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On Conformally Flat G.R.C. of Exponential (α, β)-Metrics

Mosayeb Zohrehvand [1] and Shahroud Azami [2] and Mehdi Jafari [3]
[1] Department of Mathematics, Faculty of Mathematical Sciences and Statistics, Malayer University, Malayer, Iran
[2] Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
[3] Department of Mathematics, Payame Noor University, PO BOX 19395-4697, Tehran, Iran

ABSTRACT. This paper is devoted to the study of generalized Randers change in a specific class of (α, β)-metrics of conformally flat type. These metrics are defined as $F = \alpha \exp(\beta/\alpha) + \varepsilon\beta$, where $\varepsilon \neq 0$ is a real constant, and are called the generalized Randers change(G.R.C.) of the exponential metric. We demonstrate that if F possesses a relatively isotropic mean Landsberg curvature, it must either be a Riemannian or a locally Minkowskian metric. Furthermore, if F is a non-Riemannian weak Einstein metric, it necessarily reduces to a locally Minkowskian metric.

Keywords: Conformally flat metric, Exponential (α, β)-metric, Mean Landsberg curvature, Weak Einstein metric.

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[1]Corresponding author: m.zohrehvand@malayeru.ac.ir
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1. Introduction

Conformal geometry has attracted considerable interest because of its broad relevance in various physical theories. Within the framework of general relativity, conformal transformations between pseudo-Riemannian metrics maintain the structure of null (light-like) geodesics. In the context of Finsler geometry, Weyl's theorem highlights the crucial role of conformal and projective characteristics in describing the behavior of metric structures [15, 21].

Consider a differentiable manifold M. Finsler metrics F and $\tilde{F}$ on M are said to be conformally equivalent if $F = e^{\kappa(x)}\tilde{F}$, where $\kappa(x)$ is a smooth function on M. In this case, $\kappa(x)$ is referred to as the conformal factor. When $\tilde{F}$ is a Minkowski metric, the metric F is described as conformally flat in the sense of Finsler geometry.

S. Kikuchi [14] introduced a Finsler connection that remains invariant under conformal transformations and used it to characterize conformal flatness. Later, M. Matsumoto [18] extended this concept by formulating a conformally invariant Finsler connection applicable to metrics that meet certain specific conditions. Subsequently, Ichijyō and Hashiguchi identified criteria determining when a Randers metric is conformally flat [11]. Randers metrics represent the most elementary examples of (α, β)-metrics, a notable class within Finsler geometry with applications across disciplines, including physics and biology [1, 20].

An (α, β)-metric is a Finsler metric F that is of the form $F = \alpha\phi(s)$, $s := \frac{\beta}{\alpha}$, where $\alpha(x, y) := \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(x, y) := b_i(x)y^i$ denote a Riemannian metric and a 1-form, respectively, and $\phi(s)$ denotes a C^∞ function satisfying certain conditions [17]. These metrics can describe Riemannian spaces that are influenced by external forces; they are computationally tractable and play an important role in the Finsler geometry.

L. Kang [13] investigated conformally flat Randers metrics possessing scalar flag curvature, establishing their projective flatness and presenting a full classification of such metrics. Further research on conformally flat (α, β)-metrics - those having specific geometric quantities, such as relatively isotropic mean Landsberg curvature, isotropic S-curvature and constant flag curvature- revealed that these metrics are necessarily Riemannian or locally Minkowskian [4, 5, 6]. Analogous results were obtained for weak Einstein Matsumoto and Kropina metrics of conformally flat type [12]. Collectively, these works highlight the close connection between conformal geometry and Finsler geometry, showing how the structure of Finsler metrics is influenced by curvature conditions. For additional studies, refer to [2, 19, 23, 25, 26, 31].

