (\mathfrak{x}_3, \{(0.9, 0.2, 0.2), (0.5, 0.1, 0.3), (0.7, 0.3, 0.3)\})\}.
\end{aligned}$$

$$\begin{aligned}
\digamma_1 \Cap \digamma_2 &= \{(\mathfrak{x}_1, \{(0.7, 0.2, 0.6), (0.6, 0.5, 0.4), (0.5, 0.4, 0.3)\}),\\
&\quad (\mathfrak{x}_2, \{(0.7, 0.1, 0.6), (0.6, 0.2, 0.4), (0.5, 0.5, 0.2)\}),\\
&\quad (\mathfrak{x}_3, \{(0.9, 0.5, 0.1), (0.5, 0.5, 0.5), (0.6, 0.3, 0.5)\})\}.
\end{aligned}$$

$$\begin{aligned}
\digamma_1^c &= \{(\mathfrak{x}_1, \{(0.6, 0.3, 0.8), (0.5, 0.5, 0.7), (0.5, 0.6, 0.7)\}),\\
&\quad (\mathfrak{x}_2, \{(0.6, 0.1, 0.7), (0.4, 0.2, 0.6), (0.2, 0.5, 0.5)\}),\\
&\quad (\mathfrak{x}_3, \{(0.2, 0.2, 0.9), (0.3, 0.1, 0.5), (0.3, 0.3, 0.7)\})\}.
\end{aligned}$$

We remark that definitions of union and intersection are unique up to equivalence relation on HTSFSs, that is $\digamma_1 \Cup \digamma_2 \equiv \digamma_2 \Cup \digamma_1$ and $\digamma_1 \Cap \digamma_2 \equiv \digamma_2 \Cap \digamma_1$.

Now, we can obtain the following properties.

Proposition 2.8. *Let $\digamma_1, \digamma_2$ and $\digamma_3$ be HTSFSs. Then using the notations of the above, the following statements hold:*
(i) If $\digamma_1 \Subset \digamma_2$, then $\digamma_1 \Cup \digamma_2 = \digamma_2$ and $\digamma_1 \Cap \digamma_2 = \digamma_1$.
(ii) $\digamma_1 \Cup \emptyset = \digamma_1$ and $\digamma_1 \Cap \emptyset = \emptyset$.
(iii) $\mathcal{U} \Cap \digamma_1 = \digamma_1$ and $\mathcal{U} \Cup \digamma_1 = \mathcal{U}$.
(iv) $(\digamma_1^c)^c = \digamma_1$.
(v) If $\digamma_1 \Subset \digamma_2$ and $\digamma_2 \Subset \digamma_3$ then $\digamma_1 \Subset \digamma_3$.

Proof. We prove each property separately.

(i) Assume $\mathcal{F}_1 \Subset \mathcal{F}_2$. Then for every $\mathfrak{x} \in \Psi$, we have $h_1(\mathfrak{x}) < h_2(\mathfrak{x})$ according to the comparison rule. Recall from Definition 2.4 that $h_1(\mathfrak{x}) < h_2(\mathfrak{x})$ means either $SC(h_1(\mathfrak{x})) < SC(h_2(\mathfrak{x}))$, or $SC(h_1(\mathfrak{x})) = SC(h_2(\mathfrak{x}))$ and $AC(h_1(\mathfrak{x})) < AC(h_2(\mathfrak{x}))$. Now, the union $\mathcal{F}_1 \Cup \mathcal{F}_2$ is defined such that for each $\mathfrak{x}$, $h_{\Cup}(\mathfrak{x})$ selects the larger of the two HT-SFEs based on the same comparison rule. Since $h_1(\mathfrak{x}) < h_2(\mathfrak{x})$ for all $\mathfrak{x}$, we have $h_{\Cup}(\mathfrak{x}) = h_2(\mathfrak{x})$ for every $\mathfrak{x} \in \Psi$. Hence, $\mathcal{F}_1 \Cup \mathcal{F}_2 = \mathcal{F}_2$. Similarly, by Definition 2.6, the intersection $\mathcal{F}_1 \Cap \mathcal{F}_2$ selects the smaller HT-SFE for each $\mathfrak{x}$. Since $h_1(\mathfrak{x}) < h_2(\mathfrak{x})$ for all $\mathfrak{x}$, we have $h_{\Cap}(\mathfrak{x}) = h_1(\mathfrak{x})$ for every $\mathfrak{x} \in \Psi$. Hence, $\mathcal{F}_1 \Cap \mathcal{F}_2 = \mathcal{F}_1$.

(ii) For any $\mathfrak{x} \in \Psi$, the empty HTSFS $\emptyset$ is defined as having $h(\mathfrak{x}) = \{(0,0,1)\}$. By the comparison rule, for any non-empty HTSFS $\mathcal{F}_1$, we have $SC(h_1(\mathfrak{x})) > SC(\{(0,0,1)\}) = -1$ for all $\mathfrak{x}$. Thus, by the union definition, $h_1(\mathfrak{x}) \vee \{(0,0,1)\} = h_1(\mathfrak{x})$ for all $\mathfrak{x}$, yielding $\mathcal{F}_1 \Cup \emptyset = \mathcal{F}_1$. Similarly, by the intersection definition, $h_1(\mathfrak{x}) \wedge \{(0,0,1)\} = \{(0,0,1)\}$ for all $\mathfrak{x}$, yielding $\mathcal{F}_1 \Cap \emptyset = \emptyset$.

(iii) The universal HTSFS $\mathcal{U}$ is defined as having $h(\mathfrak{x}) = \{(1,0,0)\}$ for all $\mathfrak{x} \in \Psi$. By the comparison rule, we have $SC(\{(1,0,0)\}) = 1$, which is the maximum possible score. Thus, for any HTSFS $\mathcal{F}_1$, we have $\{(1,0,0)\} > h_1(\mathfrak{x})$ for all $\mathfrak{x}$ (unless $\mathcal{F}_1 = \mathcal{U}$, in which case they are equal). By the intersection definition, $h_1(\mathfrak{x}) \wedge \{(1,0,0)\} = h_1(\mathfrak{x})$ for all $\mathfrak{x}$, yielding $\mathcal{U} \Cap \mathcal{F}_1 = \mathcal{F}_1$. By the union definition, $h_1(\mathfrak{x}) \vee \{(1,0,0)\} = \{(1,0,0)\}$ for all $\mathfrak{x}$, yielding $\mathcal{U} \Cup \mathcal{F}_1 = \mathcal{U}$.

(iv) By Definition 2.6, the complement of $\mathcal{F}_1$ is $\mathcal{F}_1^c = \{(\mathfrak{x}, h_1^c(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$, where for each $(p,q,r) \in h_1(\mathfrak{x})$, we include (r,q,p) in $h_1^c(\mathfrak{x})$. Taking the complement again, we obtain $(\mathcal{F}_1^c)^c = \{(\mathfrak{x}, (h_1^c)^c(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$, where for each $(r,q,p) \in h_1^c(\mathfrak{x})$, we include (p,q,r) in $(h_1^c)^c(\mathfrak{x})$. Since this operation simply reverses the coordinates twice, it returns the original triple (p,q,r). Hence, $(h_1^c)^c(\mathfrak{x}) = h_1(\mathfrak{x})$ for all $\mathfrak{x}$, yielding $(\mathcal{F}_1^c)^c = \mathcal{F}_1$.

(v) Assume $\mathcal{F}_1 \Subset \mathcal{F}_2$ and $\mathcal{F}_2 \Subset \mathcal{F}_3$. This means that for every $\mathfrak{x} \in \Psi$, we have

$$h_1(\mathfrak{x}) < h_2(\mathfrak{x}) \quad \text{and} \quad h_2(\mathfrak{x}) < h_3(\mathfrak{x}).$$

Consider any $\mathfrak{x} \in \Psi$. Then we have the following possibilities:

- If $SC(h_1(\mathfrak{x})) < SC(h_2(\mathfrak{x}))$ and $SC(h_2(\mathfrak{x})) < SC(h_3(\mathfrak{x}))$, then by transitivity of $<$ on real numbers, $SC(h_1(\mathfrak{x})) < SC(h_3(\mathfrak{x}))$, which implies $h_1(\mathfrak{x}) < h_3(\mathfrak{x})$.
- If $SC(h_1(\mathfrak{x})) = SC(h_2(\mathfrak{x}))$ and $AC(h_1(\mathfrak{x})) < AC(h_2(\mathfrak{x}))$, and similarly $SC(h_2(\mathfrak{x})) = SC(h_3(\mathfrak{x}))$ and $AC(h_2(\mathfrak{x})) < AC(h_3(\mathfrak{x}))$, then by transitivity of $<$ on real numbers applied to the accuracy values, we have $AC(h_1(\mathfrak{x})) < AC(h_3(\mathfrak{x}))$, and since $SC(h_1(\mathfrak{x})) = SC(h_3(\mathfrak{x}))$, this implies $h_1(\mathfrak{x}) < h_3(\mathfrak{x})$.

- Mixed cases (e.g., $SC(h_1(\mathfrak{x})) < SC(h_2(\mathfrak{x}))$ and $SC(h_2(\mathfrak{x})) = SC(h_3(\mathfrak{x}))$ with $AC(h_2(\mathfrak{x})) < AC(h_3(\mathfrak{x}))$) also yield $SC(h_1(\mathfrak{x})) < SC(h_3(\mathfrak{x}))$ by combining the inequalities appropriately.

In all cases, we obtain $h_1(\mathfrak{x}) < h_3(\mathfrak{x})$ for every $\mathfrak{x} \in \Psi$. Therefore $\digamma_1 \Subset \digamma_3$, establishing transitivity. □

Remark 2.9. The transitivity property (v) relies crucially on the fact that the comparison of HT-SFEs is ultimately reducible to comparisons of real numbers (score values and accuracy values). Since the less-than relation on real numbers is transitive, this transitivity is inherited by the subset relation $\Subset$ on HTSFSs. This demonstrates that despite the additional complexity introduced by the "hesitant" nature (where each HTSFE may contain multiple T-SFNs), the ordering defined via score values maintains the essential properties of a partial order, ensuring consistency in set-theoretic operations.

Proposition 2.10. *(De Morgan's laws) Let $\digamma_1$ and $\digamma_2$ be two HT-SFSs and for any $\mathfrak{x} \in \Psi$, $SC(h_1(\mathfrak{x})) \neq SC(h_2(\mathfrak{x}))$. Then,*

(1) $(\digamma_1 \Cup \digamma_2)^c = \digamma_1^c \Cap \digamma_2^c$.
(2) $(\digamma_1 \Cap \digamma_2)^c = \digamma_1^c \Cup \digamma_2^c$.

Proof. (1) Let $(\digamma_1 \Cup \digamma_2)^c = \{(\mathfrak{x}, g(\mathfrak{x}) : \mathfrak{x} \in \Psi)\}$ and $(\digamma_1)^c \Cap (\digamma_2)^c = \{(\mathfrak{x}, k(\mathfrak{x}) : \mathfrak{x} \in \Psi)\}$. Then, for any $\mathfrak{x} \in \Psi$, if $h_1(\mathfrak{x}) > h_2(\mathfrak{x})$ then $SC(h_1(\mathfrak{x})) > SC(h_2(\mathfrak{x})$. Hence $SC(h_1^c(\mathfrak{x})) < SC(h_2^c(\mathfrak{x}))$. This means $g(\mathfrak{x}) = h_1^c(\mathfrak{x}) = k(\mathfrak{x})$. Hence $(\digamma_1 \Cup \digamma_2)^c = \digamma_1^c \Cap \digamma_2^c$. If we consider the situations $h_2(\mathfrak{x}) > h_1(\mathfrak{x})$ and $h_1(\mathfrak{x}) = h_2(\mathfrak{x})$, then proofs can be made in a similar way.
(2) The proof is similar to the proof of (1). □

Proposition 2.11 (Distributive Laws)**.** *Let $\digamma_1, \digamma_2$ and $\digamma_3$ be HTFSs. Then*

(1) $\digamma_1 \Cap (\digamma_2 \Cup \digamma_3) \equiv (\digamma_1 \Cap \digamma_2) \Cup (\digamma_1 \Cap \digamma_3)$.
(2) $\digamma_1 \Cup (\digamma_2 \Cap \digamma_3) \equiv (\digamma_1 \Cup \digamma_2) \Cap (\digamma_1 \Cup \digamma_3)$.

