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On the solution of fuzzy Sylvester matrix equations without using Kronecker product and Vec-operation

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Abstract. It is known that using Kronecker product and Vec-operation for solving matrix equations under fuzzy uncertainty, make the dimension of the crisp transformed systems much higher than the dimension of the initial fuzzy matrix equation. In this paper, a new iterative approach is proposed for solving Sylvester type of fuzzy matrix equations without using Kronecker product and Vec-operation. Based on the proposed method, a fuzzy Sylvester matrix equation is transformed to two corresponding crisp equations and then homotopy analysis method (HAM) is applied for obtaining a fuzzy approximate solution of the original equation. The convergence of the proposed method is showed by presenting two conditions. Finally, to show the ability and efficiency of the proposed method three numerical examples are given.

Keywords: Fuzzy number, Kronecker product, Vec-operation, Fuzzy Sylvester matrix equation, Homotopy analysis method.

2020 Mathematics subject classification: 15A24, 65F45, 15B15.

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1. Introduction

It is well known that if some parameters of a matrix equation are considered by fuzzy quantities, then we arrive at a fuzzy matrix equation. There are many works that focus on developing the numerical solutions of fuzzy matrix equations, for example see [3, 7, 26]. A fuzzy linear matrix equation can be presented as follows

$$\mathbf{A}\widetilde{\mathbf{X}}\mathbf{E} + \mathbf{D}\widetilde{\mathbf{X}}\mathbf{B} = \widetilde{\mathbf{C}}, \tag{1.1}$$

where the matrices $\mathbf{A}, \mathbf{E}, \mathbf{D}$ and $\mathbf{B}$ are crisp-valued and the matrices $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{C}}$ are fuzzy-valued. If $\mathbf{E} = \mathbf{D} = \mathbf{I}$, then Eq. (1.1) is considered as Sylvester type of fuzzy matrix equations, also if in addition $\mathbf{B} = \mathbf{A}^T$, then Eq. (1.1) is considered as Lyapunov type of fuzzy matrix equations [37].

In the last few years, interest in solving various types of the fuzzy matrix equations has increased. For instance, some authors [13, 17, 18, 19, 27, 30, 35] have applied various methods to solve the simplest form of a fuzzy matrix equation as

$$\mathbf{A}\widetilde{\mathbf{X}} = \widetilde{\mathbf{B}}, \tag{1.2}$$

and also its dual type [14, 15, 23] as

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{B}} = \mathbf{C}\widetilde{\mathbf{X}} + \widetilde{\mathbf{D}}.$$

Moreover, the other form of a fuzzy linear matrix equation as

$$\mathbf{A}\widetilde{\mathbf{X}}\mathbf{B} = \widetilde{\mathbf{C}},$$

was studied in [28, 36]. In 2017, Guo et al. [16, 20] provided new numerical approaches to solve a special case of fuzzy matrix equations in the form

$$\widetilde{\mathbf{A}}\mathbf{X} = \widetilde{\mathbf{B}},$$

in real and complex states. On the other hand, one of the most famous types of fuzzy matrix equations is fuzzy Sylvester matrix equations. In the following, we introduce some of the works done in this field.

In 2011, Salkuyeh [34] used the accelerated over-relaxation (AOR) method to obtain an approximate solution of a fuzzy Sylvester matrix equation. According to the method presented by Salkuyeh [34], a fuzzy Sylvester matrix equation is converted to a crisp linear system of equations with higher dimension, by using Vec-operator and Kronecker product. In 2012, Fariborzi Araghi and Hosseinzadeh [9] applied ABS method to obtain a fuzzy solution for fuzzy Sylvester matrix equations. According to their approach, a fuzzy Sylvester matrix equation is converted to three crisp linear system. In 2013, Sadeghi et al. [33] have presented some conditions for the existence of fuzzy solutions of fuzzy

Sylvester matrix equations and also introduced a new iteration method to solve these equations. They first transformed a fuzzy Sylvester matrix equations into a fuzzy linear system and then applied the method proposed by Ding and Chen [5] to solve the obtained fuzzy linear system. Also, in 2013, Rivaz and Salary Pour Sharif Abad [32] applied the Adomian decomposition method and the Jacobi iteration method for obtaining the approximate fuzzy solution of fuzzy Sylvester matrix equations. Similar to the above-mentioned authors, they first transformed a fuzzy Sylvester matrix equation into a crisp linear system of equations with higher dimension and then used their proposed methods. In 2018, He et al. [21] provided a new approach to solve the Sylvester type of fuzzy linear matrix equations such that it keeps the constructure of the original equation. In 2025, Wan Daud et al. [39] presented a new technique to solve fully fuzzy matrix equations. Recently, Stepanovic and Tepavcevic [38] studied the fuzzified matrix space and solvability of matrix equations.

In addition to the above research works, there are also some studies where all parameters of a matrix equation are considered as fuzzy quantities, for instance see [3, 6, 7, 8, 31].

In this paper, we propose a new iteration method to solve the Sylvester type of a fuzzy linear matrix equation in the form

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}\mathbf{B} = \widetilde{\mathbf{C}}, \tag{1.3}$$

where $\mathbf{A} = (a_{ij})_{m\times m}$ and $\mathbf{B} = (b_{ij})_{n\times n}$ are crisp-valued matrices and $\widetilde{\mathbf{X}} = (\widetilde{x}_{ij})_{m\times n}$ and $\widetilde{\mathbf{C}} = (\widetilde{c}_{ij})_{m\times n}$ are fuzzy-valued matrices. Based on the suggested method, we firstly transform the fuzzy Sylvester equation (1.3) into two crisp-valued Sylvester matrix equations by using two concepts α-center and α-radius. In the second step, the well-known homotopy analysis method (HAM) is used to solve these two crisp-valued Sylvester matrix equations. To show the convergence of our proposed method, two conditions are presented. Finally, the ability of the suggested method is shown by three numerical examples. It should be noted that due to the lack of use of the Kronecker product and Vec-operation, the dimensions of two crisp-valued Sylvester linear matrix equations are equal to the dimension of original fuzzy Sylvester linear matrix equation. This is an important advantage of our proposed method over some existing approaches.

Throughout the paper, vectors and matrices will be denoted by nonbold and bold capital letters, respectively. Also, elements of vectors and matrices will be denoted by small letters and also fuzzy quantities will be specified by a bar over a symbol.

This paper is organized as follows. In Section 2, some definitions and remarks of fuzzy mathematics are given. In Section 3, our iteration method is presented to solve the fuzzy Sylvester equation (1.3) and also some conditions for the convergence of the suggested method are given. Three computational examples are presented in Section 4. Finally, the paper is ended with conclusions and future research.

2. Preliminaries

Here, we recall some of the fundamental definitions of fuzzy mathematics.

Definition 2.1. [1] We say that a fuzzy set $\widetilde{x}$ with the membership function $\mu_{\widetilde{x}} : \mathbb{R} \longrightarrow [0,1]$ is a fuzzy number if

- **(i):** $\exists a \in \mathbb{R};\ \mu_{\widetilde{x}}(a) = 1$,
- **(ii):** $\forall \beta \in [0,1]\ \&\ \forall a, b \in \mathbb{R};\ \mu_{\widetilde{x}}(\beta a + (1-\beta)b) \geqslant \min\{\mu_{\widetilde{x}}(a), \mu_{\widetilde{x}}(b)\}$,
- **(iii):** $\forall a \in \mathbb{R};\ \{b \in \mathbb{R} : \mu_{\widetilde{x}}(b) > a\}$ is an open subset of $\mathbb{R}$,
- **(iv):** The set $\overline{\{t \in \mathbb{R} : \mu_{\widetilde{x}}(t) > 0\}} \subset \mathbb{R}$ is compact.

The set of all fuzzy numbers will be indicated by $\mathbb{F}$. Clearly, any crisp real number x can be considered as a fuzzy number $\widetilde{x}$ where

$$\mu_{\widetilde{x}}(a) = \begin{cases} 1, & a = x; \\ 0, & a \neq x, \end{cases}$$

and consequently $\mathbb{R} \subset \mathbb{F}$ [2]. Moreover, for $\alpha \in (0,1]$, α-levels of $\widetilde{x}$ are specified by $[\widetilde{x}]_\alpha = \{a \in \mathbb{R} : \mu_{\widetilde{x}}(a) \geqslant \alpha\}$ and especially for $\alpha = 0$ by $[\widetilde{x}]_0 = \overline{\{a \in \mathbb{R} : \mu_{\widetilde{x}}(a) > 0\}}$. Also, based on Definition 2.1, it can be shown that α-levels of $\widetilde{x}$ are bounded closed intervals in $\mathbb{R}$ [40], or in other words, for each $\alpha \in [0,1]$

$$[\widetilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)].$$

Clearly, $\widetilde{x}$ is a crisp real number if and only if $\underline{x}(\alpha) = \overline{x}(\alpha), \forall \alpha \in [0,1]$.

In continuation, we present some conditions for sets of intervals to construct the α-levels of fuzzy numbers.

Lemma 2.2. [22] *Suppose that* $\{[\underline{x}(\alpha), \overline{x}(\alpha)]\ : \alpha \in [0,1]\}$ *be a family of closed intervals in* $\mathbb{R}$ *and also two following conditions hold*

- **(a):** *if* $\alpha_1, \alpha_2 \in [0,1]$ *and* $\alpha_1 \leqslant \alpha_2$ *then* $[\underline{x}(\alpha_2), \overline{x}(\alpha_2)] \subseteq [\underline{x}(\alpha_1), \overline{x}(\alpha_1)]$,
- **(b):** *if the non-decreasing sequence* $\{\alpha_i\}_{i=0}^{\infty}$ *converges to* α *on* $[0,1]$, *then*

  $[\lim_{i\to\infty} \underline{x}(\alpha_i), \lim_{i\to\infty} \overline{x}(\alpha_i)] = [\underline{x}(\alpha), \overline{x}(\alpha)]$,

then the set $\{[\underline{x}(\alpha), \overline{x}(\alpha)]\ : 0 \in [0,1]\}$ *constructs the* α*-levels of* $\widetilde{x} \in \mathbb{F}$.