The (α, β)-metric $F = \alpha \exp(s)$, $s := \frac{\beta}{\alpha}$, is referred to as the *exponential metric* and has been extensively investigated by various authors [7, 10, 22, 24, 27, 28, 29, 30], who have explored its geometric properties under different conditions. One notable feature of the exponential metric is its relationship to Rander's metric. Specifically, under certain transformations, the exponential metric can be reduced to Rander's metric, which has significant applications in theoretical physics, particularly in the study of space time geometries and cosmological models. On the other hand, a special form of the exponential metric as

$$F = \alpha \exp\Big(\int_0^s \frac{q\sqrt{b^2 - t^2}}{1 + qt\sqrt{b^2 - t^2}} dt\Big),$$

with $b := \|\beta\|_\alpha$ and q a constant, is an almost regular unicorn metric. Unicorn metrics refer to Finsler metrics that are Landsberg metrics but do not belong to the class of Berwald metrics [27]. Thus, the exponential metrics represent a notable subclass of (α, β)-metric that deserves more attention.

In Finsler geometry, several fundamental quantities, including the Cartan torsion $\mathbf{C}$ and the Ricci curvature $\mathbf{Ric}$, are obtained from the Finsler metric F. The horizontally covariant derivative of $\mathbf{C}$ along the geodesics of F yields a new tensor field $\mathbf{L}$, known as the Landsberg curvature. Taking contractions of the tensors $\mathbf{C}$ and $\mathbf{L}$, one obtains two tensors $\mathbf{I}$ and $\mathbf{J}$ which are refereed to as the mean Cartan torsion and the mean Landsberg curvature, respectively. A Finsler metric F is said to possess relatively isotropic mean Landsberg curvature if the quotient $\mathbf{J}/\mathbf{I}$, which describes how the mean Cartan torsion varies along the geodesics, is isotropic. Equivalently, there exists a scalar function $c = c(x)$ defined on M satisfying

$$\mathbf{J} + cF\mathbf{I} = 0. \tag{1.1}$$

A Finsler metric F on an n-dimensional manifold M is said to be a weak Einstein metric if its Ricci curvature takes the form

$$\mathbf{Ric} = (n-1)\Big(\frac{3\eta}{F} + \sigma\Big)F^2, \tag{1.2}$$

where $\eta := \eta_i(x)y^i$ denotes a 1-form and $\sigma := \sigma(x)$ is a scalar function on M. When η vanishes, the Finsler metric F is refereed to as an Einstein metric, in which case the Ricci curvature satisfies $\mathbf{Ric} = (n-1)\sigma F^2$. In the case where σ is constant, F is known as the Ricci constant Finsler metric. Furthermore, a Finsler metric F is called Ricci flat when its Ricci curvature vanishes [3].

This paper considers a special transformation of the exponential (α, β)-metrics, defined as

$$F = \alpha \exp(\beta/\alpha) + \varepsilon\beta,$$

where $\varepsilon \neq 0$ is a constant. It is referred to as the generalized Randers change (G.R.C.) of the exponential (α, β)-metrics. This metric represents a extension of the classical Rander's metric and the exponential metric, combining their features in a way that opens up new avenues for geometric exploration. We assume that these metrics are of conformally flat type and possess either a relatively isotropic mean Landsberg curvature or are weak Einstein metrics, and categorize them accordingly. Our results show that, under these assumptions, those metrics reduce to either Riemannian or locally Minkowskian.

2. Preliminaries

Let M denote an n-dimensional differentiable manifold, and $TM_0 := \bigcup_{x\in M}(T_xM\backslash\{0\})$ denotes the slit tangent bundle of M. Suppose that $F = F(x, y)$ is a Finsler metric defined on M, the symmetric bilinear form $(\mathbf{g}_y) = (g_{ij}(x, y))$ on T_xM that is expressed as

$$g_{ij}(x, y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y),$$

is called the fundamental tensor of F.