Proof. For $\mathfrak{x} \in \Psi$, consider the HT-SFEs, $h_1(\mathfrak{x}), h_2(\mathfrak{x})$ and $h_3(\mathfrak{x})$ of $\digamma_1$, $\digamma_2$ and $\digamma_3$, respectively. We state the proof for the following cases.

Case 1: Assume that $h_1(\mathfrak{x}) \leq h_2(\mathfrak{x}) \leq h_3(\mathfrak{x})$. Then, for $\mathfrak{x} \in \Psi$

$$h_1(\mathfrak{x}) \wedge (h_2(\mathfrak{x}) \vee h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.2}$$

$$(h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x})) \vee (h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \vee h_1(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.3}$$

From (2) and (3), $\digamma_1 \Cap (\digamma_2 \Cup \digamma_3) \equiv (\digamma_1 \Cap \digamma_2) \Cup (\digamma_1 \Cap \digamma_3)$.

Case 2: Assume that $h_1(\mathfrak{x}) \leq h_3(\mathfrak{x}) \leq h_2(\mathfrak{x})$. Then, for $\mathfrak{x} \in \Psi$

$$h_1(\mathfrak{x}) \wedge (h_2(\mathfrak{x}) \vee h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.4}$$

$$(h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x})) \vee (h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \vee h_1(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.5}$$

From (4) and (5), $F_1 \Cap (F_2 \Cup F_3) \equiv (F_1 \Cap F_2) \Cup (F_1 \Cap F_3)$.

Case 3: Let $h_2(\mathfrak{x}) \leq h_1(\mathfrak{x}) \leq h_3(\mathfrak{x})$. Then, for $\mathfrak{x} \in \Psi$

$$h_1(\mathfrak{x}) \wedge (h_2(\mathfrak{x}) \vee h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.6}$$

$$(h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x})) \vee (h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x})) = h_2(\mathfrak{x}) \vee h_1(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.7}$$

From (6) and (7), $F_1 \Cap (F_2 \Cup F_3) \equiv (F_1 \Cap F_2) \Cup (F_1 \Cap F_3)$.

Case 4: Let $h_2(\mathfrak{x}) \leq h_3(\mathfrak{x}) \leq h_1(\mathfrak{x})$. Then, for $\mathfrak{x} \in \Psi$

$$h_1(\mathfrak{x}) \wedge (h_2(\mathfrak{x}) \vee h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x}) = h_3(\mathfrak{x}) \tag{2.8}$$

$$(h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x})) \vee (h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x})) = h_2(\mathfrak{x}) \vee h_3(\mathfrak{x}) = h_3(\mathfrak{x}) \tag{2.9}$$

From (8) and (9), $F_1 \Cap (F_2 \Cup F_3) \equiv (F_1 \Cap F_2) \Cup (F_1 \Cap F_3)$.

Case 5: Let $h_3(\mathfrak{x}) \leq h_1(\mathfrak{x}) \leq h_2(\mathfrak{x})$. Then, for $\mathfrak{x} \in \Psi$

$$h_1(\mathfrak{x}) \wedge (h_2(\mathfrak{x}) \vee h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.10}$$

$$(h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x})) \vee (h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \vee h_3(\mathfrak{x}) = h_1(\mathfrak{x}) \tag{2.11}$$

From (10) and (11), $F_1 \Cap (F_2 \Cup F_3) \equiv (F_1 \Cap F_2) \Cup (F_1 \Cap F_3)$.

Case 6: Let $h_3(\mathfrak{x}) \leq h_2(\mathfrak{x}) \leq h_1(\mathfrak{x})$. Then, for $\mathfrak{x} \in \Psi$

$$h_1(\mathfrak{x}) \wedge (h_2(\mathfrak{x}) \vee h_3(\mathfrak{x})) = h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x}) = h_2(\mathfrak{x}) \tag{2.12}$$

$$(h_1(\mathfrak{x}) \wedge h_2(\mathfrak{x})) \vee (h_1(\mathfrak{x}) \wedge h_3(\mathfrak{x})) = h_2(\mathfrak{x}) \vee h_3(\mathfrak{x}) = h_2(\mathfrak{x}) \tag{2.13}$$

From (12) and (13), $F_1 \Cap (F_2 \Cup F_3) \equiv (F_1 \Cap F_2) \Cup (F_1 \Cap F_3)$. □

3. Hesitant T-Spherical Fuzzy Hamacher Aggregation Operators

The defined Hamacher operations, i.e. Hamacher product and Hamacher sum are specific types of triangular norms and conorms given in this section.

3.1. **Hamacher operations of HT-SFEs.** t-norm and t-conorm are important notions in fuzzy theory, which are used to define generalized union and intersection of fuzzy sets. In [8], Hamacher proposed axiomatics for continuous many-valued logical connectives "AND" and "OR" when the set of truth values is the unit interval. Hamacher operations lead to define the Hamacher product and Hamacher sum, which are examples of t-norms and t-conorms, respectively.

Definition 3.1. For any $\gamma \geq 0$ the Hamacher t-norm T_γ is defined as

$$T_\gamma(a, b) = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)}.$$

where $a, b \in [0, 1]$. Its t-conorm is

$$S_\gamma(a, b) = 1 - T_\gamma(1-a, 1-b) = \frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}.$$

It is interesting to note that $T_\gamma(a,b) = ab$ is the algebraic t-norm when $\gamma = 1$ and $T_\gamma(a,b) = \frac{ab}{a+b-ab}$ for $\gamma = 0$. Also, $T_2(a,b) = \frac{ab}{1+(1-a)(1-b)}$ is known as Einstein t-norm.

Let γ be a fixed number. Throughout the rest of the paper, for convenience, the Hamacher t-norm is considered as (Hamacher) product and we write $T_\gamma(a,b) = a \otimes b$ while the Hamacher t-conorm is known as the (Hamacher) sum and denoted by $S_\gamma(a,b) = a \oplus b$.

Definition 3.2. Suppose (p_1, q_1, r_1) and (p_2, q_2, r_2) are TSFNs. Then, we define

$$(p_1,q_1,r_1) \oplus (p_2,q_2,r_2) := \left(\sqrt[n]{p_1^n \oplus p_2^n},\ \sqrt[n]{q_1^n \otimes q_2^n},\ \sqrt[n]{r_1^n \otimes r_2^n} \right), \quad (3.1)$$

$$(p_1,q_1,r_1) \otimes (p_2,q_2,r_2) := \left(\sqrt[n]{p_1^n \otimes p_2^n},\ \sqrt[n]{q_1^n \oplus q_2^n},\ \sqrt[n]{r_1^n \oplus r_2^n} \right). \quad (3.2)$$

Also, if we put

$$1 \cdot (p_1,q_1,r_1) = (p_1,q_1,r_1)^1 = (p_1,q_1,r_1),$$

then we can inductively define

$$(\lambda+1) \cdot (p_1,q_1,r_1) = \lambda \cdot (p_1,q_1,r_1) \oplus (p_1,q_1,r_1),$$
$$(p_1,q_1,r_1)^{\lambda+1} = (p_1,q_1,r_1)^{\lambda} \otimes (p_1,q_1,r_1).$$

It is easily verified that for any natural number λ,

$$\lambda \cdot (p_1,q_1,r_1) = \begin{pmatrix} \sqrt[n]{\frac{(1+(\gamma-1)p_1^n)^\lambda - (1-p_1^n)^\lambda}{(1+(\gamma-1)p_1^n)^\lambda + (\gamma-1)(1-p_1^n)^\lambda}}, \\ \frac{\sqrt[n]{\gamma} q_1^\lambda}{\sqrt[n]{(1+(\gamma-1)(1-q_1^n))^\lambda + (\gamma-1)(q_1^n)^{2\lambda}}}, \\ \frac{\sqrt[n]{\gamma} r_1^\lambda}{\sqrt[n]{(1+(\gamma-1)(1-r_1^n))^\lambda + (\gamma-1)(r_1^n)^{2\lambda}}} \end{pmatrix},$$

$$(p_1,q_1,r_1)^\lambda = \begin{pmatrix} \frac{\sqrt[n]{\gamma} p_{1t}^\lambda}{\sqrt[n]{(1+(\gamma-1)(1-p_{1t}^n))^\lambda + (\gamma-1)(p_{1t}^n)^{2\lambda}}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)q_{1t}^n)^\lambda - (1-q_{1t}^n)^\lambda}{(1+(\gamma-1)q_{1t}^n)^\lambda + (\gamma-1)(1-q_{1t}^n)^\lambda}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)r_{1t}^n)^\lambda - (1-r_{1t}^n)^\lambda}{(1+(\gamma-1)r_{1t}^n)^\lambda + (\gamma-1)(1-r_{1t}^n)^\lambda}} \end{pmatrix}.$$

Now, we define four Hamacher operations on HTSFEs.

Definition 3.3. Let $h_1 = \{(p_{1t}, q_{1t}, r_{1t}) : 1 \le t \le \ell_{h_1}\}$ and $h_2 = \{(p_{2s}, q_{2s}, r_{2s}) : 1 \le s \le \ell_{h_2}\}$ be two HTSFEs and $\gamma, \lambda > 0$

(1) $$h_1 \oplus h_2 = \bigcup_{\substack{(p_{1t},q_{1t},r_{1t}) \in h_1 \\ (p_{2s},q_{2s},r_{2s}) \in h_2}} \{(p_{1t},q_{1t},r_{1t}) \oplus (p_{2s},q_{2s},r_{2s})\}$$

$$= \bigcup_{\substack{(p_{1t},q_{1t},r_{1t})\in h_1\\(p_{2s},q_{2s},r_{2s})\in h_2}} \left\{\left(\begin{array}{c} \sqrt[n]{\frac{p_{1t}^n+p_{2s}^n-p_{1t}^n p_{2s}^n-(1-\gamma)p_{1t}^n p_{2s}^n}{1-(1-\gamma)p_{1t}^n p_{2s}^n}}, \\ \frac{q_{1t}q_{2s}}{\sqrt[n]{\gamma+(1-\gamma)(q_{1t}^n+q_{2s}^n-q_{1t}^n q_{2s}^n)}}, \\ \frac{r_{1t}r_{2s}}{\sqrt[n]{\gamma+(1-\gamma)(r_{1t}^n+r_{2s}^n-r_{1t}^n r_{2s}^n)}} \end{array}\right)\right\}$$

(2) $$h_1 \otimes h_2 = \bigcup_{\substack{(p_{1t},q_{1t},r_{1t})\in h_1\\(p_{2s},q_{2s},r_{2s})\in h_2}} \{(p_{1t},q_{1t},r_{1t})\otimes(p_{2s},q_{2s},r_{2s})\}$$

$$= \bigcup_{\substack{(p_{1t},q_{1t},r_{1t})\in h_1\\(p_{2s},q_{2s},r_{2s})\in h_2}} \left\{\left(\begin{array}{c} \frac{p_{1t}p_{2s}}{\sqrt[n]{\gamma+(1-\gamma)(p_{1t}^n+p_{2s}^n-p_{1t}^n p_{2s}^n)}}, \\ \sqrt[n]{\frac{q_{1t}^n+q_{2s}^n-q_{1t}^n q_{2s}^n-(1-\gamma)q_{1t}^n q_{2s}^n}{1-(1-\gamma)q_{1t}^n q_{2s}^n}}, \\ \sqrt[n]{\frac{r_{1t}^n+r_{2s}^n-r_{1t}^n r_{2s}^n-(1-\gamma)r_{1t}^n r_{2s}^n}{1-(1-\gamma)r_{1t}^n r_{2s}^n}} \end{array}\right)\right\}$$