Moreover, if the bounded closed intervals $[\underline{x}(\alpha), \overline{x}(\alpha)]$, $\alpha \in [0,1]$, *construct the* α*-levels of an arbitrary fuzzy number, then the above-mentioned conditions* (a) *and* (b) *are satisfied.*

Remark 2.3. [29] The conditions presented in Lemma 2.2 are equivalent to the following conditions:

(i): The real function $\underline{x}(\cdot)$ is bounded, left-continuous and non-decreasing in the interval $[0, 1]$.
(ii): The real function $\overline{x}(\cdot)$ is bounded, left-continuous and non-increasing in the interval $[0, 1]$.
(iii): $\underline{x}(\alpha) \leqslant \overline{x}(\alpha)$, $\forall \alpha \in [0, 1]$.

In following, the α-levels and interval arithmetic are used to present the arithmetic operations on $\mathbb{F}$. Let $\widetilde{x}$ and $\widetilde{y}$ be two arbitrary fuzzy numbers and also $\beta \in \mathbb{R}$, then the α-levels of $\widetilde{y} + \widetilde{x}$ and $\beta \cdot \widetilde{y}$ is obtained by

$$[\widetilde{y} + \widetilde{x}]_\alpha = \{b + a : b \in [\widetilde{y}]_\alpha,\ a \in [\widetilde{x}]_\alpha\} = [\underline{y}(\alpha) + \underline{x}(\alpha), \overline{y}(\alpha) + \overline{x}(\alpha)],$$

$$[\beta \cdot \widetilde{y}]_\alpha = \{\beta b : b \in [\widetilde{y}]_\alpha\} = \begin{cases} [\beta \underline{y}(\alpha), \beta \overline{y}(\alpha)], & \beta \geqslant 0, \\ [\beta \overline{y}(\alpha), \beta \underline{y}(\alpha)], & \beta < 0. \end{cases}$$

One of the most commonly used types of fuzzy numbers is the trapezoidal fuzzy number. This type of fuzzy number is defined as follows.

Definition 2.4. A fuzzy number $\widetilde{x}$ is called a trapezoidal fuzzy number if its membership function is defined as

$$\mu_{\widetilde{x}}(t) = \begin{cases} \frac{t-a}{b-a}, & a \leqslant t \leqslant b, \\ 1, & b \leqslant t \leqslant c, \\ \frac{d-t}{d-c}, & c \leqslant t \leqslant d, \end{cases}$$

where $a, b, c, d \in \mathbb{R}$.

The above trapezoidal fuzzy number is denoted as an ordered quadruple in the form (a, b, c, d). The trapezoidal fuzzy number $\widetilde{x} = (a, b, c, d)$ is illustrated in Figure 1. Also, if in the trapezoidal fuzzy number $\widetilde{x} = (a, b, c, d)$ holds equality $a = b = m$, then the trapezoidal fuzzy number becomes a triangular fuzzy number $\widetilde{x} = (a, m, d)$. Therefore, triangular fuzzy numbers are a special case of trapezoidal fuzzy numbers. It can be easily shown that the α-levels of the trapezoidal fuzzy number $\widetilde{x} = (a, b, c, d)$ can be obtained as

$$[\widetilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)] = [a + (b - a)r, d - (d - c)r].$$

It should be noted that all fuzzy numbers considered in this work are of the trapezoidal type.

Definition 2.5. Suppose that $\widetilde{y}$ and $\widetilde{x}$ be two arbitrary fuzzy numbers. We say that $\widetilde{y} = \widetilde{x}$, if and only if $[\widetilde{y}]_\alpha = [\widetilde{x}]_\alpha$, $\forall \alpha \in [0, 1]$, i.e.

$$\underline{y}(\alpha) = \underline{x}(\alpha), \qquad \overline{y}(\alpha) = \overline{x}(\alpha).$$

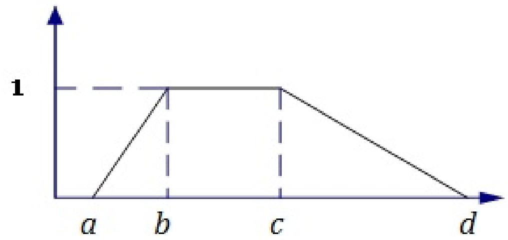

Figure 1. Representation of the trapezoidal fuzzy number $\widetilde{x} = (a, b, c, d)$.

Definition 2.6. [11, 12] α-radius and α-center of the fuzzy number $\widetilde{x}$ are denoted by $x^R(\alpha)$ and $x^C(\alpha)$, respectively, and obtained by

$$x^R(\alpha) = \frac{\overline{x}(\alpha) - \underline{x}(\alpha)}{2}, \qquad x^C(\alpha) = \frac{\underline{x}(\alpha) + \overline{x}(\alpha)}{2}, \qquad \forall \alpha \in [0, 1].$$

Remark 2.7. [12] In order to the concepts of α-center and α-radius of an arbitrary fuzzy number, the following notes can be easily showed:

(1) The α-center and α-radius are two real-valued functions on $[0, 1]$.
(2) If $x^R(\alpha) = 0$, $\forall \alpha \in [0, 1]$, then $\widetilde{x}$ is a real-valued number.
(3) If $x^C(\alpha) = \underline{x}(\alpha)$ or $x^C(\alpha) = \overline{x}(\alpha)$, $\forall \alpha \in [0, 1]$, then $\widetilde{x}$ is a real-valued number.
(4) $\widetilde{x} = \widetilde{y}$ if and only if $x^C(\alpha) = y^C(\alpha)$ and $x^R(\alpha) = y^R(\alpha)$, $\forall \alpha \in [0, 1]$.

Definition 2.8. The distance between two arbitrary fuzzy numbers $\widetilde{x}$ and $\widetilde{y}$, with the α-levels $[\widetilde{x}]_\alpha = [\underline{x}(\alpha), \overline{x}(\alpha)]$ and $[\widetilde{y}]_\alpha = [\underline{y}(\alpha), \overline{y}(\alpha)]$, can be presented as

$$d(\widetilde{x}, \widetilde{y}) = \int_0^1 |x^C(\alpha) - y^C(\alpha)| d\alpha + \int_0^1 |x^R(\alpha)) - x^R(\alpha)| d\alpha. \qquad (2.1)$$

Also, the above function $d(\cdot, \cdot)$ is a metric in $\mathbb{F}$.

In the following remark, a method for obtaining the α-center and α-radius of a linear combination of fuzzy numbers with crisp numbers as coefficients is presented.

Remark 2.9. [12] Suppose that $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n \in \mathbb{F}$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and also $\widetilde{u} = \sum_{i=1}^n \lambda_i \, \widetilde{x}_i$. Then

$$u^C(\alpha) = \sum_{i=1}^n \lambda_i \, x_i^C(\alpha), \qquad u^R(\alpha) = \sum_{i=1}^n |\lambda_i| \, x_i^R(\alpha).$$

Definition 2.10. We say that $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)^T$ is a fuzzy number vector (or briefly fuzzy vector) if $\widetilde{x}_i \in \mathbb{F}$, $i = 1, 2, \ldots, n$. Also, $[\widetilde{X}]_\alpha$ is denoted by

$$[\widetilde{X}]_\alpha = ([\widetilde{x}_1]_\alpha, [\widetilde{x}_2]_\alpha, \ldots, [\widetilde{x}_n]_\alpha)^T,$$

and consequently

$$X^C(\alpha) = \left(x_1^C(\alpha), x_2^C(\alpha), \ldots, x_n^C(\alpha)\right)^T, \quad X^R(\alpha) = \left(x_1^R(\alpha), x_2^R(\alpha), \ldots, x_n^R(\alpha)\right)^T.$$

The set of all fuzzy number vectors is denoted by $\mathbb{F}^n$.

Theorem 2.11. [12] *Let $\widetilde{X} \in \mathbb{F}^n$ and also $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ is an arbitrary crisp matrix, Then*

$$\left(\boldsymbol{B} \cdot \widetilde{X}\right)^C(\alpha) = \boldsymbol{B} \cdot X^C(\alpha),$$

and

$$\left(\boldsymbol{B} \cdot \widetilde{X}\right)^R(\alpha) = |\boldsymbol{B}| \cdot X^R(\alpha),$$

where $|\boldsymbol{B}| = (|a_{ij}|)_{n\times n}$.

In continuation, we extend the above mentioned concepts to a fuzzy number matrix.

Definition 2.12. A fuzzy number matrix (or briefly fuzzy matrix) is defined as $\widetilde{\mathbf{X}} = (\widetilde{x}_{ij})_{m\times n}$, where $\widetilde{x}_{ij} \in \mathbb{F}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Also, $[\widetilde{\mathbf{X}}]_\alpha$ is denoted by

$$[\widetilde{\mathbf{X}}]_\alpha = ([\widetilde{x}_{ij}]_\alpha)_{m\times n},$$

and consequently

$$\mathbf{X}^C(\alpha) = \left(x_{ij}^C(\alpha)\right)_{m\times n}, \qquad \mathbf{X}^R(\alpha) = \left(x_{ij}^R(\alpha)\right)_{m\times n}.$$

The set of all fuzzy number matrices with m rows and n columns is denoted by $\mathbb{F}^{m\times n}$.

In the following theorem, we present a generalization of Theorem 2.11.

Theorem 2.13. *Let $\widetilde{\boldsymbol{X}} \in \mathbb{F}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$. Then*

$$\left(\boldsymbol{B} \cdot \widetilde{\boldsymbol{X}}\right)^C(\alpha) = \boldsymbol{B} \cdot \boldsymbol{X}^C(\alpha), \qquad \left(\widetilde{\boldsymbol{X}} \cdot \boldsymbol{B}\right)^C(\alpha) = \boldsymbol{X}^C(\alpha) \cdot \boldsymbol{B},$$

and

$$\left(\boldsymbol{B} \cdot \widetilde{\boldsymbol{X}}\right)^R(\alpha) = |\boldsymbol{B}| \cdot \boldsymbol{X}^R(\alpha), \qquad \left(\widetilde{\boldsymbol{X}} \cdot \boldsymbol{B}\right)^R(\alpha) = \boldsymbol{X}^R(\alpha) \cdot |\boldsymbol{B}|,$$

where $|\boldsymbol{B}| = (|a_{ij}|)_{n\times n}$.