A smooth curve $x = x^i(t)$ defined on a Finsler space (M, F) is called a geodesic if it satisfies the subsequent system of ODEs:

$$\frac{d^2x^i}{dt^2} + G^i(x, \frac{dx}{dt}) = 0,$$

for which G^i are the coefficients of the geodesic spray that is derived from F as

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^m \partial y^l}y^m - \frac{\partial[F^2]}{\partial x^l}\Big\}.$$

The tensor field $\mathbf{R} := R^i_k dx^k \frac{\partial}{\partial x^i}$ of type (1,1) is defined by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}, \qquad (2.1)$$

and is called the Riemann curvature of (M, F). Taking contractions of the Riemann curvature tensor $\mathbf{R}$ yields the Ricci curvature $\mathbf{Ric}$, therefore

$$\mathbf{Ric} = R^m_m.$$

Within Finsler geometry, certain geometric quantities vanish in the Riemannian case and are therefore referred to as non-Riemannian quantities. Among these is the Cartan torsion $\mathbf{C}$, a symmetric trilinear form on TM_0 denoted as $\mathbf{C} := C_{ijk}dx^i \otimes dx^j \otimes dx^k$. It is defined by

$$C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

A Finsler metric F becomes Riemannian if and only if $\mathbf{C} = 0$, hence $\mathbf{C}$ characterizes the Riemannian metrics in the Finsler geometry.

The tensor field $\mathbf{I} := I_i dx^i$, is called the mean Cartan torsion, where

$$I_i := g^{jk}C_{ijk}.$$

Moreover, it satisfies

$$I_i = \frac{\partial}{\partial y^i}\Big[\ln\sqrt{\det(g_{jk})}\Big]$$

Another non-Riemannian quantity is the Landsberg curvature $\mathbf{L} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$, that is a tensor field on TM_0. It is defined as $L_{ijk} := C_{ijk;m}y^m$, where ";" represents the horizontal covariant derivative with respect to the Berwald connection associated with F. An equivalent formulation of the Landsberg curvature is given by

$$L_{ijk} = -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k}. \tag{2.2}$$

A Landsberg metric is a Finsler metric F such that its Landsberg curvature $\mathbf{L}$ vanishes.

Taking the horizontal covariant derivative of the mean Cartan torsion $\mathbf{I}$ along the geodesics determined by F, yields the mean Landsberg curvature $\mathbf{J} := J_i dx^i$. Therefore, we have

$$J_i := I_{i;m}y^m. \tag{2.3}$$

An equivalent formulation for $\mathbf{J}$ can be expressed as

$$J_i := g^{jk}L_{ijk}.$$

A weak Landsberg metric is a Finsler metric with vanishing mean Landsberg curvature.

A Finsler metric F is an (α,β)-metric if $F = \alpha\phi(s), \quad s := \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form with $||\beta_x|| < b_0$, $x \in M$ and $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \le b < b_0, \tag{2.4}$$

which yields that F is positive definite [9]. The fundamental tensor $F = \alpha\phi(\beta/\alpha)$ is represented by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_i + b_j\alpha_i) + \rho_2\alpha_i\alpha_j,$$

here $\alpha_i := \alpha^{-1} a_{ij} y^j$, and

$$\rho := \phi(\phi - s\phi'), \qquad \rho_0 := \phi\phi'' + \phi'\phi',$$
$$\rho_1 := -s\rho_0 + \phi\phi', \qquad \rho_2 := s\{-s\rho_0 + \phi\phi'\}.$$

One can see that

$$g^{ij} = \rho^{-1}\{a^{ij} - \tau b^i b^j - \eta Y^i Y^j\}, \tag{2.5}$$

where $b^i := a^{ij} b_j$ and

$$\eta := \frac{\mu}{1 + Y^2\mu}, \quad \mu := \frac{\rho_2}{\rho}, \qquad Y^2 := 1 + (\lambda + \epsilon)s + \lambda\epsilon b^2,$$
$$Y^i := y^i\alpha^{-1} + \lambda b^i, \quad \lambda := \frac{\epsilon - \delta s}{1 + \delta b^2}, \quad \epsilon := \frac{\rho_1}{\rho_2},$$
$$\delta := \frac{\rho_0 - \epsilon^2\rho_2}{\rho}, \quad \tau := \frac{\delta}{1 + \delta b^2}.$$