(3) $$\lambda\cdot h_1 = \bigcup_{(p_{1t},q_{1t},r_{1t})\in h_1} \left\{\left(\begin{array}{c} \sqrt[n]{\frac{(1+(\gamma-1)p_{1t}^n)^\lambda-(1-p_{1t}^n)^\lambda}{(1+(\gamma-1)p_{1t}^n)^\lambda+(\gamma-1)(1-p_{1t}^n)^\lambda}}, \\ \frac{\sqrt[n]{\gamma}q_{1t}^\lambda}{\sqrt[n]{(1+(\gamma-1)(1-q_{1t}^n))^\lambda+(\gamma-1)(q_{1t}^n)^{2\lambda}}}, \\ \frac{\sqrt[n]{\gamma}r_{1t}^\lambda}{\sqrt[n]{(1+(\gamma-1)(1-r_{1t}^n))^\lambda+(\gamma-1)(r_{1t}^n)^{2\lambda}}} \end{array}\right)\right\}$$

(4) $$h_1^\lambda = \bigcup_{(p_{1t},q_{1t},r_{1t})\in h_1} \left\{\left(\begin{array}{c} \frac{\sqrt[n]{\gamma}p_{1t}^\lambda}{\sqrt[n]{(1+(\gamma-1)(1-p_{1t}^n))^\lambda+(\gamma-1)(p_{1t}^n)^{2\lambda}}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)q_{1t}^n)^\lambda-(1-q_{1t}^n)^\lambda}{(1+(\gamma-1)q_{1t}^n)^\lambda+(\gamma-1)(1-q_{1t}^n)^\lambda}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)r_{1t}^n)^\lambda-(1-r_{1t}^n)^\lambda}{(1+(\gamma-1)r_{1t}^n)^\lambda+(\gamma-1)(1-r_{1t}^n)^\lambda}} \end{array}\right)\right\}$$

Example 3.4. Suppose

$$h(x_1) = h_1 = \{(0.6, 0.5, 0.3), (0.5, 0.4, 0.3), (0.8, 0.3, 0.6)\}$$
$$h(x_2) = h_2 = \{(0.5, 0.3, 0.4), (0.8, 0.3, 0.6)\}.$$

For $n = 3$, $\gamma = 2$ and $\lambda = 2$, we have

$$\begin{aligned} h_1 \oplus h_2 \;=\; & \{(0.6925, 0.1222, 0.0967), (0.8687, 0.1222, 0.1490), \\ & (0.6267, 0.0967, 0.9671), (0.8428, 0.0967, 0.1490), \\ & (0.8428, 0.0721, 0.1998), (0.9327, 0.0721, 0.3069)\}. \end{aligned}$$

$$\begin{aligned} h_1 \otimes h_2 \;=\; & \{(0.2521, 0.5331, 0.4495), (0.4309, 0.5331, 0.6228), \\ & (0.2068, 0.4495, 0.4495), (0.3553, 0.4495, 0.6228), \\ & (0.3553, 0.3779, 0.6512), (0.5960, 0.3779, 0.7445)\}. \end{aligned}$$

$$\begin{aligned} 2\cdot h_1 \;=\; & \{(0.7445, 0.2071, 0.0721), (0.6267, 0.1298, 0.0721), \\ & (0.9327, 0.0721, 0.3083)\}. \end{aligned}$$

$$\begin{aligned} h_1^2 &= \{(0.3083, 0.6267, 0.3779), (0.2071, 0.5033, 0.3779), \\ &\quad (0.6124, 0.3779, 0.7445)\}. \end{aligned}$$

In the following proposition, some elementary properties of these operators are given.

Proposition 3.5. *Let h, h_1 and h_2 be HT-SFEs. Then*

(1) *$h_1 \oplus h_2$, $h_1 \otimes h_2$, λh and h^λ are HT-SFEs.*
(2) *$h_1 \oplus h_2 = h_2 \oplus h_1$ and $h_1 \otimes h_2 = h_2 \otimes h_1$.*

Proof. The proof is clear from the Definitions 3.2 and 3.3. □

Definition 3.6. Let $\digamma_1 = \{(\mathfrak{x}, h_1(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$ and $\digamma_2 = \{(\mathfrak{x}, h_2(\mathfrak{x})) : \mathfrak{x} \in \Psi\}$ be two HT-SFSs over Ψ. Then, we define

$$\begin{aligned} \digamma_1 \oplus \digamma_2 &= \{(\mathfrak{x}, h_1(\mathfrak{x}) \oplus h_2(\mathfrak{x})) : \mathfrak{x} \in \Psi\}, \\ \digamma_1 \otimes \digamma_2 &= \{(\mathfrak{x}, h_1(\mathfrak{x}) \otimes h_2(\mathfrak{x})) : \mathfrak{x} \in \Psi\}. \end{aligned}$$

3.2. Hesitant T-spherical Fuzzy Hamacher Weighted Arithmetic Aggregation Operator.

Definition 3.7. Suppose $\Omega^m = \{h_k = \{(p_{kj}, q_{kj}, r_{kj}) : 1 \leq j \leq \ell_{h_k}, k = 1, 2, ..., m\}$ is a collection of HT-SFEs. A function $HTSFHWAA : \Omega^m \to \Omega$ for HTSFHWAA operator is defined:

$$\begin{aligned} HTSFHWAA(h_1, h_2, h_3, ..., h_m) &= \bigoplus_{z=1}^{m} (\varpi_z h_z) \\ &= (\varpi_1 h_1) \oplus (\varpi_2 h_2) \oplus ... \oplus (\varpi_m h_m), \end{aligned}$$

where ϖ_z is a weight vector of h_k, $0 \leq \varpi_z \leq 1$ and $\sum_{z=1}^{m} \varpi_z = 1$.

Theorem 3.8. *If $h_1, \ldots, h_m \in \Omega^m$, then*

$$HTSFHWAA(h_1, h_2, ..., h_m) = \bigoplus_{z=1}^{m} (\varpi_z h_z)$$

$$= \bigcup_{\substack{(p_1, q_1, r_1) \in h_1 \\ (p_2, q_2, r_2) \in h_2 \\ \cdots \\ (p_m, q_m, r_m) \in h_m}} \left(\begin{array}{c} \sqrt[n]{\frac{\Pi_{z=1}^{m}[1+(\gamma-1)p_z^n]^{\varpi_z} - \Pi_{z=1}^{m}[1-p_z^n]^{\varpi_z}}{\Pi_{z=1}^{m}[1+(\gamma-1)p_z^n]^{\varpi_z} + (\gamma-1)\Pi_{z=1}^{m}[1-p_z^n]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}(\Pi_{z=1}^{m} q_z^{\varpi_z})}{\sqrt[n]{\Pi_{z=1}^{m}[1+(\gamma-1)(1-q_z^n)]^{\varpi_z} + (\gamma-1)\Pi_{z=1}^{m} q_z^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}(\Pi_{z=1}^{m} r_z^{\varpi_z})}{\sqrt[n]{\Pi_{z=1}^{m}[1+(\gamma-1)(1-r_z^n)]^{\varpi_z} + (\gamma-1)\Pi_{z=1}^{m} r_z^{n\varpi_z}}} \end{array} \right)$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$ and $\sum_{k=1}^{m} \varpi_z = 1$.

Proof. The proof of the theorem is made by induction as follows:

(1) For $m = 2$, we obtain

$$\varpi_1 h_1 = \bigcup_{(p_1,q_1,r_1)\in\mathfrak{h}_1} \left(\begin{array}{c} \sqrt[n]{\frac{[1+(\gamma-1)p_1^n]^{\varpi_1}-[1-p_1^n]^{\varpi_1}}{([1+(\gamma-1)p_1^n]^{\varpi_1}+(\gamma-1)[1-p_1^n]^{\varpi_1}}}, \\ \frac{\sqrt[n]{\gamma}q_1^{\varpi_1}}{\sqrt[n]{[1+(\gamma-1)(1-q_1^n)]^{\varpi_1}+(\gamma-1)q_1^{n\varpi_1}}}, \\ \frac{\sqrt[n]{\gamma}r_1^{\varpi_1}}{\sqrt[n]{[1+(\gamma-1)(1-r_1^n)]^{\varpi_1}+(\gamma-1)r_1^{n\varpi_1}}} \end{array} \right)$$

$$\varpi_2 h_2 = \bigcup_{(p_2,q_2,r_2)\in\mathfrak{h}_2} \left(\begin{array}{c} \sqrt[n]{\frac{[1+(\gamma-1)p_2^n]^{\varpi_2}-[1-p_2^n]^{\varpi_2}}{[1+(\gamma-1)p_2^n]^{\varpi_2}+(\gamma-1)[1-p_2^n]^{\varpi_2}}}, \\ \frac{\sqrt[n]{\gamma}q_2^{\varpi_2}}{\sqrt[n]{[1+(\gamma-1)(1-q_2^n)]^{\varpi_2}+(\gamma-1)q_2^{n\varpi_2}}}, \\ \frac{\sqrt[n]{\gamma}r_2^{\varpi_2}}{\sqrt[n]{[1+(\gamma-1)(1-r_2^n)]^{\varpi_2}+(\gamma-1)r_2^{n\varpi_2}}} \end{array} \right)$$

Hence $\varpi_1 h_1 \oplus \varpi_2 h_2$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1 \\ (p_2,q_2,r_2)\in h_2}} \left(\begin{array}{c} \sqrt[n]{\frac{[1+(\gamma-1)p_1^n]^{\varpi_1}[1+(\gamma-1)p_2^n]^{\varpi_2}-[1-p_1^n]^{\varpi_1}[1-p_2^n]^{\varpi_2}}{[1+(\gamma-1)p_1^n]^{\varpi_1}[1+(\gamma-1)p_2^n]^{\varpi_2}+(\gamma-1)[1-p_1^n]^{\varpi_1}[1-p_2^n]^{\varpi_2}}}, \\ \frac{\sqrt[n]{\gamma}q_1^{\varpi_1}q_2^{\varpi_2}}{\sqrt[n]{([1+(\gamma-1)(1-q_1^n)]^{\varpi_1}[1+(\gamma-1)(1-q_2^n)]^{\varpi_2})+(\gamma-1)q_1^{n\varpi_1}q_2^{n\varpi_2}}}, \\ \frac{\sqrt[n]{\gamma}r_1^{\varpi_1}r_2^{\omega_2}}{\sqrt[n]{([1+(\gamma-1)(1-r_1^n)]^{\varpi_1}[1+(\gamma-1)(1-r_2^n)]^{\varpi_2})+(\gamma-1)r_1^{n\varpi_1}r_2^{n\varpi_2}}} \end{array} \right)$$

which is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1 \\ (p_2,q_2,r_2)\in h_2}} \left(\begin{array}{c} \sqrt[n]{\frac{\Pi_{z=1}^2[1+(\gamma-1)p_z^n]^{\varpi_z}-\Pi_{z=1}^2[1-p_z^n]^{\varpi_z}}{\Pi_{z=1}^2[1+(\gamma-1)p_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^2[1-p_z^n]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^2 q_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^2[1+(\gamma-1)(1-q_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^2 q_z^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^2 r_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^2[1+(\gamma-1)(1-r_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^2 r_z^{n\varpi_z}}}. \end{array} \right)$$

Then the theorem holds for $m = 2$.