Proof. The proof can be presented by using Theorem 2.11. □

One of the most important types of fuzzy matrix equations is the fuzzy Sylvester matrix equations that are specified as follows.

Definition 2.14. Linear matrix equation

$$\mathbf{A}\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}\mathbf{B} = \widetilde{\mathbf{C}}, \tag{2.2}$$

where the coefficient matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ and also $\widetilde{\mathbf{C}} \in \mathbb{F}^{m\times n}$, $\widetilde{\mathbf{X}} \in \mathbb{F}^{m\times n}$ is called a fuzzy Sylvester matrix equation (FSME).

In following, we present a general method for solving FSME (2.2) by using Kronecker product and Vec-operation.

Definition 2.15. The Kronecker product of two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$ is marked by $\mathbf{A}\otimes\mathbf{B}$ and computed as the matrix $\mathbf{A}\otimes\mathbf{B} = (a_{ij}\mathbf{B}) \in \mathbb{R}^{mp\times nq}$. Also, the Kronecker product can be defined for the fuzzy matrices in the same way.

Definition 2.16. Let $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$ and A_i is the i-th column of matrix $\mathbf{A}$. Then $Vec(\mathbf{A})$ is defined to be a vector as follows

$$Vec(\mathbf{A}) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}_{mn\times 1}.$$

Also, the Vec-operation $Vec(\cdot)$ can be defined for the fuzzy matrices in the same way.

Using Definitions 2.15 and 2.16, the fuzzy Sylvester matrix equation (2.2) can be rewritten in the form of a system of fuzzy linear equation as follows

$$(\mathbf{I}_n \otimes \mathbf{A}) \cdot Vec(\widetilde{\mathbf{X}}) + (\mathbf{B}^T \otimes \mathbf{I}_m) \cdot Vec(\widetilde{\mathbf{X}}) = Vec(\widetilde{\mathbf{C}}), \tag{2.3}$$

where $\mathbf{I}_m$ and $\mathbf{I}_n$ are the identity matrices of orders m and n, respectively. It should be noted that in general, the system (2.3) can not be replaced to the following fuzzy linear system

$$\big(\mathbf{I}_n \otimes \mathbf{A} + \mathbf{B}^T \otimes \mathbf{I}_m\big) \cdot Vec(\widetilde{\mathbf{X}}) = Vec(\widetilde{\mathbf{C}}).$$

Now, based on the approach proposed by Friedman et. al. [10], the fuzzy system (2.3) is transformed to the crisp system

$$\begin{pmatrix} (\mathbf{I}_n \otimes \mathbf{A}^+) + (\mathbf{B}^{T^+} \otimes \mathbf{I}_m) & (\mathbf{I}_n \otimes \mathbf{A}^-) + (\mathbf{B}^{T^-} \otimes \mathbf{I}_m) \\ (\mathbf{I}_n \otimes \mathbf{A}^-) + (\mathbf{B}^{T^-} \otimes \mathbf{I}_m) & (\mathbf{I}_n \otimes \mathbf{A}^+) + (\mathbf{B}^{T^+} \otimes \mathbf{I}_m) \end{pmatrix} \begin{pmatrix} \underline{Vec(\widetilde{\mathbf{X}})} \\ \overline{Vec(\widetilde{\mathbf{X}})} \end{pmatrix} = \begin{pmatrix} \underline{Vec(\widetilde{\mathbf{C}})} \\ \overline{Vec(\widetilde{\mathbf{C}})} \end{pmatrix}, \tag{2.4}$$

where the matrices $\mathbf{A}^+$ and $\mathbf{B}^{T^+}$ contain only the positive elements of matrices $\mathbf{A}$ and $\mathbf{B}^T$, and also the matrices $\mathbf{A}^-$ and $\mathbf{B}^{T^-}$ contain only the negative elements of matrices $\mathbf{A}$ and $\mathbf{B}^T$, respectively, i.e.

$$\mathbf{A}^+ = \left(a_{ij}^+\right)_{m\times m} \qquad \Rightarrow \qquad a_{ij}^+ = \begin{cases} a_{ij}, & \text{if } a_{ij} > 0; \\ 0, & \text{if } a_{ij} \leqslant 0, \end{cases}$$

$$\mathbf{A}^- = \left(a_{ij}^-\right)_{m\times m} \qquad \Rightarrow \qquad a_{ij}^- = \begin{cases} a_{ij}, & \text{if } a_{ij} < 0; \\ 0, & \text{if } a_{ij} \geqslant 0, \end{cases}$$

and also

$$\mathbf{B}^{T^+} = \left(b_{ij}^+\right)_{n\times n} \qquad \Rightarrow \qquad b_{ij}^+ = \begin{cases} b_{ji}, & \text{if } b_{ji} > 0; \\ 0, & \text{if } b_{ji} \leqslant 0, \end{cases}$$

$$\mathbf{B}^{T^-} = \left(b_{ij}^-\right)_{n\times n} \qquad \Rightarrow \qquad b_{ij}^- = \begin{cases} b_{ji}, & \text{if } b_{ji} < 0; \\ 0, & \text{if } b_{ji} \geqslant 0. \end{cases}$$

Obviously, the system (2.4) is a system of crisp linear equations with the dimension of $2mn \times 2mn$. Unfortunately, if m or n is large, the above-mentioned approach is not feasible for numerical solving of Eq. (2.2) [21].

In the next section, we propose a method by using some defined concepts to transform the fuzzy Sylvester matrix equation (2.2) into two crisp Sylvester matrix equations with the dimensions equal to the original problem. Finally we use the homotopy analysis method (HAM) to solve the obtained crisp Sylvester matrix equations approximately.

3. The proposed method

Based on the part 4 of Remark 2.7, it is concluded that if $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \mathbb{F}$, then $\widetilde{\mathbf{X}} = \widetilde{\mathbf{Y}}$ if and only if $\mathbf{X}^C(\alpha) = \mathbf{Y}^C(\alpha)$ and $\mathbf{X}^R(\alpha) = \mathbf{Y}^R(\alpha)$. In light of this finding and Theorem 2.13, the subsequent theorem is straightforwardly established.

Theorem 3.1. *The fuzzy Sylvester matrix equation (2.2) holds if and only if*

$$\boldsymbol{A} \cdot \boldsymbol{X}^C(\alpha) + \boldsymbol{X}^C(\alpha) \cdot \boldsymbol{B} = \boldsymbol{C}^C(\alpha), \tag{3.1}$$

$$|\boldsymbol{A}| \cdot \boldsymbol{X}^R(\alpha) + \boldsymbol{X}^R(\alpha) \cdot |\boldsymbol{B}| = \boldsymbol{C}^R(\alpha), \tag{3.2}$$

for each $\alpha \in [0, 1]$, *where* $|\boldsymbol{A}| = (|a_{ij}|)_{m\times m}$ *and* $|\boldsymbol{B}| = (|b_{ij}|)_{n\times n}$ *and also*

$$[\boldsymbol{X}^C(\alpha) - \boldsymbol{X}^R(\alpha), \boldsymbol{X}^C(\alpha) + \boldsymbol{X}^R(\alpha)], \tag{3.3}$$

constructs the α*-levels of a fuzzy matrix.*

It should be noted that in Theorem 3.1, the equations (3.1) and (3.2) are two crisp Sylvester matrix equations with the dimensions equal to the original fuzzy Sylvester matrix equation (2.2).

According to our proposed method, for obtaining the fuzzy solution of equation (2.2), we should first solve two crisp equations (3.1) and (3.2) and obtain the crisp matrices $\mathbf{X}^C(\alpha)$ and $\mathbf{X}^R(\alpha)$. Now, if $[\mathbf{X}^C(\alpha) - \mathbf{X}^R(\alpha), \mathbf{X}^C(\alpha) + \mathbf{X}^R(\alpha)]$ constructs the α-levels of a fuzzy matrix, then the solution of Eq. (2.2) can be easily presented by Eq. (3.3). Otherwise, the Eq. (2.2) does not have a fuzzy solution matrix.

In continuation, we apply the homotopy analysis method (HAM) to solve the crisp Sylvester matrix equations (3.1) and (3.2) approximately. To do this, we will first present a brief introduction to the HAM. Let us consider the following general equation

$$N[x(t)] = 0,$$

where N is an operator, t denotes independent variable, $x(t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [24, 25] constructs the so-called zero-order deformation equation

$$(1-p)\, L[\varphi(t;p) - x_0(t)] = p\, \hbar\, H(t)\, N[\varphi(t;p)], \tag{3.4}$$

where $p \in [0,1]$ is called homotopy-parameter, $\hbar$ is a non-zero auxiliary parameter which is called convergence-control parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $x_0(t)$ is an initial guess of $x(t)$, $\varphi(t;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\varphi(t;0) = x_0(t), \qquad \varphi(t;1) = x(t),$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $x_0(t)$ to the solution $x(t)$. Expanding $\varphi(t;p)$ in Maclaurin series with respect to p, we have

$$\varphi(t;p) = x_0(t) + \sum_{m=1}^{\infty} x_m(t)\, p^m, \tag{3.5}$$

where

$$x_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t;p)}{\partial p^m}|_{p=0}. \tag{3.6}$$

Here, the series (3.5) is called homotopy-series and the Eq. (3.6) is called the mth-order homotopy-derivative of φ [25]. If the auxiliary linear operator, the initial guess, the convergence-control parameter $\hbar$, and the auxiliary function are so properly chosen, the homotopy-series

(3.5) converges at $p = 1$, then using the relationship $\varphi(t; 1) = x(t)$, one has the so-called homotopy-series solution [25]

$$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t), \tag{3.7}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [25].

According to the definition (3.6), the governing equation can be deducted from the zero-order deformation equation (3.4). Define the vector

$$\overrightarrow{x}_n = \{x_0(t), x_1(t), \ldots, x_n(t)\}.$$

Differentiating equation (3.4) m times with respect to the homotopy-parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[x_m(t) - \chi_m x_{m-1}(t)] = \hbar\, H(t)\, R_m(\overrightarrow{x}_{m-1}), \tag{3.8}$$

where

$$R_m(\overrightarrow{x}_{m-1}) = \frac{1}{(m-1)!}\, \frac{\partial^{m-1} N[\varphi(t; p)]}{\partial p^{m-1}}\,|_{p=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.9}$$

It should be emphasized that $x_m(t)$ for $m \geq 1$ is governed by the linear equation (3.8) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as MAPLE, MATHEMATICA and MATLAB.