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$$

where $b_{i|j}$ represents the coefficients of the covariant derivative of β with respect to α. Furthermore we define

$$r_{00} := r_{ij} y^i y^j, \qquad r_{i0} := r_{im} y^m, \qquad r_i := b^m r_{mi},$$
$$r_0 := r_i y^i, \qquad r^i{}_j := a^{im} r_{mj}, \qquad s_{i0} := s_{im} y^m,$$
$$s^i{}_j := a^{im} s_{mj}, \qquad s_i := b^m s_{mi}, \qquad s_0 := s_i y^i.$$

Suppose that G^i and G^i_α represent the geodesic coefficients of $F = \alpha\phi(s)$ and α, respectively. The relationship between G^i and G^i_α, is expressed as:

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\Big\{\Psi b^i + \Theta\alpha^{-1} y^i\Big\}. \tag{2.6}$$

Here

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$\Psi := \frac{\phi''}{2[\phi - s\phi' + (b^2 - s^2)\phi'']},$$
$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[\phi - s\phi' + (b^2 - s^2)\phi'']}.$$

For more details, see [9].

As we known that, a Randers metric $F = \alpha + \beta$ can be obtained as a transformation of Riemannian metric α. For a Finsler space (M, F) and a one form $\beta = b_i(x)y^i$ defined on M, one can consider the following transformation

$$F \to \tilde{F} = F + \epsilon\beta,$$

where ϵ is a non-zero constant. If $F = \alpha$ and $\epsilon = 1$, then $\tilde{F}$ is a Randers metric, thus it is called generalized Randers change (G.R.C.) of F.

3. Main results

This section considers the generalized Randers change of exponential (α, β)-metrics of conformally flat type. They are expressed as

$$F = \alpha \exp(\beta/\alpha) + \varepsilon\beta,$$

where $\varepsilon \neq 0$ is a constant. One can see that, if $\|\beta_x\| < \frac{1}{\varepsilon}$, then F is positive definite.

The next Lemma is fundamental for the study of (α, β)-metrics of conformally flat type.

Lemma 3.1. ([1]) *For an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$,*

F is locally Minkowskian $\Longleftrightarrow$ α is parallel w.r.t. α.

Firstly, we study conformally flat G.R.C. of exponential (α, β)-metric whose mean Landsberg curvature is relatively isotropic. We begin with a Lemma that computes the mean Cartan torsion corresponding to an (α, β)-metric.

Lemma 3.2. ([8])*Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric. Its mean Cartan torsion can be expressed as*

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi')h_i, \tag{3.1}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := (1 + n\Delta + sQ)(sQ' - Q) - (b^2 - s^2)(sQ + 1)Q'',$$
$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\right]',$$
$$h_j := b_j - \alpha^{-1}sy_j.$$

By combining equation (3.1) with Deicke's theorem, we obtain the subsequent lemma.

Lemma 3.3. *For an (α, β)-metric $F = \alpha\phi(\beta/\alpha)$,*

F is Riemannian $\Longleftrightarrow$ $\Phi = 0$.

Combining relations (2.3) and (3.1), it follows that the mean Landsberg curvature corresponding to an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, can be expressed as

$$\begin{aligned} J_j =&\frac{1}{2\alpha^4\Delta}\bigg\{\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](s_0+r_0)h_j \\ &+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Qs_0)h_j \\ &+\alpha\Big[-\alpha^2Q's_0h_j+\alpha Q(\alpha^2s_j-y_js_0)+\alpha^2\Delta s_{j0} \\ &+\alpha^2(r_{j0}-2\alpha Qs_j)-(r_{00}-2\alpha Qs_0)y_j\Big]\frac{\Phi}{\Delta}\bigg\}, \end{aligned} \tag{3.2}$$

where $y_j:=a_{ij}y^i$. Further information can be found in [8, 16].