(2) Suppose that the theorem is correct for $z = k$, that is $\oplus_{z=1}^k(\varpi_z \mathfrak{h}_z)$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1 \\ (p_2,q_2,r_2)\in h_2 \\ \cdots \\ (p_k,q_k,r_k)\in h_k}} \left(\begin{array}{c} \sqrt[n]{\frac{\Pi_{z=1}^k[1+(\gamma-1)p_z^n]^{\varpi_z}-\Pi_{z=1}^k[1-p_z^n]^{\varpi_z}}{\Pi_{z=1}^k[1+(\gamma-1)p_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k[1-p_z^n]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^k q_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^k[1+(\gamma-1)(1-q_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^k q_z^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^k r_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^k[1+(\gamma-1)(1-r_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^k r_z^{n\varpi_z}}} \end{array} \right)$$

When $z = k+1$, $\oplus_{z=1}^{k}(\varpi_z h_z) \oplus \varpi_{k+1} h_{k+1}$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_k,q_k,r_k)\in h_k}} \begin{pmatrix} \sqrt[n]{\frac{\Pi_{z=1}^{k}[1+(\gamma-1)p_z^n]^{\varpi_z}-\Pi_{z=1}^{k}[1-p_z^n]^{\varpi_z}}{\Pi_{z=1}^{k}[1+(\gamma-1)p_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k}[1-p_z^n]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k}q_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^{k}[1+(\gamma-1)(1-q_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^{k}q_z^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k}r_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^{k}[1+(\gamma-1)(1-r_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^{k}r_z^{n\varpi_z}}} \end{pmatrix} \oplus \varpi_{k+1}h_{k+1}$$

$\oplus_{z=1}^{k}(\varpi_z h_z) \oplus \varpi_{k+1} h_{k+1}$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_k,q_k,r_k)\in h_k}} \begin{pmatrix} \sqrt[n]{\frac{\Pi_{z=1}^{k}[1+(\gamma-1)p_z^n]^{\varpi_z}-\Pi_{z=1}^{k}[1-p_z^n]^{\varpi_z}}{\Pi_{z=1}^{k}[1+(\gamma-1)p_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k}[1-p_z^n]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k}q_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^{k}[1+(\gamma-1)(1-q_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^{k}q_z^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k}r_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^{k}[1+(\gamma-1)(1-r_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^{k}r_z^{n\varpi_z}}} \end{pmatrix} \oplus$$

$$\begin{pmatrix} \sqrt[n]{\frac{[1+(\gamma-1)p_1^n]^{\varpi_{k+1}}-[1-p_1^n]^{\varpi_{k+1}}}{[1+(\gamma-1)p_1^n]^{\varpi_{k+1}}+(\gamma-1)[1-p_1^n]^{\varpi_{k+1}}}}, \\ \frac{\sqrt[n]{\gamma}q_1^{\varpi_{k+1}}}{\sqrt[n]{([1+(\gamma-1)(1-q_1^n)]^{\varpi_{k+1}})+(\gamma-1)q_1^{n\varpi_{k+1}}}}, \\ \frac{\sqrt[n]{\gamma}r_1^{\varpi_{k+1}}}{\sqrt[n]{([1+(\gamma-1)(1-r_1^n)]^{\varpi_{k+1}})+(\gamma-1)r_1^{n\varpi_{k+1}}}} \end{pmatrix}$$

Hence $\oplus_{z=1}^{k+1}(\varpi_z h_z)$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_{k+1},q_{k+1},r_{k+1})\in h_{k+1}}} \begin{pmatrix} \sqrt[n]{\frac{\Pi_{z=1}^{k+1}[1+(\gamma-1)p_z^n]^{\varpi_z}-\Pi_{z=1}^{k+1}[1-p_z^n]^{\varpi_z}}{\Pi_{z=1}^{k+1}[1+(\gamma-1)p_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k+1}[1-p_z^n]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k+1}q_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^{k+1}[1+(\gamma-1)(1-q_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^{k+1}q_z^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k+1}r_z^{\varpi_z}}{\sqrt[n]{(\Pi_{z=1}^{k+1}[1+(\gamma-1)(1-r_z^n)]^{\varpi_z})+(\gamma-1)\Pi_{z=1}^{k+1}r_z^{n\varpi_z}}} \end{pmatrix}$$

This completes the proof. □

Example 3.9. Let $h_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.9)\}$, $h_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\}$ and $h_3 = \{(0.5, 0.4, 0.3)\}$. For $n = 3$, $\gamma = 2$ and $\varpi = (0.5, 0.3, 0.2)$, by using Theorem 3.8, we obtain

$$\begin{aligned} HTSFHWAA(h_1, h_2, h_3) = \{&(0.5217, 0.7376, 0.4044), (0.5148, 0.5776, 0.4509), \\ &(0.4025, 0.5838, 0.6295), (0.3904, 0.4476, 0.6935), \\ &(0.5217, 0.3752, 0.6295), (0.5148, 0.2835, 0.6935)\}. \end{aligned}$$

3.3. **Hesitant T-spherical Fuzzy Hamacher Weighted Geometric Aggregation Operator.**

Definition 3.10. Suppose $\Omega^m = \{h_k = \{(p_{kj}, q_{kj}, r_{kj}) : 1 \leq j \leq \ell_{h_k}, k = 1 \text{ to } m\}$ is a collection of HT-SFEs. A function $HTSFHWGA : \Omega^m \to \Omega$ for HTSFHWGA operator is defined:

$$\begin{aligned} HTSFHWGA(h_1, h_2, h_3, ..., h_m) &= \bigotimes_{z=1}^{m} (h_z^{\varpi_z}) \\ &= (h_1)^{\varpi_1} \otimes (h_2)^{\varpi_2} \otimes ... \otimes (h_m)^{\varpi_m}, \end{aligned}$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$ and $\sum_{z=1}^{m} \varpi_z = 1$.

By using the Hamacher operations on HT-SFEs, we can prove the following.

Theorem 3.11. *Let $h_k \in \Omega^m$. Then*

$$HTSFHWGA(h_1, h_2, ..., h_m) = \bigotimes_{z=1}^{m} (h_z^{\varpi_z})$$

$$= \bigcup_{\substack{(p_1,q_1,r_1)\in h_1 \\ (p_2,q_2,r_2)\in h_2 \\ \cdots \\ (p_m,q_m,r_m)\in h_m}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{m} p_z^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^{m}[1+(\gamma-1)(1-p_z^n)]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{m} p_z^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^{m}[1+(\gamma-1)q_z^n]^{\varpi_z}-\Pi_{z=1}^{m}[1-q_z^n]^{\varpi_z}}{\Pi_{z=1}^{m}[1+(\gamma-1)q_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{m}[1-q_z^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^{m}[1+(\gamma-1)r_z^n]^{\varpi_z}-\Pi_{z=1}^{m}[1-r_z^n]^{\varpi_z}}{\Pi_{z=1}^{m}[1+(\gamma-1)r_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{m}[1-r_z^n]^{\varpi_z}}} \end{array} \right).$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$, and $\sum_{z=1}^{m} \varpi_z = 1$.

Proof. The proof of the theorem proceed by mathematical induction as follows:

(1) For $m = 2$, we obtain

$$h_1^{\varpi_1} = \bigcup_{(p_1,q_1,r_1)\in h_1} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma} p_1^{\varpi_1}}{\sqrt[n]{[1+(\gamma-1)(1-p_1^n)]^{\varpi_1}+(\gamma-1)p_1^{n\varpi_1}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)q_1^n]^{\varpi_1}-[1-q_1^n]^{\varpi_1}}{[1+(\gamma-1)q_1^n]^{\varpi_1}+(\gamma-1)[1-q_1^n]^{\varpi_1}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)r_1^n]^{\varpi_1}-[1-r_1^n]^{\varpi_1}}{[1+(\gamma-1)r_1^n]^{\varpi_1}+(\gamma-1)[1-r_1^n]^{\varpi_1}}} \end{array} \right)$$

$$h_2^{\varpi_2} = \bigcup_{(p_2,q_2,r_2)\in h_2} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma} p_2^{\varpi_2}}{\sqrt[n]{[1+(\gamma-1)(1-p_2^n)]^{\varpi_2}+(\gamma-1)p_2^{n\varpi_2}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)q_2^n]^{\varpi_2}-[1-q_2^n]^{\varpi_2}}{[1+(\gamma-1)q_2^n]^{\varpi_2}+(\gamma-1)[1-q_2^n]^{\varpi_2}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)r_2^n]^{\varpi_2}-[1-r_2^n]^{\varpi_2}}{[1+(\gamma-1)r_2^n]^{\varpi_2}+(\gamma-1)[1-r_2^n]^{\varpi_2}}} \end{array} \right)$$

$h_1^{\varpi_1} \otimes h_2^{\varpi_2}$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma} p_1^{\varpi_1} p_2^{\varpi_2}}{\sqrt[n]{[1+(\gamma-1)(1-p_1^n)]^{\varpi_1}[1+(\gamma-1)(1-p_2^n)]^{\varpi_2}+(\gamma-1)p_1^{n\varpi_1}p_2^{n\varpi_2}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)q_1^n]^{\varpi_1}[1+(\gamma-1)q_2^n]^{\varpi_2}-[1-q_1^n]^{\varpi_1}[1-q_2^n]^{\varpi_2}}{[1+(\gamma-1)q_1^n]^{\varpi_1}[1+(\gamma-1)q_2^n]^{\varpi_2}+(\gamma-1)[1-q_1^n]^{\varpi_1}[1-q_2^n]^{\varpi_2}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)r_1^n]^{\varpi_1}[1+(\gamma-1)r_2^n]^{\varpi_2}-[1-r_1^n]^{\varpi_1}[1-r_2^n]^{\varpi_2}}{[1+(\gamma-1)r_1^n]^{\varpi_1}[1+(\gamma-1)r_2^n]^{\varpi_2}+(\gamma-1)[1-r_1^n]^{\varpi_1}[1-r_2^n]^{\varpi_2}}} \end{array} \right)$$

$$= \bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}\Pi_{z=1}^2 p_z^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^2[1+(\gamma-1)(1-p_z^n)]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^2 p_z^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^2[1+(\gamma-1)q_z^n]^{\varpi_z}-\Pi_{z=1}^2[1-q_z^n]^{\varpi_z}}{\Pi_{z=1}^2[1+(\gamma-1)q_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^2[1-q_z^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^2[1+(\gamma-1)r_z^n]^{\varpi_z}-\Pi_{z=1}^2[1-r_z^n]^{\varpi_z}}{\Pi_{z=1}^2[1+(\gamma-1)r_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^2[1-r_z^n]^{\varpi_z}}} \end{array} \right)$$

Thus, the theorem is correct for $m = 2$.

(2) Consider it holds when $z = k$, that is $\bigotimes_{z=1}^k (h_z^{\varpi_z})$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_k,q_k,r_k)\in h_k}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}\Pi_{z=1}^k p_z^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^k[1+(\gamma-1)(1-p_z^n)]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k p_z^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^k[1+(\gamma-1)q_z^n]^{\varpi_z}-\Pi_{z=1}^k[1-q_z^n]^{\varpi_z}}{\Pi_{z=1}^k[1+(\gamma-1)q_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k[1-q_z^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^k[1+(\gamma-1)r_z^n]^{\varpi_z}-\Pi_{z=1}^k[1-r_z^n]^{\varpi_z}}{\Pi_{z=1}^k[1+(\gamma-1)r_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k[1-r_z^n]^{\varpi_z}}} \end{array} \right)$$

When $z = k+1$, $\otimes_{z=1}^k (h_z)^{\varpi_z} \otimes h_{k+1}^{\varpi_{k+1}}$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_k,q_k,r_k)\in h_k}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}\Pi_{z=1}^k p_z^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^k[1+(\gamma-1)(1-p_z^n)]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k p_z^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^k[1+(\gamma-1)q_z^n]^{\varpi_z}-\Pi_{z=1}^k[1-q_z^n]^{\varpi_z}}{\Pi_{z=1}^k[1+(\gamma-1)q_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k[1-q_z^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^k[1+(\gamma-1)r_z^n]^{\varpi_z}-\Pi_{z=1}^k[1-r_z^n]^{\varpi_z}}{\Pi_{z=1}^k[1+(\gamma-1)r_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^k[1-r_z^n]^{\varpi_z}}} \end{array} \right) \otimes h_{k+1}^{\varpi_{k+1}}$$