Finally an nth-order approximate solution is given by

$$\mathbf{x}^*_n(t) = x_0(t) + \sum_{m=1}^{n} x_m(t), \tag{3.10}$$

and the exact solution is $x(t) = \lim_{n\to\infty} \mathbf{x}^*_n(t)$.

Now, we explain the application of the homotopy analysis method in solving the Eq. (3.1). The Eq. (3.2) can be solved by the homotopy analysis method similarly.

According to the homotopy analysis method the zero-order deformation equation can be presented as

$$(1-p)\, L[\mathbf{V}(\alpha; p) - \mathbf{X}_0^C(\alpha)] = p\, \hbar\, \mathbf{Q} \cdot N[\mathbf{V}(\alpha; p)], \tag{3.11}$$

where $p \in [0, 1]$ is homotopy-parameter [25], $\hbar$ is non-zero convergence-control parameter [25], $\mathbf{Q}$ is an auxiliary $m \times m$ matrix, $\mathbf{X}_0^C(\alpha)$ is an initial approximation of $\mathbf{X}^C(\alpha)$ and $\mathbf{V}(\alpha; p)$ is a parametric unknown $m \times n$ matrix.

If in the zero-order deformation equation (3.11), we set $\mathbf{Q} = \mathbf{I}_m$ and

$$L[\mathbf{V}(\alpha;p)] = \mathbf{V}(\alpha;p),$$

and

$$N[\mathbf{V}(\alpha;p)] = \mathbf{A}\cdot\mathbf{V}(\alpha;p) + \mathbf{V}(\alpha;p)\cdot\mathbf{B} - \mathbf{C}^C(\alpha),$$

then we have

$$(1-p)\,\mathbf{V}(\alpha;p) - \mathbf{X}_0^C(\alpha) = p\,\hbar\,\left(\mathbf{A}\cdot\mathbf{V}(\alpha;p) + \mathbf{V}(\alpha;p)\cdot\mathbf{B} - \mathbf{C}^C(\alpha)\right). \tag{3.12}$$

Obviously, for $p = 0$ and $p = 1$, the following results are obtained

$$\mathbf{V}(\alpha;0) = \mathbf{X}_0^C(\alpha), \qquad\qquad \mathbf{V}(\alpha;1) = \mathbf{X}^C(\alpha),$$

respectively. Therefore, when p moves from 0 to 1, the parametric matrix $\mathbf{V}(\alpha;p)$ changes from $\mathbf{X}_0^C(\alpha)$ to $\mathbf{X}^C(\alpha)$. Computing the Taylor series of $\mathbf{V}(\alpha;p)$ in terms of p, we conclude

$$\mathbf{V}(\alpha;p) = \mathbf{X}_0^C(\alpha) + \sum_{i=1}^{\infty}\mathbf{X}_i^C(\alpha)\,p^i, \tag{3.13}$$

where

$$\mathbf{X}_i^C(\alpha) = \frac{1}{i!}\frac{\partial^i\mathbf{V}(\alpha;p)}{\partial p^i}|_{p=0}. \tag{3.14}$$

Eqs. (3.13) and (3.14) are called as homotopy-series and ith-order homotopy-derivative of $\mathbf{V}(\alpha;p)$ [25]. If the initial approximation and the convergence-control parameter $\hbar$ are so properly chosen, then the Eq. (3.13) converges at $p = 1$, and so by using $\mathbf{V}(\alpha;1) = \mathbf{X}^C(\alpha)$, we have

$$\mathbf{X}^C(\alpha) = \mathbf{X}_0^C(\alpha) + \sum_{i=1}^{\infty}\mathbf{X}_i^C(\alpha).$$

By deriving i-times in Eq. (3.12) with respect to p and then substituting $p = 0$ and finally dividing them by $i!$, the following results can be obtained

$$\mathbf{X}_1^C(\alpha) = \hbar\left(\mathbf{A}\cdot\mathbf{X}_0^C(\alpha) + \mathbf{X}_0^C(\alpha)\cdot\mathbf{B} - \mathbf{C}^C(\alpha)\right), \tag{3.15}$$

$$\mathbf{X}_i^C(\alpha) = \mathbf{X}_{i-1}^C(\alpha) + \hbar\left(\mathbf{A}\cdot\mathbf{X}_{i-1}^C(\alpha) + \mathbf{X}_{i-1}^C(\alpha)\cdot\mathbf{B}\right), \qquad i \geqslant 2. \tag{3.16}$$

For later numerical computations, we set

$$\mathbf{K}_s(\alpha) = \sum_{i=0}^{s}\mathbf{X}_i^C(\alpha), \qquad s = 0,1,2,\ldots, \tag{3.17}$$

to denote the sth-order approximation of the solution $\mathbf{X}^C(\alpha)$.

Similarly, we can obtain the sth-order approximation of $\mathbf{X}^R(\alpha)$ as follows

$$\mathbf{R}_s(\alpha) = \sum_{i=0}^{s} \mathbf{X}_i^R(\alpha), \qquad s = 0, 1, 2, \ldots, \tag{3.18}$$

where $\mathbf{X}_0^R(\alpha)$ is an initial approximation of $\mathbf{X}^R(\alpha)$ and also

$$\mathbf{X}_1^R(\alpha) = \hbar\left(|\mathbf{A}| \cdot \mathbf{X}_0^R(\alpha) + \mathbf{X}_0^R(\alpha) \cdot |\mathbf{B}| - \mathbf{C}^R(\alpha)\right), \tag{3.19}$$

$$\mathbf{X}_i^R(\alpha) = \mathbf{X}_{i-1}^R(\alpha) + \hbar\left(|\mathbf{A}| \cdot \mathbf{X}_{i-1}^R(\alpha) + \mathbf{X}_{i-1}^R(\alpha) \cdot |\mathbf{B}|\right), \qquad i \geqslant 2. \tag{3.20}$$

Finally, if the family

$$\{[\mathbf{K}_s(\alpha) - \mathbf{R}_s(\alpha), \mathbf{K}_s(\alpha) + \mathbf{R}_s(\alpha)] \quad : \quad 0 \leqslant \alpha \leqslant 1\},$$

constructs the α-levels of a fuzzy matrix, then the sth-order approximation solution of the Eq. (2.2) is represented as

$$[\widetilde{\mathbf{X}}_s]_\alpha = [\mathbf{K}_s(\alpha) - \mathbf{R}_s(\alpha), \mathbf{K}_s(\alpha) + \mathbf{R}_s(\alpha)], \qquad s = 0, 1, 2, \ldots, \quad \alpha \in [0, 1]. \tag{3.21}$$

Remark 3.2. The iterative formulas (3.15), (3.16) and (3.19), (3.20) can be replaced by

$$\begin{cases} \mathbf{X}_1^C(\alpha) = \hbar\left(\mathbf{A} \cdot \mathbf{X}_0^C(\alpha) + \mathbf{X}_0^C(\alpha) \cdot \mathbf{B} - \mathbf{C}^C(\alpha)\right), \\ \mathbf{X}_i^C(\alpha) = \left(\frac{1}{2}\mathbf{I}_m + \hbar\mathbf{A}\right) \cdot \mathbf{X}_{i-1}^C(\alpha) + \mathbf{X}_{i-1}^C(\alpha) \cdot \left(\frac{1}{2}\mathbf{I}_n + \hbar\mathbf{B}\right), \qquad i \geqslant 2, \end{cases} \tag{3.22}$$

and

$$\begin{cases} \mathbf{X}_1^R(\alpha) = \hbar\left(|\mathbf{A}| \cdot \mathbf{X}_0^R(\alpha) + \mathbf{X}_0^R(\alpha) \cdot |\mathbf{B}| - \mathbf{C}^R(\alpha)\right), \\ \mathbf{X}_i^R(\alpha) = \left(\frac{1}{2}\mathbf{I}_m + \hbar|\mathbf{A}|\right) \cdot \mathbf{X}_{i-1}^R(\alpha) + \mathbf{X}_{i-1}^R(\alpha) \cdot \left(\frac{1}{2}\mathbf{I}_n + \hbar|\mathbf{B}|\right), \qquad i \geqslant 2. \end{cases} \tag{3.23}$$

In following, the convergence theorem of our proposed approach is presented.

Theorem 3.3. *Suppose that $\|\cdot\|$ is an arbitrary matrix norm and also $\|\frac{1}{2}\boldsymbol{I}_m + \hbar\boldsymbol{A}\| < \frac{1}{2}$, $\|\frac{1}{2}\boldsymbol{I}_n + \hbar\boldsymbol{B}\| < \frac{1}{2}$, $\|\frac{1}{2}\boldsymbol{I}_m + \hbar|\boldsymbol{A}|\,\| < \frac{1}{2}$ and $\|\frac{1}{2}\boldsymbol{I}_n + \hbar|\boldsymbol{B}|\,\| < \frac{1}{2}$, then the sequences $\{\boldsymbol{K}_s\}_{s=0}^{\infty}$ and $\{\boldsymbol{R}_s\}_{s=0}^{\infty}$ obtained from Eqs. (3.22) and (3.23), are convergent.*

Proof. Here, we show that the sequences $\{\mathbf{K}_s\}_{s=0}^{\infty}$ and $\{\mathbf{R}_s\}_{s=0}^{\infty}$ are two Chauchy sequences with respect to the matrix norm $\|\cdot\|$.