Using of (3.1) and (3.2), we get

$$\begin{aligned} J_j+c(x)FI_j =& -\frac{1}{2\alpha^4\Delta}\bigg\{\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](s_0+r_0)h_j \\ &+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Qs_0)h_j+\alpha\Big[-\alpha^2Q's_0h_j \\ &+\alpha Q(\alpha^2s_j-y_js_0)+\alpha^2\Delta s_{j0}+\alpha^2(r_{j0}-2\alpha Qs_j) \\ &-(r_{00}-2\alpha Qs_0)y_j\Big]\frac{\Phi}{\Delta}+c(x)\alpha^4\Phi(\phi-s\phi')h_j\bigg\}. \end{aligned} \tag{3.3}$$

Now, consider a conformally flat (α,β)-metric $F=\alpha\phi(\beta/\alpha)$, defined on a manifold M. By definition, $\tilde{F}=e^{\kappa(x)}F$ where $\tilde{F}$ is a locally Minkowski metric on M. Since the metric $F=\alpha\phi(\beta/\alpha)$ constructed from α and β, the conformal relation induces corresponding scaled quantities. Consequently

$$\tilde{F}=\tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha}),$$

where

$$\tilde{\alpha}=e^{\kappa(x)}\alpha, \qquad \tilde{\beta}=e^{\kappa(x)}\beta. \tag{3.4}$$

From (3.4), we have

$$\tilde{a}_{ij}=e^{2\kappa(x)}a_{ij}, \qquad \tilde{b}_i=e^{\kappa(x)}b_i.$$

The Christoffel symbols corresponding to the Levi-Civita connections of α and $\tilde{\alpha}$ are connected through the relation

$$\tilde{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\kappa_k+\delta^i_k\kappa_j-\kappa^ia_{jk},$$

where $\kappa_i := \frac{\partial \kappa}{\partial x^i}$ and $\kappa^i := a^{ij}\kappa_j$. As a consequence, the components of the covariant derivative of $\tilde{\beta}$ with respect to $\tilde{\alpha}$, satisfy

$$\tilde{b}_{i\|j} = \frac{\partial \tilde{b}_i}{\partial x^j} - \tilde{b}_s \tilde{\Gamma}^i_{jk} = e^{\kappa}(b_{i|j} - b_j\kappa_i + b_r\kappa^r a_{ij}). \tag{3.5}$$

Since $\tilde{F}$ is a locally Minkowski metric, Lemma 3.1 implies that $\tilde{b}_{i\|j} = 0$. Substituting into the (3.5), yields

$$b_{i|j} = b_j\kappa_i - b_r\kappa^r a_{ij}. \tag{3.6}$$

From (3.6) we obtain

$$r_{ij} = \frac{1}{2}(\kappa_i b_j + \kappa_j b_i) - b_r\kappa^r a_{ij}, \qquad r_j = -\frac{1}{2}(b^r\kappa_r)b_j + \frac{1}{2}\kappa_j b^2, \tag{3.7}$$

$$r_{i0} = \frac{1}{2}[\kappa_i\beta + (\kappa_r y^r)b_i] - \kappa_r b^r y_i, \qquad s_{ij} = \frac{1}{2}(\kappa_i b_j - \kappa_j b_i), \tag{3.8}$$

$$s_j = \frac{1}{2}(b^r\kappa_r)b_j - \kappa_j b^2, \qquad s_{i0} = \frac{1}{2}[\kappa_i\beta - (\kappa_r y^r)b_i]. \tag{3.9}$$

Furthermore, we have

$$r_{00} = (\kappa_r y^r)\beta - (\kappa_r b^r)\alpha^2, \tag{3.10}$$

$$r_0 = \frac{1}{2}(\kappa_r y^r)b^2 - \frac{1}{2}(\kappa_r b^r)\beta, \tag{3.11}$$

$$s_0 = \frac{1}{2}(\kappa_r b^r)\beta - \frac{1}{2}(\kappa_r y^r)b^2. \tag{3.12}$$

Equations (3.11) and (3.12) together imply that, for an (α, β)-metric of conformally flat type, the condition $r_0 + s_0 = 0$ holds. This condition is equivalent to saying that the length of β with respect to α be constant.