$\otimes_{z=1}^{k}(h_z)^{\varpi_z} \otimes h_{k+1}^{\varpi_{k+1}}$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_k,q_k,r_k)\in h_k}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k} p_z^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^{k}[1+(\gamma-1)(1-p_z^n)]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k} p_z^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^{k}[1+(\gamma-1)q_z^n]^{\varpi_z}-\Pi_{z=1}^{k}[1-q_z^n]^{\varpi_z}}{\Pi_{z=1}^{k}[1+(\gamma-1)q_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k}[1-q_z^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^{k}[1+(\gamma-1)r_z^n]^{\varpi_z}-\Pi_{z=1}^{k}[1-r_z^n]^{\varpi_z}}{\Pi_{z=1}^{k}[1+(\gamma-1)r_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k}[1-r_z^n]^{\varpi_z}}} \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}p_1^{\varpi_{k+1}}}{\sqrt[n]{[1+(\gamma-1)(1-p_1^n)]^{\omega_{k+1}}+(\gamma-1)p_1^{n\varpi_{k+1}}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)q_1^n]^{\varpi_{k+1}}-[1-q_1^n]^{\varpi_{k+1}}}{[1+(\gamma-1)q_1^n]^{\varpi_{k+1}}+(\gamma-1)[1-q_1^n]^{\varpi_{k+1}}}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)r_1^n]^{\varpi_1}-[1-r_1^n]^{\varpi_{k+1}}}{[1+(\gamma-1)r_1^n]^{\varpi_{k+1}}+(\gamma-1)[1-r_1^n]^{\varpi_{k+1}}}} \end{array} \right)$$

Hence $\otimes_{z=1}^{k+1}(h_z)^{\varpi_z}$ is equal to

$$\bigcup_{\substack{(p_1,q_1,r_1)\in h_1\\(p_2,q_2,r_2)\in h_2\\ \cdots \\(p_{k+1},q_{k+1},r_{k+1})\in h_{k+1}}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma}\Pi_{z=1}^{k+1} p_z^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^{k+1}[1+(\gamma-1)(1-p_z^n)]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k+1} p_z^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^{k+1}[1+(\gamma-1)q_z^n]^{\varpi_z}-\Pi_{z=1}^{k+1}[1-q_z^n]^{\varpi_z}}{\Pi_{z=1}^{k+1}[1+(\gamma-1)q_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k+1}[1-q_z^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\Pi_{z=1}^{k+1}[1+(\gamma-1)r_z^n]^{\varpi_z}-\Pi_{z=1}^{k+1}[1-r_z^n]^{\varpi_z}}{\Pi_{z=1}^{k+1}[1+(\gamma-1)r_z^n]^{\varpi_z}+(\gamma-1)\Pi_{z=1}^{k+1}[1-r_z^n]^{\varpi_z}}} \end{array} \right)$$

Then the theorem is proved since it holds for $z = k + 1$. □

Example 3.12. Suppose that

$$h_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\},$$
$$h_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\},$$
$$h_3 = \{(0.5, 0.4, 0.3)\}$$

for $\gamma = 1$, $n = 3$ and $\varpi = (0.5, 0.3, 0.2)$, by 3.11, we obtain

$$HTSFHWGA(h_1, h_2, h_3) = \{(0.4730, 0.8030, 0.4217), (0.4207, 0.6730, 0.5238), (0.3842, 0.7977, 0.7759), (0.3409, 0.4556, 0.8040), (0.4730, 0.6593, 0.7759), (0.4207, 0.3303, 0.8040)\}.$$

3.4. **Hesitant T-spherical Fuzzy Hamacher Ordered Weighted Arithmetic Aggregation Operator.**

Definition 3.13. Let $\Omega^m = \{h_k = \{(p_{kj}, q_{kj}, r_{kj}) : 1 \leq j \leq \ell_{h_k}, (k = 1 \; to \; m)\}$ be a collection of HT-SFEs. A function $HTSFHOWAA : \Omega^m \to \Omega$ for HTSFHOWAA operator is defined:

$$HTSFHOWAA(h_1, h_2, h_3, ..., h_m) = \bigoplus_{z=1}^{m} (\varpi_z h_{\sigma(z)})$$
$$= (\varpi_1 h_{\sigma(1)}) \oplus (\varpi_2 h_{\sigma(2)}) \oplus ... \oplus (\varpi_m h_{\sigma(m)}),$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$, and $\sum_{z=1}^{m} \varpi_z = 1$.

By using the Hamacher operations on HT-SFEs, we obtain:

Theorem 3.14. *Let $h_k \in \Omega^m$. Then*
$HTSFHOWAA(h_1, h_2, ..., h_m) = \bigoplus_{z=1}^{m} (\varpi_z h_{\sigma(z)}) =$

$$\bigcup_{\substack{(p_\sigma(1), q_\sigma(1), r_\sigma(1)) \in h_1 \\ (p_\sigma(2), q_\sigma(2), r_\sigma(2)) \in h_2 \\ \cdots \\ (p_\sigma(m), q_\sigma(m), r_\sigma(m)) \in h_m}} \left(\begin{array}{c} \sqrt[n]{\frac{\Pi_{z=1}^{m}[1+(\gamma-1)p_{\sigma(z)}^{n}]^{\varpi_z} - \Pi_{z=1}^{m}[1-p_{\sigma(z)}^{n}]^{\varpi_z}}{\Pi_{z=1}^{m}[1+(\gamma-1)p_{\sigma(z)}^{n}]^{\varpi_z} + (\gamma-1)\Pi_{z=1}^{m}[1-p_{\sigma(z)}^{n}]^{\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma} \Pi_{z=1}^{m} q_{\sigma(z)}^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^{m}[1+(\gamma-1)(1-q_{\sigma(z)}^{n})]^{\varpi_z} + (\gamma-1)\Pi_{z=1}^{m} q_{\sigma(z)}^{n\varpi_z}}}, \\ \frac{\sqrt[n]{\gamma} \Pi_{z=1}^{m} r_{\sigma(z)}^{\varpi_z}}{\sqrt[n]{\Pi_{z=1}^{m}[1+(\gamma-1)(1-r_{\sigma(z)}^{n})]^{\varpi_z} + (\gamma-1)\Pi_{z=1}^{m} r_{\sigma(z)}^{n\varpi_z}}} \end{array} \right),$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$, and $\sum_{z=1}^{m} \varpi_z = 1$.

Proof. The proof is similar to proof of Theorem 3.11. □

Example 3.15. Let $h_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\}$, $h_2 = \{(0.3, 0.9, 0.5),$
$(0.2, 0.4, 0.7)\}$ and $h_3 = \{(0.5, 0.4, 0.3)\}$ for $\gamma = 1$, $n = 3$, and $\varpi = (0.5, 0.3, 0.2)$
By using Eq. 2.1 and Theorem 3.14, we obtain

$$SC(h_1) = -0.2133, SC(h_2) = -0.2165 \;,\; SC(h_3) = 0.089$$

Then
$SC(h_3) > SC(h_1) > SC(h_2)$ and $h_{\sigma(1)} = h_3 = \{(0.5, 0.4, 0.3)\}$,
$h_{\sigma(2)} = h_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\}$,
$h_{\sigma(3)} = h_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\}$.

$$\begin{aligned} HTSFHOWAA(h_1, h_2, h_3) = \{&(0.5108, 0.5956, 0.3629), \\ &(0.4435, 0.5126, 0.4776), (0.5108, 0.3921, 0.4334), \\ &(0.5060, 0.4378, 0.3905), (0.4370, 0.3736, 0.5127), \\ &(0.5060, 0.2835, 0.4658)\}. \end{aligned}$$

3.5. **Hesitant T-Spherical Fuzzy Hamacher Ordered Weighted Geometric Aggregation Operator.**

Definition 3.16. Let $\Omega^m = \{h_k = \{(p_{kj}, q_{kj}, r_{kj}) : 1 \leq j \leq \ell_{h_k}, k = 1, \dots, m\}$ be a set of HT-SFEs. A function $HTSFHOWGA : \Omega^m \to \Omega$ for HTSFHOWGA operator is defined:

$$\begin{aligned} HTSFHOWGA(h_1, h_2, h_3, ..., h_m) &= \bigotimes_{\kappa=1}^{m} (h_{\sigma(z)}^{\varpi_z}) \\ &= (h_{\sigma(1)})^{\varpi_1} \otimes (h_{\sigma(2)})^{\varpi_2} \otimes ... \otimes (h_{\sigma(m)})^{\varpi_m}, \end{aligned}$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$, and $\sum_{z=1}^{m} \varpi_z = 1$.

By using the Hamacher operations on HT-SFEs, we get:

Theorem 3.17. *Let $h_\kappa \in \Omega^m$. Then,*

$$HTSFHOWGA(h_1, h_2, ..., h_m) = \bigotimes_{\kappa=1}^{m} (h_{\sigma(\kappa)}^{\varpi_\kappa}) = \bigcup_{\substack{(p_{\sigma(1)}, q_{\sigma(1)}, r_{\sigma(1)}) \in h_{\sigma(1)} \\ (p_{\sigma(2)}, q_{\sigma(2)}, r_{\sigma(2)}) \in h_{\sigma(2)} \\ \cdots \\ (p_{\sigma(m)}, q_{\sigma(m)}, r_{\sigma(m)}) \in h_{\sigma(m)}}} \left(\begin{array}{c} \frac{\sqrt[n]{\gamma} \Pi_{\kappa=1}^{m} p_{\sigma(\kappa)}^{\varpi_\kappa}}{\sqrt[n]{\Pi_{\kappa=1}^{m} [1+(\gamma-1)(1-p_{\sigma(\kappa)}^{n})]^{\varpi_\kappa} + (\gamma-1)\Pi_{\kappa=1}^{m} p_{\sigma(\kappa)}^{n\varpi_\kappa}}}, \\ \sqrt[n]{\frac{\Pi_{\kappa=1}^{m} [1+(\gamma-1)q_\sigma(\kappa)^n]^{\varpi_\kappa} - \Pi_{\kappa=1}^{m} [1-q_\sigma(\kappa)^n]^{\varpi_\kappa}}{\Pi_{\kappa=1}^{m} [1+(\gamma-1)q_\sigma(\kappa)^n]^{\varpi_\kappa} + (\gamma-1)\Pi_{\kappa=1}^{m} [1-q_\sigma(\kappa)^n]^{\varpi_\kappa}}}, \\ \sqrt[n]{\frac{\Pi_{\kappa=1}^{m} [1+(\gamma-1)r_\sigma(\kappa)^n]^{\varpi_\kappa} - \Pi_{\kappa=1}^{m} [1-r_\sigma(\kappa)^n]^{\varpi_\kappa}}{\Pi_{\kappa=1}^{m} [1+(\gamma-1)r_\sigma(\kappa)^n]^{\varpi_\kappa} + (\gamma-1)\Pi_{\kappa=1}^{m} [1-r_\sigma(\kappa)^n]^{\varpi_\kappa}}} \end{array} \right)$$

where ϖ_z is a weight vector of h_z, $0 \leq \varpi_z \leq 1$, and $\sum_{z=1}^{m} \varpi_z = 1$.