At first, we show that the sequences $\{\mathbf{K}_s\}_{s=0}^{\infty}$ is a Chauchy sequence. For this, from Remark 3.2 we conclude

$$\mathbf{X}_i^C(\alpha) = \hbar \sum_{j=0}^{i-1} \binom{i-1}{j} \cdot \left(\frac{1}{2}\mathbf{I}_m + \hbar|\mathbf{A}|\right)^{i-j-1} \cdot \left(\mathbf{A} \cdot \mathbf{X}_0^C(\alpha) + \mathbf{X}_0^C(\alpha) \cdot \mathbf{B} - \mathbf{C}\right) \cdot \left(\frac{1}{2}\mathbf{I}_n + \hbar\mathbf{B}\right)^j,$$

where $\begin{pmatrix} i-1 \\ j \end{pmatrix}$ is a j-combination of $(i-1)$, i.e., $\begin{pmatrix} i-1 \\ j \end{pmatrix} = \frac{(i-1)!}{j!\,(i-j-1)!}$. If we set

$$\lambda = \max\{\|\frac{1}{2}\mathbf{I}_m + \hbar\mathbf{A}\|,\ \|\frac{1}{2}\mathbf{I}_n + \hbar\mathbf{B}\|\},$$

and

$$\theta = \max\{\|\mathbf{A}\|,\ \|\mathbf{B}\|,\ \|\mathbf{C}\|,\ \|\mathbf{X}_0^C(\alpha)\|\},$$

then we have

$$\|\mathbf{K}_{s+t} - \mathbf{K}_s\| \leqslant |\hbar|(\theta^2 + \theta) \sum_{i=s+1}^{s+t} \left(\lambda^{i-1} \sum_{j=0}^{i-1} \begin{pmatrix} i-1 \\ j \end{pmatrix} \right).$$

On the other hand, it's well known that

$$\sum_{j=0}^{i-1} \begin{pmatrix} i-1 \\ j \end{pmatrix} = 2^{i-1}.$$

Therefore

$$\|\mathbf{K}_{s+t} - \mathbf{K}_s\| \leqslant |\hbar|(\theta^2 + \theta) \sum_{i=s+1}^{s+t} (2\lambda)^{i-1},$$

or in other words

$$\|\mathbf{K}_{s+t} - \mathbf{K}_s\| \leqslant |\hbar|(\theta^2 + \theta)\,(2\lambda)^s \sum_{i=1}^{t} (2\lambda)^{i-1}.$$

Regarding to the assumptions of theorem, since $2\lambda \neq 1 (2\lambda < 1)$, then we have

$$\|\mathbf{K}_{s+t} - \mathbf{K}_s\| \leqslant |\hbar|(\theta^2 + \theta) \frac{(2\lambda)^t - 1}{2\lambda - 1} (2\lambda)^s,$$

and consequently

$$\lim_{s \longrightarrow \infty} \|\mathbf{K}_{s+t} - \mathbf{K}_s\| = 0.$$

This completes the proof of convergence of the sequence $\{\mathbf{K}_s\}_{s=0}^{\infty}$. The proof of convergence of the sequence $\{\mathbf{R}_s\}_{s=0}^{\infty}$ can be presented similarly. $\square$

Theorem 3.4. *If $\rho(\frac{1}{2}\boldsymbol{I}_m + \hbar\boldsymbol{A}) < \frac{1}{2}$, $\rho(\frac{1}{2}\boldsymbol{I}_n + \hbar\boldsymbol{B}) < \frac{1}{2}$, $\rho(\frac{1}{2}\boldsymbol{I}_m + \hbar|\boldsymbol{A}|) < \frac{1}{2}$ and $\rho(\frac{1}{2}\boldsymbol{I}_n + \hbar|\boldsymbol{B}|) < \frac{1}{2}$, then the sequences $\{\boldsymbol{K}_s\}_{s=0}^{\infty}$ and $\{\boldsymbol{R}_s\}_{s=0}^{\infty}$ obtained from Eqs. (3.22) and (3.23), are convergent.*

Proof. For a square matrix $\mathbf{A}$, we know

$$\rho(\mathbf{A}) = \inf_{\|\cdot\|}\{\|\mathbf{A}\|\},$$

where $\|\cdot\|$ is a matrix norm. Therefore

$$\rho(\frac{1}{2}\mathbf{I}_m + \hbar\mathbf{A}) < \frac{1}{2},$$

implies that there exists a matrix norm denoted as $\|\cdot\|_1$ with the property that $\|\frac{1}{2}\mathbf{I}_m + \hbar\mathbf{A}\|_1 < \frac{1}{2}$. Similarly

$$\rho(\frac{1}{2}\mathbf{I}_n + \hbar\mathbf{B}) < \frac{1}{2} \qquad \Rightarrow \qquad \exists \|\cdot\|_2 \ : \ \|\frac{1}{2}\mathbf{I}_n + \hbar\mathbf{B}\|_2 < \frac{1}{2},$$

$$\rho(\frac{1}{2}\mathbf{I}_m + \hbar|\mathbf{A}|) < \frac{1}{2} \qquad \Rightarrow \qquad \exists \|\cdot\|_3 \ : \ \|\frac{1}{2}\mathbf{I}_m + \hbar|\mathbf{A}|\,\|_3 < \frac{1}{2},$$

$$\rho(\frac{1}{2}\mathbf{I}_n + \hbar|\mathbf{B}|) < \frac{1}{2} \qquad \Rightarrow \qquad \exists \|\cdot\|_4 \ : \ \|\frac{1}{2}\mathbf{I}_n + \hbar|\mathbf{B}|\,\|_4 < \frac{1}{2}.$$

Now, we define

$$\|\cdot\|^* = \min\{\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_3, \|\cdot\|_4\}.$$

It can be easily showed that $\|\cdot\|^*$ is a matrix norm. According to the above mentioned, we conclude that

$$\|\frac{1}{2}\mathbf{I}_m + \hbar\mathbf{A}\|^* < \frac{1}{2}, \qquad \|\frac{1}{2}\mathbf{I}_n + \hbar\mathbf{B}\|^* < \frac{1}{2},$$

$$\|\frac{1}{2}\mathbf{I}_m + \hbar|\mathbf{A}|\,\|^* < \frac{1}{2}, \qquad \|\frac{1}{2}\mathbf{I}_n + \hbar|\mathbf{B}|\,\|^* < \frac{1}{2},$$

and finally by Theorem 3.3 the proof is completed. □

Finally, the algorithm of the proposed method for solving the Eq. (1.3) can be presented as follows:

step 1:: Construct the crisp Sylvester matrix equations (3.1) and (3.2).

step 2:: Choose the initial approximations $X_0^C(\alpha)$ and $X_0^R(\alpha)$.

step 3:: Obtain the sth-order approximation of the solution $\mathbf{X}^C(\alpha)$, i.e., $\mathbf{K}_s(\alpha)$ by Eqs. (3.17) and (3.22).

step 4:: Obtain the sth-order approximation of the solution $\mathbf{X}^R(\alpha)$, i.e., $\mathbf{R}_s(\alpha)$ by Eqs. (3.18) and (3.23).

step 5:: Present the sth-order approximation solution of the Eq. (1.3) as follows

$$[\widetilde{\mathbf{X}}_s]_\alpha = [\mathbf{K}_s(\alpha) - \mathbf{R}_s(\alpha), \mathbf{K}_s(\alpha) + \mathbf{R}_s(\alpha)], \qquad s = 0, 1, 2, \ldots, \qquad \alpha \in [0, 1].$$

Remark 3.5. The convergence conditions presented in Theorems 3.3 and 3.4 are necessary but not sufficient for obtaining a fuzzy solution. This means that the solution derived from Eq. (3.21) must construct the α-levels of a fuzzy matrix. To ensure this, the matrix obtained from Eq. (3.18) must be a nonnegative matrix.

Remark 3.6. Based on Eqs. (3.16) and (3.20), we conclude that the computational complexity of the proposed method is $O(mn(m+n))$. In contrast, according to Eq. (2.4), the computational complexity of methods based on the Kronecker product is $O(m^2n^2)$. Therefore, it is evident that employing the proposed method leads to a reduction in computational complexity.

4. Numerical examples

In this section, the relative error

$$E(s) = \frac{\|\widetilde{\mathbf{X}}_s - \widetilde{\mathbf{X}}\|_\infty}{\|\widetilde{\mathbf{X}}\|_\infty}, \qquad s \geqslant 1, \tag{4.1}$$

is used to compute the accuracy of the obtained approximate solutions, where $\widetilde{\mathbf{X}}$ is the exact solution and $\widetilde{\mathbf{X}}_s$ represents s-th order approximation solution obtained from Eq. (3.21). Also

$$\|\widetilde{\mathbf{X}}_s - \widetilde{\mathbf{X}}\|_\infty = \max_{i=1,2,\ldots,m} \sum_{j=1}^{n} d(\widetilde{x}_{ij_s}, \widetilde{x}_{ij}),$$

where the function $d(\cdot,\cdot)$ is the metric defined in Eq. (2.1). In this section, MATLAB is used to compute all numerical results.

Example 4.1. Consider the fuzzy Sylvester matrix equation

$$\mathbf{A} \cdot \widetilde{\mathbf{X}} + \widetilde{\mathbf{X}} \cdot \mathbf{B} = \widetilde{\mathbf{C}}, \tag{4.2}$$

where

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & -3 \\ 2 & 5 & 4 \\ 1 & -2 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & -7 \\ 1 & 6 \end{pmatrix},$$

and

$$[\widetilde{\mathbf{C}}]_\alpha = \begin{pmatrix} [27r-32, 37-26r] & [35r-47, 70-55r] \\ [35r-13, 50-16r] & [26r+5, 87-43r] \\ [36r-15, 37-11r] & [22r-75, 15-58r] \end{pmatrix},$$

with the exact solution

$$[\widetilde{\mathbf{X}}]_\alpha = \begin{pmatrix} [3r-2, 4-r] & [2r-1, 5-3r] \\ [r-1, 2-r] & [r+3, 6-2r] \\ [5r-1, 5-r] & [r-3, 1-2r] \end{pmatrix}. \tag{4.3}$$

According to the Theorem 3.1, the Eq. (4.2) is equivalent to the following crisp Sylvester matrix equations

$$\mathbf{A} \cdot \mathbf{X}^C(\alpha) + \mathbf{X}^C(\alpha) \cdot \mathbf{B} = \mathbf{C}^C(\alpha), \tag{4.4}$$