To streamline the subsequent calculations, we choose at any point x an α-orthonormal basis such that $\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}$ and $\beta = by^1$, where $b := \|\beta_x\|_\alpha$. We then introduce the coordinate change

$$\psi : (s, v^A) \longrightarrow (y^i),$$

in T_xM, which

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = v^A, \quad A := 2, \dots, n, \tag{3.13}$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^n (v^A)^2}$. One can see that

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{3.14}$$

Then, by (3.6)-(3.14) one can obtain

$$r_{00} = -b\kappa_1\bar{\alpha}^2 + \frac{bs\bar{\kappa}_0\bar{\alpha}}{\sqrt{b^2-s^2}}, \qquad r_0 = -s_0 = \frac{1}{2}b^2\bar{\kappa}_0, \tag{3.15}$$

$$r_{A0} = \frac{1}{2}\frac{\kappa_A bs\bar{\alpha}}{\sqrt{b^2-s^2}} - (b\kappa_1)v_A, \quad r_{10} = \frac{1}{2}b\bar{\kappa}_0, \tag{3.16}$$

$$s_A = -\frac{1}{2}\kappa_A b^2, \qquad s_1 = 0, \tag{3.17}$$

$$s_{A0} = \frac{1}{2}\frac{\kappa_A bs\bar{\alpha}}{\sqrt{b^2-s^2}}, \qquad s_{10} = -\frac{1}{2}b\bar{\kappa}_0, \tag{3.18}$$

$$h_A = -\frac{\sqrt{b^2-s^2}\,sv_A}{b\bar{\alpha}}, \qquad h_1 = b - \frac{s^2}{b}. \tag{3.19}$$

where $\bar{\kappa}_0 := \kappa_A v^A$.

We may now proceed to demonstrate the theorem stated below.

Theorem 3.4. *Let $F = \alpha\exp(s) + \varepsilon\beta$, $s := \beta/\alpha$ be an (α,β)-metric of conformally flat type on an n-dimensional differentiable manifold M, $(n \geq 3)$. Suppose that F has relatively isotropic mean Landsberg curvature. Then F reduces either to a Riemannian or to a locally Minkowski metric.*

Proof. Since $\tilde{b}_{i\|j} = 0$, it follows that $\tilde{b}$ is constant. In the case $\tilde{b} = 0$, the relation $F = e^{k(x)}\tilde{\alpha}$ shows that F is Riemannian. Therefore, we suppose that F is a non-Riemannian (α,β)-metric of conformally flat type that possesses relatively isotropic mean Landsberg curvature. Using (1.1), (3.3) and $r_0 + s_0 = 0$, we have

$$\begin{aligned}\frac{\alpha^2}{b^2-s^2}&\Big\{\Psi_1 + s\frac{\Phi}{\Delta}\Big\}(r_{00} - 2\alpha Q s_0)h_j + \alpha\big\{-\alpha^2 Q' s_0 h_j \\ &+ \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) \\ &- (r_{00} - 2\alpha Q s_0)y_j\big\}\frac{\Phi}{\Delta} - c(x)\alpha^4\Phi(\phi - s\phi')h_j = 0.\end{aligned} \tag{3.20}$$

Putting $j = 1$ in (3.20), yields

$$\begin{aligned}\frac{\alpha^2}{b^2-s^2}&\Big\{\Psi_1 + s\frac{\Phi}{\Delta}\Big\}(r_{00} - 2\alpha Q s_0)h_1 + \alpha\big\{-\alpha^2 Q' s_0 h_1 \\ &+ \alpha Q(\alpha^2 s_1 - y_1 s_0) + \alpha^2\Delta s_{10} + \alpha^2(r_{10} - 2\alpha Q s_1) \\ &- (r_{00} - 2\alpha Q s_0)y_1\big\}\frac{\Phi}{\Delta} - c(x)\alpha^4\Phi(\phi - s\phi')h_1 = 0.\end{aligned} \tag{3.21}$$

By replacing (3.14)-(3.19) into (3.21) and afterward multiplying the resulting relation with $-2\Delta(b^2-s^2)^{3/2}$ we have

$$b^2\bar{\alpha}^3\Big\{2\sqrt{b^2-s^2}\Delta\big[bc\Phi(\phi-s\phi')+\Psi_1\sigma_1\big]\bar{\alpha}-\bar{\kappa}_0\big[b^2\Phi Q'(b^2-s^2) + \Phi b^2(sQ+1)+\Delta\Phi b^2-2\Psi_1\Delta(b^2Q+s)\big]\Big\}=0. \quad (3.22)$$