Proof. The proof can be made by the similar way to proof of Theorem 3.11 □

Example 3.18. Let

$$\begin{aligned} h_1 &= \{(0.60, 0.80, 0.40), (0.40, 0.50, 0.90), (0.60, 0.20, 0.70)\}, \\ h_2 &= \{(0.30, 0.90, 0.50), (0.20, 0.40, 0.70)\}, \\ h_3 &= \{(0.50, 0.40, 0.30)\}, \end{aligned}$$

for $\gamma = 1$, $n = 3$, $\varpi = (0.50, 0.30, 0.20)$, Eq. 2.1 and Theorem 3.17, give

$$SC(h_1) = -0.2133, SC(h_2) = -0.2165, SC(h_3) = 0.089.$$

Then, $SC(h_3) > SC(h_1) > SC(h_2)$ and

$$\begin{aligned} h_{\sigma(1)} &= h_3 = \{(0.5, 0.4, 0.3)\}, \\ h_{\sigma(2)} &= h_1 = \{(0.6, 0.8, 0.4), (0.4, 0.5, 0.9), (0.6, 0.2, 0.7)\}, \\ h_{\sigma(3)} &= h_2 = \{(0.3, 0.9, 0.5), (0.2, 0.4, 0.7)\}. \end{aligned}$$

$$\begin{aligned} HTSFHOWGA(h_1, h_2, h_3) = \{&(0.4978, 0.4686, 0.7086), \\ &(0.4343, 0.3840, 0.6168), (0.4978, 0.4686, 0.5855), \\ &(0.4567, 0.4628, 0.5695), (0.3991, 0.3752, 0.3767), \\ &(0.4567, 0.4628, 0.2608)\}. \end{aligned}$$

Theorem 3.19. *(Idempotency) Let $h_1, h_2, \ldots, h_m$ be a collection of HTSFEs. If $h_k = h$ for all $k = 1, \ldots, m$, where h is an HTSFE of length one, then for any weight vector $\varpi = (\varpi_1, \ldots, \varpi_m)$ with $\varpi_z \in (0, 1]$ and $\sum_{z=1}^{m} \varpi_z = 1$, the following hold:*

$$\begin{aligned} HTSFHWAA(h_1, h_2, \ldots, h_m) &= h, \\ HTSFHWGA(h_1, h_2, \ldots, h_m) &= h, \\ HTSFHOWAA(h_1, h_2, \ldots, h_m) &= h, \\ HTSFHOWGA(h_1, h_2, \ldots, h_m) &= h. \end{aligned}$$

Proof. Since all h_k are identical and equal to h, and h is of length one, we can write $h = \{(p, q, r)\}$. We prove each case separately.
Case 1: HTSFHWAA. By Theorem 3.8,

$$\begin{aligned} \text{HTSFHWAA}(h_1, \ldots, h_m) &= \bigoplus_{z=1}^{m} (\varpi_z h) \\ &= \begin{pmatrix} \sqrt[n]{\dfrac{\prod_{z=1}^{m}[1+(\gamma-1)p^n]^{\varpi_z} - \prod_{z=1}^{m}[1-p^n]^{\varpi_z}}{\prod_{z=1}^{m}[1+(\gamma-1)p^n]^{\varpi_z} + (\gamma-1)\prod_{z=1}^{m}[1-p^n]^{\varpi_z}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{z=1}^{m} q^{\varpi_z}}{\sqrt[n]{\prod_{z=1}^{m}[1+(\gamma-1)(1-q^n)]^{\varpi_z} + (\gamma-1)\prod_{z=1}^{m} q^{n\varpi_z}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{z=1}^{m} r^{\varpi_z}}{\sqrt[n]{\prod_{z=1}^{m}[1+(\gamma-1)(1-r^n)]^{\varpi_z} + (\gamma-1)\prod_{z=1}^{m} r^{n\varpi_z}}} \end{pmatrix} \end{aligned}$$

Since $\sum_{z=1}^{m} \varpi_z = 1$, each product simplifies to a single factor:

$$\prod_{z=1}^{m} [1 + (\gamma - 1)p^n]^{\varpi_z} = [1 + (\gamma - 1)p^n],$$

$$\prod_{z=1}^{m} [1 - p^n]^{\varpi_z} = [1 - p^n],$$

$$\prod_{z=1}^{m} q^{\varpi_z} = q,$$

$$\prod_{z=1}^{m} [1 + (\gamma - 1)(1 - q^n)]^{\varpi_z} = [1 + (\gamma - 1)(1 - q^n)],$$

$$\prod_{z=1}^{m} q^{n\varpi_z} = q^n,$$

with analogous expressions for r. Substituting these simplifications:

$$\text{HTSFHWAA}(h_1, \ldots, h_m) = \begin{pmatrix} \sqrt[n]{\frac{[1+(\gamma-1)p^n]-[1-p^n]}{[1+(\gamma-1)p^n]+(\gamma-1)[1-p^n]}}, \\ \frac{\sqrt[n]{\gamma}q}{\sqrt[n]{[1+(\gamma-1)(1-q^n)]+(\gamma-1)q^n}}, \\ \frac{\sqrt[n]{\gamma}r}{\sqrt[n]{[1+(\gamma-1)(1-r^n)]+(\gamma-1)r^n}} \end{pmatrix} = (p, q, r) = h.$$

Case 2: HTSFHWGA. By Theorem 3.11,

$$\begin{aligned} \text{HTSFHWGA}(h_1, \ldots, h_m) &= \bigotimes_{z=1}^{m} (h^{\varpi_z}) \\ &= \begin{pmatrix} \frac{\sqrt[n]{\gamma}\prod_{z=1}^{m} p^{\varpi_z}}{\sqrt[n]{\prod_{z=1}^{m}[1+(\gamma-1)(1-p^n)]^{\varpi_z}+(\gamma-1)\prod_{z=1}^{m} p^{n\varpi_z}}}, \\ \sqrt[n]{\frac{\prod_{z=1}^{m}[1+(\gamma-1)q^n]^{\varpi_z}-\prod_{z=1}^{m}[1-q^n]^{\varpi_z}}{\prod_{z=1}^{m}[1+(\gamma-1)q^n]^{\varpi_z}+(\gamma-1)\prod_{z=1}^{m}[1-q^n]^{\varpi_z}}}, \\ \sqrt[n]{\frac{\prod_{z=1}^{m}[1+(\gamma-1)r^n]^{\varpi_z}-\prod_{z=1}^{m}[1-r^n]^{\varpi_z}}{\prod_{z=1}^{m}[1+(\gamma-1)r^n]^{\varpi_z}+(\gamma-1)\prod_{z=1}^{m}[1-r^n]^{\varpi_z}}} \end{pmatrix} \end{aligned}$$

Again using $\sum_{z=1}^{m} \varpi_z = 1$ to simplify the products:

$$\text{HTSFHWGA}(h_1, \ldots, h_m) = \begin{pmatrix} \frac{\sqrt[n]{\gamma}p}{\sqrt[n]{[1+(\gamma-1)(1-p^n)]+(\gamma-1)p^n}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)q^n]-[1-q^n]}{[1+(\gamma-1)q^n]+(\gamma-1)[1-q^n]}}, \\ \sqrt[n]{\frac{[1+(\gamma-1)r^n]-[1-r^n]}{[1+(\gamma-1)r^n]+(\gamma-1)[1-r^n]}} \end{pmatrix} = (p, q, r) = h.$$

Case 3: HTSFHOWAA. For the ordered weighted arithmetic averaging operator, when all inputs are identical, we have $h_{\sigma(z)} = h$ for every position $z = 1, \ldots, m$, regardless of the permutation σ determined by the score values. Consequently,

$$\begin{aligned}\mathrm{HTSFHOWAA}(h_1,\ldots,h_m) &= \bigoplus_{z=1}^{m}(\varpi_z h_{\sigma(z)}) \\ &= \bigoplus_{z=1}^{m}(\varpi_z h) \quad (\text{since } h_{\sigma(z)} = h \text{ for all } z) \\ &= \mathrm{HTSFHWAA}(h_1,\ldots,h_m) = h.\end{aligned}$$

Case 4: HTSFHOWGA. Similarly, for the ordered weighted geometric averaging operator, when all inputs are identical, we have $h_{\sigma(z)} = h$ for all z. Therefore,

$$\begin{aligned}\mathrm{HTSFHOWGA}(h_1,\ldots,h_m) &= \bigotimes_{z=1}^{m}(h_{\sigma(z)}^{\varpi_z}) \\ &= \bigotimes_{z=1}^{m}(h^{\varpi_z}) \quad (\text{since } h_{\sigma(z)} = h \text{ for all } z) \\ &= \mathrm{HTSFHWGA}(h_1,\ldots,h_m) = h.\end{aligned}$$

Thus, in all four cases, the aggregation of identical singleton HTSFEs returns the original HTSFE h. □

Remark 3.20. The idempotency property holds for all four operators under the following conditions:

(i) All inputs are identical: $h_k = h$ for all k.

(ii) The common HTSFE h is of length one (a singleton). This condition is necessary because if h contained multiple T-SFNs, the aggregation would produce all possible combinations of elements from h, resulting in a set of cardinality ℓ^m (where ℓ is the length of h), which would not equal the original set h.

(iii) The weights sum to one, $\sum_{z=1}^{m} \varpi_z = 1$, ensuring the averaging property.

(iv) For the ordered weighted operators, when all inputs are identical, the ordering becomes irrelevant because $h_{\sigma(z)} = h$ for all z, regardless of the permutation σ.

The idempotency property is essential for the consistency and reliability of any aggregation operator in multi-criteria group decision-making. It guarantees that if all decision-makers provide identical evaluations for a given alternative across all criteria, or if all criteria are assessed with the same value, the aggregated result remains unchanged. This ensures that

(i) The operator does not artificially amplify or diminish consensus evaluations;

(ii) Redundant or repeated identical information does not skew the final decision;
(iii) The aggregation process is stable and predictable, producing outcomes that align with intuitive expectations.

Without idempotency, an aggregation operator could produce anomalous results even in cases of perfect agreement among experts or criteria, undermining the credibility of the decision-making process. The confirmation that all four proposed Hamacher operators satisfy idempotency demonstrates their suitability for practical applications where consistency and stability are paramount.

4. Multiple Criteria Group Decision Method (MCGDM) Under HT-SF environment

In real-world group decision-making scenarios, such as selecting a project manager, it is common for experts to hesitate among several possible evaluations for a candidate under a given criterion. The proposed HT-SFS framework is specifically designed to model this complexity by allowing each expert's assessment to be represented as a set of T-spherical fuzzy numbers, thereby capturing intra-expert hesitation. Furthermore, the interactions and trade-offs between different selection criteria are often non-linear and cannot be adequately captured by simple averaging or algebraic operators. By employing the generalized Hamacher t-norm and t-conorm in our aggregation operators, we introduce a flexible parameter γ that can model a spectrum of decision-making strategies, from pessimistic to optimistic, making the framework adaptable to various real-world contexts.

Let $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, ..., \mathfrak{B}_l\}$ be a set of alternatives, $\epsilon = \{\epsilon_1, \epsilon_2, ..., \epsilon_s\}$ be a collection of criteria, and $\rho = \{\rho_1, \rho_2, ..., \rho_t\}$ be a group of decision-makers. $\varpi = (\varpi_1, \varpi_2, ...\varpi_s)$ is weight vector of criteria determined by decision-makers such that $\varpi_j \in (0, 1]$ and $\sum_{j=1}^{s} \varpi_j = 1$. The MCGDM method has many steps:

Step 1: The evaluation provided by each decision-maker ρ_y for each criterion ϵ_j concerning alternative $\mathfrak{B}i$ is represented by the matrix $DM\mathfrak{B}_i$, defined as follows:

$$DM_{\mathfrak{B}_i} = [\zeta_{yj}]_{t\times s} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1s} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2s} \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_{t1} & \zeta_{t2} & \cdots & \zeta_{ts} \end{pmatrix}.$$

Here ζ_{yj} denotes the HTSFE related to criteria ϵ_j determined by decision maker ρ_y.

Step 2: For $i = 1, 2, ..., l$, $HTSF_i$ obtained as:

$$HTSF_i = \Big\{(\epsilon_j, h_{\epsilon_j}) : j = 1, ..., s\Big\}.$$

Here $h_{\epsilon_j} = \cup_{y=1}^{t}\{\zeta_{yj}\}$.
Step 3:The HT-SF element for κ_i with $i = 1, ..., l$, denoted by $\mathfrak{B}_i$:

$$\mathfrak{B}_i = \bigoplus_{j=1}^{s} w_j \mathfrak{h}_{\epsilon_j}.$$

Step 4: Find SC($\mathfrak{B}_i$) where $(i = 1, ..., l)$.
Step 5: List the alternatives $\mathfrak{B}_i$ in ascending order based on their score values.
Step 6: Select the option with the highest possible score.
Flowchart of the algorithm is given in Figure 1.

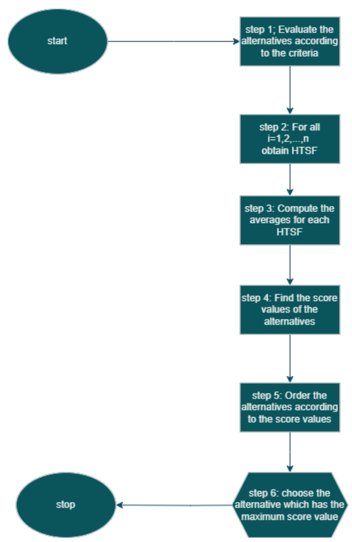

Figure 1. Flowchart of the proposed method

5. Illustrative Example

Consider the need to fill a project manager position in children's protection at an international organization. After posting the job opening, five candidates—denoted as $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4$, and $\mathfrak{B}_5$—applied for the role. The organization's director selects three experts to assess the candidates based on specific criteria. To evaluate the options, we will use three criteria: ϵ_1 for experience, ϵ_2 for quick response, and ϵ_3 for effectiveness of management. After conducting interviews, the experts have established the following weight vector for these criteria:

$(0.300, 0.250, 0.450)^T$.