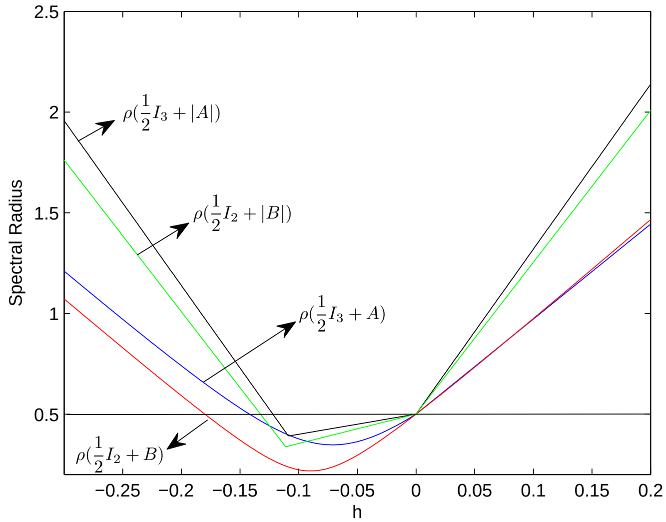

FIGURE 2. The valid region of $\hbar$ for Example 4.1.

where

$$\mathbf{C}^C(\alpha)=\begin{pmatrix} 31r-39.5 & 53.5-40.5r \\ 30.5r-4 & 68.5-29.5r \\ 2r-45 & 26-34.5r \end{pmatrix},$$

and

$$|\mathbf{A}|\cdot\mathbf{X}^R(\alpha)+\mathbf{X}^R(\alpha)\cdot|\mathbf{B}|=\mathbf{C}^R(\alpha), \tag{4.5}$$

where

$$|\mathbf{A}|=\begin{pmatrix} 4 & 1 & 3 \\ 2 & 5 & 4 \\ 1 & 2 & 3 \end{pmatrix}, \qquad |\mathbf{B}|=\begin{pmatrix} 3 & 7 \\ 1 & 6 \end{pmatrix},$$

and

$$\mathbf{C}^R(\alpha)=\begin{pmatrix} 4r-7.5 & 16.5-14.5r \\ -4.5r+9 & 18.5-13.5r \\ -7r-30 & -11-23.5r \end{pmatrix}.$$

To find the valid region of $\hbar$, the curves of $\rho(\frac{1}{2}\mathbf{I}_3+\hbar\mathbf{A})$, $\rho(\frac{1}{2}\mathbf{I}_2+\hbar\mathbf{B})$, $\rho(\frac{1}{2}\mathbf{I}_3+\hbar|\mathbf{A}|)$ and $\rho(\frac{1}{2}\mathbf{I}_2+\hbar|\mathbf{B}|)$ are plotted in Figure 2. Based on Theorem 3.4 and Figure 2, if $\hbar\in(-0.122,0)$, then the sequences $\{\mathbf{K}_s\}_{s=0}^{\infty}$ and $\{\mathbf{R}_s\}_{s=0}^{\infty}$ obtained from Eq. (3.22) and (3.23) are convergent with arbitrary initial approximations $\mathbf{X}_0^C(\alpha)$ and $\mathbf{X}_0^R(\alpha)$, respectively.

Here, by starting $\mathbf{X}_0^C(\alpha)=\mathbf{X}_0^R(\alpha)=zeros(3,2)$ and choosing $\hbar=-0.08$, 30th-order approximate solutions of $\mathbf{X}^C(\alpha)$ and $\mathbf{X}^R(\alpha)$ are presented as follows

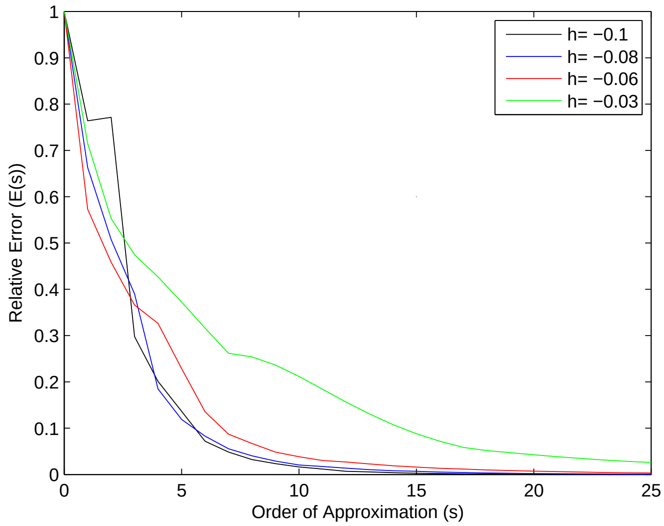

FIGURE 3. The relative error in terms of the order of approximation for the various values of $\hbar$ in Example 4.1.

$$\mathbf{K}_{30}^{C}(\alpha) = \begin{pmatrix} 1.0000r + 1.0000 & 2.0000 - 0.5000r \\ 0.0000r + 0.5000 & 4.5000 - 0.5000r \\ 2.0000r + 2.0000 & -1.0000 - 0.5000r \end{pmatrix},$$
$$\mathbf{R}_{30}^{C}(\alpha) = \begin{pmatrix} -2.0045r + 3.0038 & 2.9942 - 2.4931r \\ -1.0034r + 1.5029 & 1.4956 - 1.4948r \\ -2.9944r + 2.9953 & 2.0073 - 1.5086r \end{pmatrix},$$

respectively. Consequently 30th-order approximate solution of $[\widetilde{\mathbf{X}}]_\alpha = [\underline{X}(\alpha), \overline{X}(\alpha)]$ can be obtained by the Eq. (3.21) as follows

$$[\widetilde{\mathbf{X}}_{30}]_\alpha = \begin{pmatrix} [3.0045r - 2.0038, 4.0038 - 1.0045r] & [1.9931r - 0.9942, 4.9942 - 2.9931r] \\ [1.0034r - 1.0029, 2.0029 - 1.0034r] & [0.9948r + 3.0044, 5.9956 - 1.9948r] \\ [4.9944r - 0.9953, 4.9953 - 0.9944r] & [1.0086r - 3.0073, 1.0073 - 2.0086r] \end{pmatrix},$$

with the relative error

$$E(30) = \frac{\|\widetilde{X}_{30} - \widetilde{X}\|_\infty}{\|\widetilde{X}\|_\infty} = 7.5271e - 004.$$

Also, in Figure 3 the graph of the relative error in terms of the order of approximation (the number of iteration) for the various values of $\hbar$ is plotted. In this figure, the initial approximations $\mathbf{X}_0^C(\alpha)$ and $\mathbf{X}_0^R(\alpha)$ are chosen always zero matrices. Obviously the results show the convergence of method.

Example 4.2. Consider the fuzzy Sylvester matrix equation

$$\mathbf{A} \cdot \widetilde{\mathbf{X}} + \widetilde{\mathbf{X}} \cdot \mathbf{B} = \widetilde{\mathbf{C}}, \tag{4.6}$$

where

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 & -2 & -1 \\ 1 & 5 & -2 & 3 & 1 \\ 1 & -2 & 4 & -1 & -2 \\ -2 & 0 & -1 & 5 & 1 \\ 3 & 1 & -1 & 2 & 5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 4 & -2 & 1 & -1 \\ -1 & 4 & 2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & -1 & 2 & 3 \end{pmatrix},$$

and

$$[\widetilde{\mathbf{C}}]_\alpha = \begin{pmatrix} [21r-41, 32-21r] & [21r-52, 5-25r] & [29r-20, 50-18r] & [29r-13, 47-18r] \\ [24r-12, 45-25r] & [26r-23, 42-34r] & [33r-33, 35-22r] & [36r-21, 55-22r] \\ [19r-28, 35-20r] & [21r-61, 3-23r] & [19r-55, 16-20r] & [21r-50, 31-22r] \\ [19r+6, 63-27r] & [30r+27, 87-26r] & [19r+12, 69-22r] & [18r+12, 57-20r] \\ [30r-39, 53-26r] & [42r+30, 125-26r] & [31r-11, 80-24r] & [33r-6, 89-35r] \end{pmatrix},$$

with the exact solution

$$[\widetilde{\mathbf{X}}]_\alpha = \begin{pmatrix} [r-2, 3-r] & [r, 2-r] & [2r-1, 4-r] & [2r+3, 6-r] \\ [r-1, 1-r] & [r-4, -1-2r] & [2r-2, 1-r] & [3r-3, 2-r] \\ [r-1, 2-r] & [r-3, -r] & [r-4, -r] & [r-4, 2-r] \\ [r+3, 6-2r] & [2r+5, 9-2r] & [r, 3-r] & [r+4, 6-r] \\ [2r-2, 3-r] & [3r+4, 10-r] & [r-1, 3-r] & [2r-2, 5-3r] \end{pmatrix}. \tag{4.7}$$

According to the Theorem 3.1, the Eq. (4.6) is equivalent to the following crisp Sylvester matrix equations

$$\mathbf{A} \cdot \mathbf{X}^C(\alpha) + \mathbf{X}^C(\alpha) \cdot \mathbf{B} = \mathbf{C}^C(\alpha), \tag{4.8}$$

where

$$\mathbf{C}^C(\alpha) = \begin{pmatrix} -4.5 & -2r-23.5 & 5.5r+15 & 5.5r+17 \\ -0.5r+16.5 & -4r+9.5 & 5.5r+1 & 7r+17 \\ -0.5r+3.5 & -r-29 & -0.5r-19.5 & -0.5r-9.5 \\ -4r+34.5 & 2r+57 & -1.5r+40.5 & -r+34.5 \\ 2r+7 & 8r+77.5 & 3.5r+34.5 & -r+41.5 \end{pmatrix},$$

and

$$|\mathbf{A}| \cdot \mathbf{X}^R(\alpha) + \mathbf{X}^R(\alpha) \cdot |\mathbf{B}| = \mathbf{C}^R(\alpha), \tag{4.9}$$

where

$$|\mathbf{A}| = \begin{pmatrix} 4 & 2 & 1 & 2 & 1 \\ 1 & 5 & 2 & 3 & 1 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 0 & 1 & 5 & 1 \\ 3 & 1 & 1 & 2 & 5 \end{pmatrix}, \qquad |\mathbf{B}| = \begin{pmatrix} 4 & 2 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 3 \end{pmatrix},$$