From (3.22), we obtain

$$\Delta\big[bc\Phi(\phi-s\phi')+\Psi_1\kappa_1\big]=0, \quad (3.23)$$

$$\bar{\kappa}_0\big[b^2\Phi Q'(b^2-s^2)+\Phi b^2(sQ+1)+\Delta\Phi b^2 - 2\Psi_1\Delta(b^2Q+s)\big]=0. \quad (3.24)$$

Since $\Delta=Q'(b^2-s^2)+sQ+1$, one can see that (3.24) simplify as follow

$$(b^2\Psi_1\Delta Q+\Psi_1\Delta s)\bar{\kappa}_0=0.$$

This means that

$$\Psi_1\Delta(b^2Q+s)\bar{\kappa}_0=0. \quad (3.25)$$

Now let $j=A$ in (3.20), thus we obtain

$$\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Qs_0)h_A+\alpha\big[-\alpha^2Q's_0h_A + \alpha Q(\alpha^2s_A-y_As_0)+\alpha^2\Delta s_{A0}+\alpha^2(r_{A0}-2\alpha Qs_A) - (r_{00}-2\alpha Qs_0)y_A\big]\frac{\Phi}{\Delta}+c(x)\alpha^4\Phi(\phi-s\phi')h_A=0. \quad (3.26)$$

By substituting (3.14)-(3.19) into (3.26) and applying the same argument as in the situation $j=1$, while using the identity $\Delta=Q'(b^2-s^2)+sQ+1$, we obtain

$$(s\Delta+s+b^2Q)b^2\Phi\kappa_A\bar{\alpha}^2-\big[(s\Delta+s+b^2Q)b^2\Phi + 2s(b^2Q+s)\Psi_1\Delta\big]\bar{\kappa}_0v_A=0, \quad (3.27)$$

$$s\sqrt{b^2-s^2}[\Phi bc(\phi-s\phi')+\Psi_1\kappa_1]\Delta v_A=0. \quad (3.28)$$

It is clear that (3.28) is equivalent to (3.23). Furthermore, multiplying (3.27) by v^A yields

$$s(b^2Q+s)\Psi_1\Delta\bar{\kappa}_0\bar{\alpha}^2=0. \quad (3.29)$$

It is easy to see that (3.29) is equivalent to (3.25). Consequently, we have demonstrated that an (α,β)-metric of conformally flat type that has relatively isotropic mean Landsberg curvature satisfies both equations (3.23) and (3.25).

If $b^2Q + s = 0$, then it follows that $\phi = k\sqrt{b^2 - s^2}$ for some constant k. However, this is incompatible with the assumption $\phi = \exp(2s)/s$. Therefore $b^2Q + s \neq 0$. Consequently, from (3.25) it follows that either $\Psi_1 = 0$ or $\kappa_A = 0$.

If $\Psi_1 = 0$, then substituting this into (3.23) yields $\Phi = 0$. By Lemma 3.3, this condition implies that F reduces to a Riemannian metric.

When $\Psi_1 \neq 0$, it follows that $\kappa_A = 0$. Under this assumption , we show that $\kappa_1 = 0$ as well. Simplifying (3.23) and then multiplying both sides by Δ^2, we arrive at the following equation

$$\Big\{[-s\Phi + (b^2 - s^2)\Phi']\Delta - \frac{3}{2}(b^2 - s^2)\Phi\Delta'\Big\}\kappa_1 - cb\Delta^2\Phi(\phi - s\phi') = 0. \tag{3.30}$$