Step 1: The alternatives are assessed by experts using HT-SFNs that correlate to linguistic factors as outlined in Table 1, with a total of four criteria being considered.

TABLE 1. A review of the candidates using linguistic variables

degrees	HT-SFNs
Very Low (VL)	(0.075,0.600,0.925)
Low (L)	(0.230,0.643,0.770)
Medium Low (ML)	(0.370,0.567,0.630)
Medium (M)	(0.5,0.5,0.5)
Medium High (MH)	(0.630,0.446,0.370)
High (H)	(0.770,0.375,0.230)
Very High (VH)	(0.925,0.405,0.075)

		ϵ_1	ϵ_2	ϵ_3
$D_{\mathfrak{B}_1}$	ρ_1	(0.630,0.446,0.370)	(0.230,0.643,0.770)	*
	ρ_2	(0.075,0.600,0.925)	(0.925,0.405,0.075)	(0.500,0.500,0.500)
	ρ_3	*	(0.770,0.375,0.230)	*

		ϵ_1	ϵ_2	ϵ_3
$D_{\mathfrak{B}_2}$	ρ_1	*	(0.230,0.643,0.770)	*
	ρ_2	(0.230,0.643,0.770)	(0.370,0.567,0.630)	(0.500,0.500,0.500)
	ρ_3	(0.770,0.375,0.230)	(0.075,0.600,0.925)	(0.630,0.446,0.370)

		ϵ_1	ϵ_2	ϵ_3
$D_{\mathfrak{B}_3}$	ρ_1	(0.370,0.567,0.630)	(0.630,0.446,0.370)	(0.075,0.600,0.925)
	ρ_2	(0.925,0.405,0.075)	(0.770,0.375,0.230)	*
	ρ_3	(0.630,0.446,0.370)	(0.075,0.600,0.925)	(0.500,0.500,0.500)

		ϵ_1	ϵ_2	ϵ_3
$D_{\mathfrak{B}_4}$	ρ_1	(0.925,0.405,0.075)	(0.500,0.500,0.500)	(0.075,0.600,0.925)
	ρ_2	(0.925,0.405,0.075)	*	(0.500,0.500,0.500)
	ρ_3	*	(0.075,0.600,0.925)	(0.500,0.500,0.500)

		ϵ_1	ϵ_2	ϵ_3
$D_{\mathfrak{B}_5}$	ρ_1	(0.370,0.567,0.630)	(0.500,0.500,0.500)	(0.075,0.600,0.925)
	ρ_2	(0.075,0.600,0.925)	(0.370,0.567,0.630)	(0.630,0.446,0.370)
	ρ_3	(0.925,0.405,0.075)	(0.075,0.600,0.925)	(0.500,0.500,0.500)

Step 2: The $HTSF_i$ $(i = 1\ to\ 5)$ are obtained by applying the HT-SF decision matrices provided in Step 1.

follows:

$$\begin{aligned}
HTSF_1 = \Big\{ &\Big(\epsilon_1, \{(0.630, 0.446, 0.370), (0.075, 0.600, 0.925)\}\Big),\\
&\Big(\epsilon_2, \{(0.230, 0.643, 0.770), (0.925, 0.405, 0.075), (0.770, 0.375, 0.230)\}\Big),\\
&\Big(\epsilon_3, \{(0.500, 0.500, 0.500)\}\Big)\Big\},\\
HTSF_2 = \Big\{ &\Big(\epsilon_1, \{(0.230, 0.643, 0.770), (0.770, 0.375, 0.230)\}\Big),\\
&\Big(\epsilon_2, \{(0.230, 0.643, 0.770), (0.370, 0.567, 0.630), (0.075, 0.600, 0.925)\}\Big),\\
&\Big(\epsilon_3, \{(0.500, 0.500, 0.500), (0.630, 0.446, 0.370)\}\Big)\Big\},\\
HTSF_3 = \Big\{ &\Big(\epsilon_1, \{(0.370, 0.567, 0.630), (0.925, 0.405, 0.075), (0.630, 0.446, 0.370)\}\Big),\\
&\Big(\epsilon_2, \{(0.630, 0.446, 0.370), (0.770, 0.375, 0.230), (0.075, 0.600, 0.925)\}\Big),\\
&\Big(\epsilon_3, \{(0.075, 0.600, 0.925), (0.500, 0.500, 0.500)\}\Big)\Big\},\\
HTSF_4 = \Big\{ &\Big(\epsilon_1, \{(0.925, 0.405, 0.075), (0.925, 0.405, 0.075)\}\Big),\\
&\Big(\epsilon_2, \{(0.500, 0.500, 0.500), (0.075, 0.600, 0.925)\}\Big),\\
&\Big(\epsilon_3, \{(0.075, 0.600, 0.925), (0.500, 0.500, 0.500), (0.500, 0.500, 0.500)\}\Big)\Big\},\\
HTSF_5 = \Big\{ &\Big(\epsilon_1, \{(0.370, 0.567, 0.630), (0.075, 0.600, 0.925), (0.925, 0.405, 0.075)\}\Big),\\
&\Big(\epsilon_2, \{(0.500, 0.500, 0.500), (0.370, 0.567, 0.630), (0.075, 0.600, 0.925)\}\Big),\\
&\Big(\epsilon_3, \{(0.075, 0.600, 0.925), (0.630, 0.446, 0.370), (0.500, 0.500, 0.500)\}\Big)\Big\}
\end{aligned}$$

Step 3: (HTSFHWAA,HTSFHWGA) values of $HTSF_i$, $(i = 1\ to\ 5)$ are obtained as in Table 2, for $\gamma = 1$ and $n = 4$.

TABLE 2. HTSFHWAA and HTSFHWGA values of $HTSF_i\,(i = 1\ to\ 5)$

	HTSFHWAA	HTSFHWGA
$\mathfrak{B}_1$	{(0.526, 0.515, 0.513),(0.740, 0.459, 0.285),(0.638, 0.450, 0.377) (0.412, 0.563, 0.684), (0.711, 0.502, 0.388),(0.588, 0.492, 0.510)}	{(0.474, 0.537, 0.605),(0.676, 0.465, 0.427),(0.639, 0.460, 0.429), (0.250, 0.576, 0.795),(0.364, 0.521, 0.737),(0.342, 0.518, 0.738)
$\mathfrak{B}_2$	{(0.415, 0.575, 0.640), (0.428, 0.557, 0.606),(0.413, 0.565, 0.676), (0.520, 0.547, 0.562), (0.527, 0.530, 0.532),(0.518, 0.537, 0.596), (0.610, 0.490, 0.446), (0.614, 0.474, 0.422),(0.609, 0.481, 0.474), (0.652, 0.465, 0.389), (0.655, 0.450, 0.368)(0.651, 0.457, 0.414)}	{(0.350, 0.592, 0.700),(0.394, 0.570, 0.651),(0.265, 0.579, 0.802), (0.384, 0.579, 0.687),(0.432, 0.556, 0.634),(0.291, 0.565, 0.793), (0.506, 0.527, 0.599),(0.568, 0.495, 0.515),(0.385, 0.508, 0.743), (0.554, 0.508, 0.575),(0.620, 0.472, 0.474),(0.422, 0.487, 0.731)}
$\mathfrak{B}_3$	{(0.461, 0.548, 0.676), (0.319, 0.582, 0.761),(0.274, 0.590, 0.837), (0.520, 0.505, 0.498), (0.443, 0.536, 0.769),(0.429, 0.544, 0.636), (0.746, 0.496, 0.364),(0.726, 0.527, 0.417), (0.723, 0.535, 0.469), (0.760, 0.456, 0.263),(0.742, 0.485, 0.302), (0.739, 0.492, 0.340), (0.543, 0.511, 0.580), (0.478, 0.542, 0.659),(0.467, 0.550, 0.733), (0.582, 0.470, 0.424), (0.532, 0.499, 0.485),(0.525, 0.506, 0.544)}	{(0.224, 0.561, 0.775),(0.195, 0.583, 0.794),(0.131, 0.591, 0.882), (0.518, 0.513, 0.531),(0.453, 0.540, 0.585),(0.305, 0.550, 0.771), (0.305, 0.526, 0.754),(0.266, 0.551, 0.775),(0.178, 0.561, 0.870), (0.690, 0.463, 0.425),(0.609, 0.499, 0.513),(0.414, 0.512, 0.743), (0.264, 0.532, 0.757),(0.230, 0.557, 0.777),(0.154, 0.566, 0.872), (0.606, 0.473, 0.442),(0.532, 0.507, 0.523),(0.360, 0.519, 0.746)}
$\mathfrak{B}_4$	{(0.732, 0.510, 0.393), (0.748, 0.470, 0.284),(0.748, 0.470, 0.284), (0.723, 0.535, 0.469),(0.739, 0.492, 0.340), (0.739, 0.492, 0.340), (0.732, 0.510, 0.393), (0.748, 0.470, 0.284),(0.748, 0.470, 0.284), (0.723, 0.535, 0.469), (0.739, 0.492, 0.340),(0.739, 0.492, 0.340)}	{(0.287, 0.535, 0.761),(0.654, 0.477, 0.457),(0.654, 0.477, 0.457), (0.178, 0.561, 0.870),(0.414, 0.512, 0.743),(0.414, 0.512, 0.743), (0.287, 0.535, 0.761),(0.654, 0.477, 0.457),(0.654, 0.477, 0.457), (0.178, 0.561, 0.870),(0.414, 0.512, 0.743),(0.414, 0.512, 0.743)}
$\mathfrak{B}_5$	{(0.382, 0.564 0.723), (0.551, 0.494, 0.4469),(0.471, 0.519, 0.536), (0.319, 0.582, 0.761), (0.534, 0.509, 0.498),(0.443, 0.536, 0.569), (0.274, 0.590, 0.837), (0.527, 0.517, 0.559),(0.429, 0.544, 0.636), (0.354, 0.574, 0.814), (0.543, 0.502, 0.539),(0.457, 0.528, 0.614), (0.262, 0.592, 0.852), (0.525, 0.518, 0.571),(0.426, 0.545, 0.650), (0.075, 0.600, 0.925), (0.517, 0.526, 0.639),(0.410, 0.553, 0.722), (0.732, 0.510, 0.393), (0.770, 0.446, 0.248),(0.748, 0.470, 0.284), (0.726, 0.527, 0.417), (0.765, 0.460, 0.263),(0.742, 0.485, 0.302), (0.723, 0.535, 0.469), (0.762, 0.467, 0.297),(0.739, 0.492, 0.340)}	{(0.211, 0.569, 0.781),(0.535, 0.504, 0.519),(0.488, 0.523, 0.550), (0.195, 0.583, 0.794),(0.497, 0.523, 0.560),(0.453, 0.540, 0.585), (0.131, 0.591, 0.882),(0.335, 0.534, 0.761),(0.305, 0.550, 0.771), (0.131, 0.580, 0.856),(0.334, 0.519, 0.738),(0.304, 0.536, 0.748), (0.121, 0.592, 0.865),(0.310, 0.536, 0.755),(0.281, 0.552, 0.763), (0.081, 0.600, 0.925),(0.208, 0.547, 0.858),(0.189, 0.562, 0.863), (0.287, 0.535, 0.761),(0.711, 0.451, 0.395),(0.654, 0.477, 0.457), (0.266, 0.551, 0.775),(0.664, 0.477, 0.472),(0.609, 0.499, 0.513), (0.178, 0.561, 0.870),(0.455, 0.492, 0.731),(0.414, 0.512, 0.743)}

Step 4: Table 3 contains the score values for the $\mathfrak{B}_i, (i = 1\ to\ 5)$ under the scoring function:

TABLE 3. According to HTSFHWAA and HTSFHWGA values, the score values for $\mathfrak{B}_i$

	HTSFHWAA	HTSFHWGA
$SC(\mathfrak{B}_1)$	0.1117	-0.16077
$SC(\mathfrak{B}_2)$	0.031839	-0.17264
$SC(\mathfrak{B}_3)$	0.039224	-0.30197
$SC(\mathfrak{B}_4)$	0.329244	-0.24055
$SC(\mathfrak{B}_5)$	-0.00964	-0.31718

Step 5: The candidates are ranked according to their scores using Eq. 2.1 as shown in Table 4.