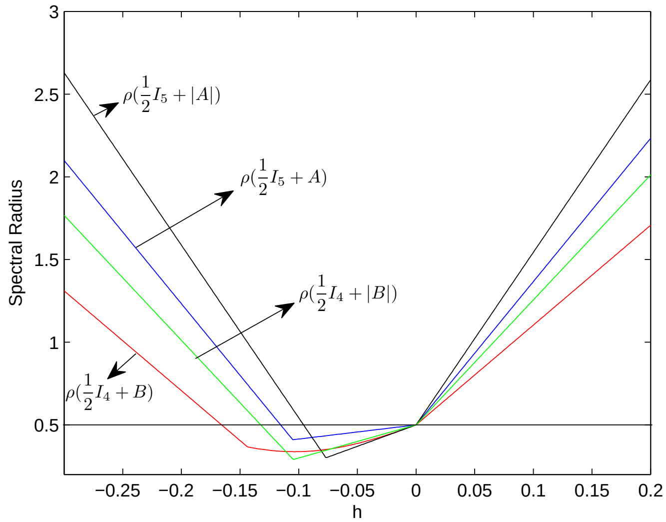

FIGURE 4. The valid region of $\hbar$ for Example 4.2.

and

$$\mathbf{C}^R(\alpha) = \begin{pmatrix} -21r+36.5 & -23r+28.5 & -23.5r+35 & -23.5r+30 \\ -24.5r+28.5 & -30r+32.5 & -27.5r+34 & -29r+38 \\ -19.5r+31.5 & -22r+32 & -19.5r+35.5 & -21.5r+40.5 \\ -23r+28.5 & -28r+30 & -20.5r+28.5 & -19r+22.5 \\ -28r+46 & -34r+47.5 & -27.5r+45.5 & -34r+47.5 \end{pmatrix}.$$

To find the valid region of $\hbar$, the curves of $\rho(\frac{1}{2}\mathbf{I}_5 + \hbar\mathbf{A})$, $\rho(\frac{1}{2}\mathbf{I}_4 + \hbar\mathbf{B})$, $\rho(\frac{1}{2}\mathbf{I}_5 + \hbar|\mathbf{A}|)$ and $\rho(\frac{1}{2}\mathbf{I}_4 + \hbar|\mathbf{B}|)$ are plotted in Figure 4. Based on Theorem 3.4 and Figure 4, if $\hbar \in (-0.096, 0)$, then the sequences $\{\mathbf{K}_s\}_{s=0}^{\infty}$ and $\{\mathbf{R}_s\}_{s=0}^{\infty}$ obtained from Eqs. (3.22) and (3.23) are convergent with an arbitrary initial approximations $\mathbf{X}_0^C(\alpha)$ and $\mathbf{X}_0^R(\alpha)$, respectively.

Here, by starting $\mathbf{X}_0^C(\alpha) = \mathbf{X}_0^R(\alpha) = zeros(5,4)$ and choosing $\hbar = -0.09$, 30th-order approximate solutions of $\mathbf{X}^C(\alpha)$ and $\mathbf{X}^R(\alpha)$ are presented as follows

$$\mathbf{K}_{30}^C(\alpha) = \begin{pmatrix} 0.0000r+0.5001 & 0.0000r+1.0000 & 0.5000r+1.5001 & 0.5000r+4.5002 \\ 0.0000r-0.0001 & -0.5000r-2.5000 & 0.5000r-0.5001 & 1.0000r-0.5002 \\ 0.0000r+0.4998 & 0.0000r-1.5000 & 0.0000r-2.0002 & 0.0000r-1.0003 \\ -0.5000r+4.5000 & 0.0000r+7.0000 & 0.0000r+1.5000 & 0.0000r+5.0001 \\ 0.5000r+4.9999 & 1.0000r+7.0000 & 0.0000r+0.9999 & -0.5000r+1.4998 \end{pmatrix},$$

$$\mathbf{R}_{30}^C(\alpha) = \begin{pmatrix} 2.5000 - 1.0000r & 1.0000 - 1.0000r & 2.5000 - 1.5000r & 1.5000 - 1.5000r \\ 1.0000 - 1.0000r & 1.5000 - 1.5000r & 1.5000 - 1.5000r & 2.5000 - 2.0000r \\ 1.5000 - 1.0000r & 1.5000 - 1.0000r & 2.0000 - 1.0000r & 3.0000 - 1.0000r \\ 1.5000 - 1.5000r & 2.0000 - 2.0000r & 1.5000 - 1.0000r & 1.0000 - 1.0000r \\ 2.5000 - 1.5000r & 3.0000 - 2.0000r & 2.0000 - 1.0000r & 3.5000 - 2.5000r \end{pmatrix},$$

respectively. Consequently 30th-order approximate solution of $[\widetilde{\mathbf{X}}]_\alpha = [\underline{X}(\alpha), \overline{X}(\alpha)]$ can be obtained by the Eq. (3.21) as follows

$$[\widetilde{\mathbf{X}}_{30}]_\alpha = [\underline{X}_{30}(\alpha), \overline{X}_{30}(\alpha)] = [\mathbf{K}_{30}^C(\alpha) - \mathbf{R}_{30}^C(\alpha)\,,\, \mathbf{K}_{30}^C(\alpha) + \mathbf{R}_{30}^C(\alpha)],$$

where

$$\underline{X}_{30}(\alpha) = \begin{pmatrix} 1.0000r - 1.9999 & 1.0000r + 0.0000 & 2.0000r - 0.9999 & 2.0000r + 3.0002 \\ 1.0000r - 1.0001 & 1.0000r - 4.0000 & 2.0000r - 2.0001 & 3.0000r - 3.0002 \\ 1.0000r - 1.0002 & 1.0000r - 3.0000 & 1.0000r - 4.0002 & 1.0000r - 4.0003 \\ 1.0000r + 3.0000 & 2.0000r + 5.0000 & 1.0000r + 0.0000 & 1.0000r + 4.0001 \\ 2.0000r - 2.0001 & 3.0000r + 4.0000 & 1.0000r - 1.0001 & 2.0000r - 2.0002 \end{pmatrix},$$

and

$$\overline{X}_{30}(\alpha) = \begin{pmatrix} 3.0001 - 1.0000r & 2.0000 - 1.0000r & 4.0001 - 1.0000r & 6.0002 - 1.0000r \\ 0.9999 - 1.0000r & -1.0000 - 2.0000r & 0.0009 - 1.0000r & 1.9998 - 1.0000r \\ 1.9998 - 1.0000r & 0.0000 - 1.0000r & -0.0002 - 1.0000r & 1.9997 - 1.0000r \\ 6.0000 - 2.0000r & 9.0000 - 2.0000r & 3.0000 - 1.0000r & 6.0001 - 1.0000r \\ 2.9999 - 1.0000r & 10.0000 - 1.0000r & 2.9999 - 1.0000r & 4.9998 - 3.0000r \end{pmatrix},$$

with the relative error

$$E(30) = \frac{\|\widetilde{X}_{30} - \widetilde{X}\|_\infty}{\|\widetilde{X}\|_\infty} = 3.1968e - 005.$$

Also, in Figure 5 the graph of the relative error in terms of the order of approximation (the number of iteration) for the various values of $\hbar$ is plotted. In this figure, the initial approximations $\mathbf{X}_0^C(\alpha)$ and $\mathbf{X}_0^R(\alpha)$ are chosen always zero matrices. Obviously the results show the convergence of method.

In continuation, we are going to compare our proposed approach with the Jacobi iteration method. Based on the Eq. (2.4), we first covert the fuzzy equation (4.6) into a system of crisp linear equations by the Kronecker product and the Vec-operation as follows

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{4.10}$$

where

$$S_1 = (\mathbf{I}_4 \otimes \mathbf{A}^+) + (\mathbf{B}^{T^+} \otimes \mathbf{I}_5),$$
$$S_2 = (\mathbf{I}_4 \otimes \mathbf{A}^-) + (\mathbf{B}^{T^-} \otimes \mathbf{I}_5),$$

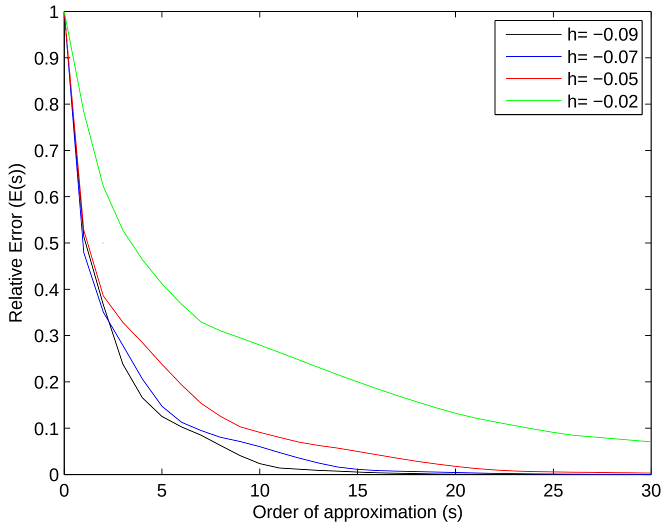

FIGURE 5. The relative error in terms of the order of approximation for the various values of $\hbar$ in Example 4.2.

are two 20×20 real matrices,

$$y_1 = \underline{Vec(\widetilde{\mathbf{X}})},$$

$$y_2 = \overline{Vec(\widetilde{\mathbf{X}})},$$

and also

$$w_1 = \underline{Vec(\widetilde{\mathbf{C}})},$$

$$w_2 = \overline{Vec(\widetilde{\mathbf{C}})},$$

are four 20×1 column vectors. Therefore, obviously the Eq. (4.10) is a 40×40 crisp linear system. By considering the initial approximation as a zero vector, i.e.

$$\begin{pmatrix} y_1^{(0)} \\ y_2^{(0)} \end{pmatrix} = \begin{pmatrix} zeros(20,1) \\ zeros(20,1) \end{pmatrix},$$

the values of relative error obtained by the Jacobi iteration method for various iterations, are presented in Table 1. Also, in Figure 6 the graph of the relative error in terms of the order of approximation (the number of iterations) obtained by the Jacobi iteration method is plotted. From Table 1 and Figure 6, we conclude that the values of relative error increase with the increase in the number of iterations. The numerical results show that unlike our proposed approach, the Jacobi iteration

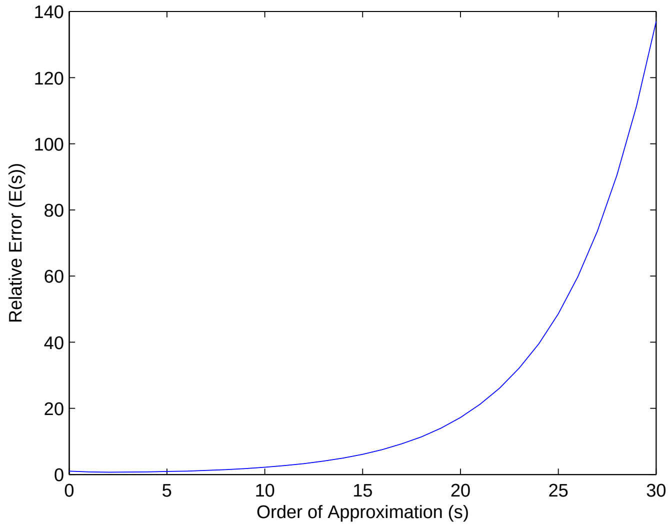

FIGURE 6. The relative error in terms of the order of approximation obtained by the Jacobi iteration method for the Eq. (4.10).

method is not convergence for the fuzzy Sylvester matrix equation (4.6).

The number of iterations	The values of relative error
0	1.0000
1	0.7774
2	0.7245
3	0.7473
4	0.7981
5	0.9113
10	2.2031
15	6.1310
20	17.2568
30	136.9959

Table 1: The values of relative error for various iterations obtained by the Jacobi iteration method for the Eq. (4.10).

Example 4.3. Consider the fuzzy Sylvester matrix equation

$$\mathbf{A} \cdot \widetilde{\mathbf{X}} + \widetilde{\mathbf{X}} \cdot \mathbf{B} = \widetilde{\mathbf{C}}, \tag{4.11}$$

where

$$\mathbf{A}=\begin{pmatrix} 4 & 1 & -4 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \end{pmatrix}, \qquad \mathbf{B}=\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix},$$

and

$$[\widetilde{\mathbf{C}}]_\alpha=\begin{pmatrix} [12r-30,19-19r] & [20r-5,39-17r] \\ [6r-11,8-8r] & [9r-4,16-9r] \\ [12r-12,18-8r] & [12r-23,9-13r] \end{pmatrix},$$

with the exact solution

$$[\widetilde{\mathbf{X}}]_\alpha=\begin{pmatrix} [-2+r,1-r] & [r,4-2r] \\ [-1+r,1-r] & [-1+r,1-r] \\ [-3+3r,4-r] & [-4+r,-3r] \end{pmatrix}. \tag{4.12}$$

In this example, we are going to compare the results obtained by our proposed method with ones obtained by Jacobi and Gauss-Seidel iteration methods. By applying these methods on the above fuzzy Sylvester matrix equation, the values of relative errors obtained by the homotopy analysis method (with $\hbar=-0.06$), Jacobi iteration method and Gauss-Seidel iteration method, for some iterations are presented in Table 2. Also, in Figures 7 to 9, the graphs of the relative errors in terms of the order of approximation (the number of iterations) obtained by these three methods are plotted. The results show that the values of relative error obtained by HAM, unlike the Jacobi and Gauss-Seidel methods, decrease with the increase in the number of iterations. It should be noted that in all above methods, the initial approximations were chosen as zero matrices.

Iterations	HAM	Jacobi method	Gauss-Seidel method
0	1.0000	1.0000	1.0000
5	0.3789	6.4761	9.1019
10	0.3085	31.1741	31.6688
15	0.2623	27.8804	99.4906
20	0.2287	198.5246	302.2995

Table 2: The relative errors obtained by HAM ($\hbar=-0.06$), Jacobi and Gauss-Seidel methods for Example 4.3.

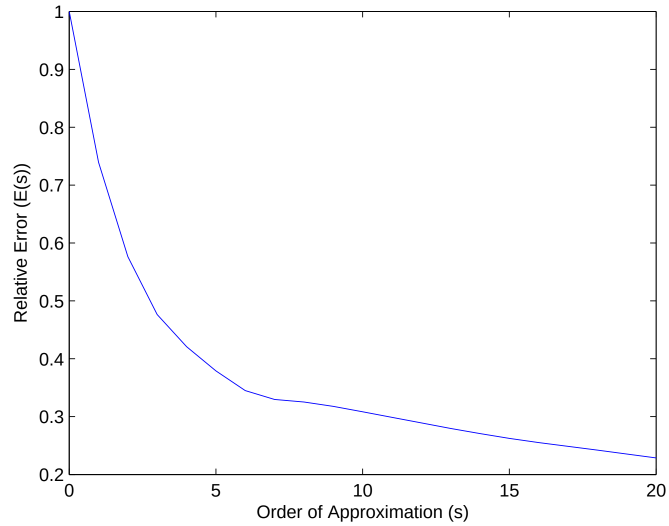

FIGURE 7. The relative error obtained by the homotopy analysis method (with $\hbar = -0.06$) for Example 4.3.

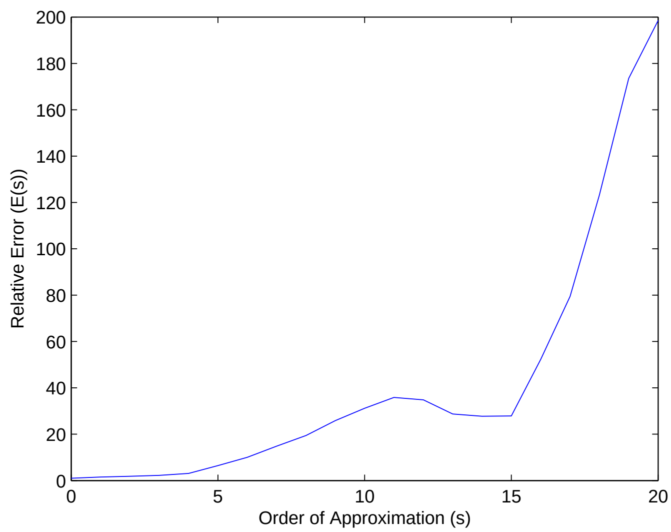

FIGURE 8. The relative error obtained by the Jacobi iteration method for Example 4.3.

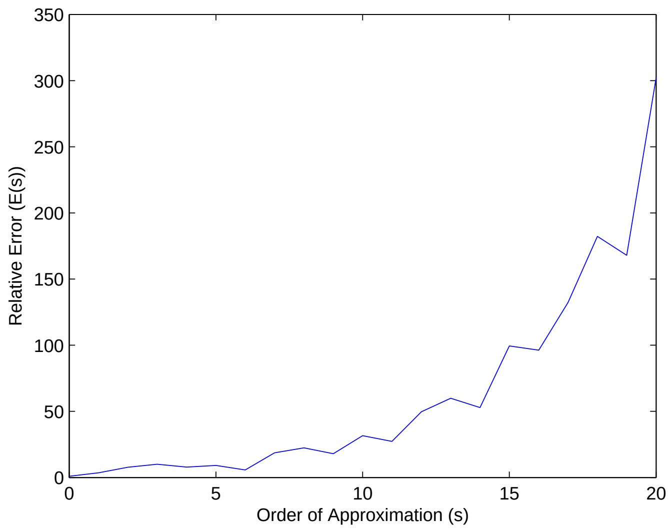

FIGURE 9. The relative error obtained by the Gauss-Seidel iteration method for Example 4.3.

5. Conclusions

In this work, a novel approach is presented to solve a fuzzy Sylvester matrix equation approximately. In our proposed approach, the original fuzzy problem is transformed into two crisp problem without using Kronecker product and Vec-operation. In the next step, the homotopy analysis method (HAM) is applied to solve the crisp Sylvester matrix equations. Also, sufficient conditions for the convergence of the method are given and finally three numerical examples are illustrated. The second and third examples show that unlike Jacobi and Gauss-Seidel methods, the results obtained by our proposed approach may be convergence.

As one of the applications of the proposed method in the numerical solution of fuzzy integral equations, consider the fuzzy linear first-kind Volterra integral equation

$$\int_a^t K(t,\tau)\cdot\widetilde{x}(\tau)d\tau = \widetilde{c}(t), \qquad t\in[a,b],$$

where the kernel $K(t,\tau)$ is a real function on the region $\Delta = \{(t,\tau)\ :\ t\in[a,b],\ \ \tau\in[a,t]\}$, $\widetilde{x}$ and $\widetilde{c}$ are two fuzzy functions on the interval $[a,b]$. In the above integral equation, assume that $\widetilde{c}(a) = \widetilde{x}(a) = \widetilde{0}$ and that $\{t_0, t_1, t_2, ..., t_n\}$ is a uniform partition with step length $h = \frac{b-a}{n}$ on the interval [a,b]. Now, by applying the trapezoidal rule of integration and substituting " = " for " ≈ ", we can obtain

$$\mathbf{A} \cdot \widetilde{\mathbf{X}} = \widetilde{\mathbf{C}}, \tag{5.1}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{h}{2}K(t_1,t_1) & 0 & 0 & 0 & 0 \\ hK(t_2,t_1) & \frac{h}{2}K(t_2,t_2) & 0 & 0 & 0 \\ hK(t_3,t_1) & hK(t_3,t_2) & \frac{h}{2}K(t_3,t_3) & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ hK(t_n,t_1) & hK(t_n,t_2) & hK(t_n,t_3) & \dots & \frac{h}{2}K(t_n,t_n) \end{pmatrix},$$

and

$$\widetilde{\mathbf{X}} = \begin{pmatrix} \widetilde{x}(t_1) \\ \widetilde{x}(t_2) \\ \widetilde{x}(t_3) \\ \vdots \\ \widetilde{x}(t_n) \end{pmatrix}, \qquad \widetilde{\mathbf{C}} = \begin{pmatrix} \widetilde{c}(t_1) \\ \widetilde{c}(t_2) \\ \widetilde{c}(t_3) \\ \vdots \\ \widetilde{c}(t_n) \end{pmatrix}.$$

Obviously, Eq. (5.1) can be considered as a fuzzy Sylvester matrix equation with $\mathbf{B} = \mathbf{0}$ and consequently solved by the proposed method.

Also, as potential future research, our proposed approach can be used to solve the linear fuzzy differential equations.

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