TABLE 4. HTSFHWAA and HTSFHWGA operators were used to calculate the score values for $\mathfrak{B}_i$

	Ordering
HTSFHWAA	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
HTSFHWGA	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

Step 6: Table 4, illustrates $\mathfrak{B}_4$ and $\mathfrak{B}_1$ are the optimum alternatives, respectively.

6. The Influences of Parameter γ on the Outcomes

We assign different values to the parameter γ, ranging from 1 to 10, and rank candidates based on their scores calculated using the HTSFHWAA and HTSFHWGA methods. This approach demonstrates the impact of the γ variable on the findings of MCGDM. Table 5, displays the ranking positions of the candidates according to their score values, as well as their standings based on the HTSFHWAA and HTSFHWGA operators. The same individual consistently emerges as the top candidate, and the ranking of competitors remains stable. The order of scores can be expressed as follows: $SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$ for $\gamma = 1$ and $SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$ for the case when $\gamma \neq 1$. According to Table 6, the ranking order of candidates for the HTSFHWGA operator follows the pattern $SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$, which remains unchanged when the value of γ is adjusted, again except for the case where $\gamma = 1$. Furthermore, the best choice for values of $1 \leq \gamma \leq 10$ is consistent across the board.

Table 5. Different γ values in the HTSFHWAA.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	0.1117	0.031839	0.039224	0.329244	-0.00964	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
2	0.09017	0.022169	0.011491	0.27861	-0.03338	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
3	0.0777	0.016294	-0.00446	0.24829	-0.04837	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
4	0.06925	0.012061	-0.01524	0.22742	-0.0584	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
5	0.06301	0.008758	-0.02318	0.2119	-0.06573	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
6	0.05816	0.006057	-0.02934	0.19979	-0.0714	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
7	0.05424	0.003779	-0.03432	0.18999	-0.07595	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
8	0.051	0.001814	-0.03845	0.18187	-0.07971	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
9	0.04824	-0.00009	-0.04194	0.17498	-0.08289	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
10	0.04587	-0.0014456	-0.04496	0.16906	-0.08563	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

Graphical representation of the Table 5 is given in Figure 2.

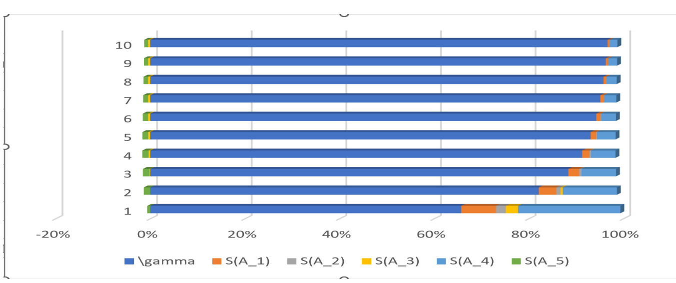

Figure 2.

TABLE 6. Different values of γ in the HTSFHWGA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	-0.16077	-0.17264	-0.30197	-0.24055	-0.31718	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
2	-0.12191	-0.16577	-0.24969	-0.19564	-0.27856	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
3	-0.09703	-0.15517	-0.22429	-0.16759	-0.25747	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
4	-0.0792	-0.14563	-0.20723	-0.14689	-0.2428	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
5	-0.06556	-0.13746	-0.19448	-0.13066	-0.23169	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
6	-0.05468	-0.13046	-0.18437	-0.11742	-0.22286	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
7	-0.04572	-0.12443	-0.17607	-0.10633	-0.2156	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
8	-0.03819	-0.11918	-0.16908	-0.09685	-0.20949	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
9	-0.03175	-0.11456	-0.16307	-0.08862	-0.20426	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
10	-0.02615	-0.11046	-0.15782	-0.08137	-0.1997	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

Graphical representation of the Table 6 is given in Figure 3.

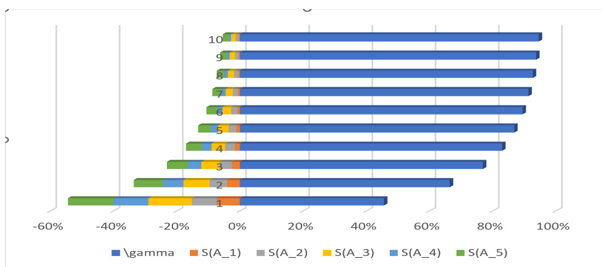

FIGURE 3.

In our previous study [73], we proposed the hesitant T-spherical Dombi aggregation operators (HTSDFWAA and HTSDFWGA), and when we apply these two operators to the data of this example, we will see the results shown in Table 7 and Table 8.

TABLE 7. Different values of γ in the HTSDFWAA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	0.18379	0.07650	0.12782	0.45877	0.07773	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
2	0.26804	0.12568	0.20279	0.59952	0.15227	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
3	0.30389	0.14955	0.23103	0.64651	0.17939	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
4	0.32317	0.16320	0.245640	0.66909	0.19293	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
5	0.33496	0.17189	0.25453	0.68229	0.20099	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
6	0.34296	0.17786	0.26049	0.69092	0.20632	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
7	0.34871	0.18220	0.26476	0.69701	0.21011	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
8	0.35303	0.18548	0.26796	0.70153	0.21294	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
9	0.35640	0.18805	0.27045	0.70502	0.21513	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$
10	0.35910	0.19012	0.27244	0.70780	0.21687	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5) > SC(\mathfrak{B}_2)$

TABLE 8. Different values of γ in the HTSDFWGA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	-0.23549	-0.23741	-0.37040	-0.32678	-0.38958	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
2	-0.32107	-0.32350	-0.43954	-0.40929	-0.46380	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
3	-0.35491	-0.36033	-0.46542	-0.43699	-0.49118	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
4	-0.37234	-0.38003	-0.47872	-0.45038	-0.50502	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
5	-0.38283	-0.39214	-0.48679	-0.45823	-0.51333	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
6	-0.38981	-0.40030	-0.49219	-0.46338	-0.51886	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
7	-0.39478	-0.40616	-0.49604	-0.46701	-0.52279	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
8	-0.39849	-0.41057	-0.49892	-0.46971	-0.52573	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
9	-0.40138	-0.41399	-0.50116	-0.47180	-0.52801	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
10	-0.40368	-0.41674	-0.50295	-0.47346	-0.52983	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

When we compare the results of (HTSFHWAA) and (HTSDFWAA), we will notice that the optimum result is $\mathfrak{B}_4$ for both operators. Similarly, when we examine the results of (HTSFHWGA) and (HTSDFWGA), we will see that the optimum result is $\mathfrak{B}_1$ in both cases.

Tables 9,10,11,12 analyze the influence of parameters γ (Hamacher coefficient) and n (dimensionality) on two HT-SF aggregation operators. For the arithmetic operator (HTSFHWAA, Tables 9 & 11), alternative $\mathfrak{B}_4$ consistently ranks highest, demonstrating robust preference regardless of γ or n. In contrast, the geometric operator (HTSFHWGA, Tables 10 & 12) shows greater sensitivity: rankings shift between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as γ increases, and the order changes notably when n varies. This highlights the arithmetic operator's stability versus the geometric operator's flexibility under parameter adjustments in MCGDM.

TABLE 9. When $n = 5$ in the HTSFHWAA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	0.09774	0.0291	0.045	0.147	0.0035	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
2	0.08031	0.0234	0.0225	0.2159	-0.0188	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
3	0.06992	0.0198	0.0095	0.1893	-0.0314	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
4	0.06274	0.0171	0.0007	0.1713	-0.04	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
5	0.05737	0.015	-0.0059	0.1581	-0.0462	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
6	0.05315	0.0132	-0.011	0.148	-0.051	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
7	0.04972	0.0117	-0.0151	0.1398	-0.0549	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
8	0.04686	0.0104	-0.0185	0.1331	-0.0582	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
9	0.04443	0.0093	-0.0214	0.1275	-0.0609	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
10	0.04232	0.0082	-0.0239	0.1226	-0.0632	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$

TABLE 10. When $n = 5$ values in the HTSFHWGA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	-0.1458	-0.1453	-0.2698	-0.2225	-0.2842	$SC(\mathfrak{B}_2) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
2	-0.1155	-0.141	-0.2263	-0.1874	-0.2509	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
3	-0.0952	-0.1328	-0.2045	-0.1645	-0.232	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
4	-0.0801	-0.125	-0.1895	-0.1471	-0.2186	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
5	-0.0684	-0.1182	-0.1781	-0.1332	-0.2082	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
6	-0.0587	-0.1123	-0.169	-0.1217	-0.2	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
7	-0.0507	-0.1071	-0.1614	-0.1119	-0.1928	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
8	-0.0438	-0.1025	-0.155	-0.1348	-0.1869	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
9	-0.0378	-0.0984	-0.1498	-0.0961	-0.1818	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
10	-0.0326	-0.0947	-0.1444	-0.0895	-0.1773	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

TABLE 11. When $n = 6$ in the HTSFHWAA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	0.0853	0.0247	0.0472	0.2585	0.0102	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
2	0.0723	0.0214	0.0289	0.2208	-0.0085	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_5)$
3	0.0639	0.0192	0.0182	0.1968	-0.0192	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
4	0.0579	0.0175	0.0108	0.1797	-0.0265	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
5	0.0533	0.0162	0.0054	0.1666	-0.0319	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
6	0.0496	0.015	0.0011	0.1562	-0.0361	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
7	0.0465	0.014	-0.0024	0.1476	-0.0395	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
8	0.044	0.0131	-0.0053	0.1404	-0.0423	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
9	0.0418	0.0123	-0.0079	0.1343	-0.0447	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
10	0.0399	0.0116	-0.01	0.1289	-0.0468	$SC(\mathfrak{B}_4) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

TABLE 12. When $n = 6$ in the HTSFHWGA operator are ranked in descending order.

γ	$SC(\mathfrak{B}_1)$	$SC(\mathfrak{B}_2)$	$SC(\mathfrak{B}_3)$	$SC(\mathfrak{B}_4)$	$SC(\mathfrak{B}_5)$	Ranking order
1	-0.1317	-0.1221	-0.2421	-0.2046	-0.2556	$SC(\mathfrak{B}_2) > SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
2	-0.1079	-0.1194	-0.2056	-0.1768	-0.2272	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
3	-0.0951	-0.113	-0.1869	-0.1581	-0.2106	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
4	-0.0793	-0.1069	-0.1738	-0.1438	-0.1985	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
5	-0.0695	-0.1014	-0.1639	-0.1322	-0.1819	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
6	-0.0614	-0.0966	-0.1557	-0.1224	-0.1814	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
7	-0.0546	-0.0923	-0.149	-0.1141	-0.175	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
8	-0.0487	-0.0885	-0.1432	-0.1069	-0.1695	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
9	-0.0435	-0.0851	-0.1382	-0.1005	-0.1647	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$
10	-0.0389	-0.0821	-0.1337	-0.0948	-0.1604	$SC(\mathfrak{B}_1) > SC(\mathfrak{B}_4) > SC(\mathfrak{B}_2) > SC(\mathfrak{B}_3) > SC(\mathfrak{B}_5)$

7. CONCLUSION

The concept of Hesitant T-Spherical fuzzy sets and its operations, such as complement, union, and intersection, has been defined in this paper. To make it more comprehensible, some examples are provided related to the defined operations. Based on Hamacher t-norm and t-conorm operations, we defined arithmetic operations for HT-SFEs and presented several aggregation operators, including HTSFHWAA, HTSFHWGA, HTSFHOWAA, and HTSFHOWGA. Moreover, we developed an MCGDM method and applied it to the problem of selecting the best candidate for a project manager position in child protection at an international organization. In the future, we aim to study other aggregation operators, similarity measures, distance measures, and decision-making methods according to VIKOR, AHP, TOPSIS, etc.

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