Let $A_1 := (s - 1)$ and $A_2 := (b^2 - s^2 - s + 1)$. Putting $\phi = \exp(s) + \varepsilon s$ into (3.30) and multiplying by $2\phi A_1^3 A_2^2$ and using Maple program, we obtain

$$E_0 + E_1 e^s + E_2 e^{2s} + E_3 e^{3s} = 0, \tag{3.31}$$

where E_i, $(0 \leq i \leq 3)$ are polynomials of s. One can see that

$$E_2 := \kappa_1\zeta_1(s) + 2cb\zeta_2(s), \tag{3.32}$$

$$E_3 := 2cb\zeta_3(s), \tag{3.33}$$

where

$$\begin{aligned}
\zeta_3(s) :=\ & 2ns^7 + 5ns^6 - 3(nb^2 + n + 1)s^5 - [(9n - 2)b^2 + 10n + 5]s^4 \\
& + [2(3nb^2 + 6n + 5)b^2 + 5n + 5]s^3 + [(3n - 4)b^4 + (9n + 2)b^2 \\
& + 6n + 5]s^2 - [(2nb^4 + 9nb^2 + 7b^2 + 12n + 12)b^2 + 5n + 5]s \\
& + (n + 2)b^6 + (3n + 5)b^4 + (3n + 4)b^2 + n + 1, \\
\zeta_2(s) :=\ & (1 - 5n)s^8 + (1 - 14n)s^7 + [(15n - 3)b^2 + 4n + 4]s^6 \\
& + [(27n - 6)b^2 + 28n + 13]s^5 - [3(5n - 1)b^4 + 6(4n + 3)b^2 \\
& + 5n + 5]s^4 - [3(4n - 3)b^4 + 3(11n + 4)b^2 + 20n + 17]s^3 \\
& + [(5n - 1)b^6 + 3(7n + 5)b^4 + 24(n + 1)b^2 + 8n + 8]s^2 \\
& - [(n + 4)b^6 + 3(2 - n)b^2 - 2n - 2]s - (n + 1)[b^6 + 3b^4 + 3b^2 + 1], \\
\zeta_1(s) :=\ & (12 - 9n)s^5 + 3[2(n - 2)b^2 - 3]s^4 + 2[(7n - 5)b^2 + 8n + 5]s^3 \\
& - [8(n - 2)b^4 + (17n - 1)b^2 + 11n + 11]s^2 - [(n - 2)b^4 \\
& - 2(n - 4)b^2 - 2n - 2]s + (n - 2)(2b^2 + 3)b^4 + (n + 1)b^2.
\end{aligned}$$

We see that $\zeta_j(s) \neq 0$, $(1 \leq j \leq 3)$. From (3.31) it follows that $E_i = 0$, $(0 \leq i \leq 3)$, and using (3.32) and (3.33), we obtain that $c = \kappa_1 = 0$.

Since $\kappa_1=\kappa_A=0$, the function κ must be constant. Consequently, the metric F is locally Minkowskian. $\square$

Now, we study the weak Einstein conformally flat G.R.C. of exponential (α,β)-metrics.

The formula for Ricci curvature of an (α,β)-metric of conformally flat type has been obtained in [4]. In fact, the subsequent Lemma holds.

Lemma 3.5. *Suppose that $F=\alpha\phi(s)$, $s:=\beta/\alpha$ be an (α,β)-metric of conformally flat type defined on a manifold M. Thus, $F=e^{\kappa(x)}\tilde{F}$, where $\tilde{F}=\tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ is a locally Minkowski metric and $\kappa(x)$ is a scalar function on M. In this case, the Ricci curvature of F is expressed as*

$$\begin{aligned}\mathbf{Ric}=&c_1\|\nabla\kappa\|^2_{\tilde{\alpha}}\tilde{\alpha}^2+c_2\kappa_0^2+c_3\kappa_0 f\tilde{\alpha}+c_4 f^2\tilde{\alpha}^2\\&+c_5 f_1\tilde{\alpha}+c_6\tilde{\alpha}^2+c_7\kappa_{00}.\end{aligned}\tag{3.34}$$

where

$$\begin{aligned}c_6:&=c_{61}\tilde{a}^{ij}\kappa_{ij}+c_{62}f_2,\\ c_{61}:&=-\frac{\phi}{\phi-s\phi'},\\ c_{62}:&=\frac{\phi\phi''}{(\phi-s\phi')\big[(\phi-s\phi')(b^2-s^2)\phi''\big]}.\end{aligned}$$

and $\|\nabla\kappa\|^2_{\tilde{\alpha}}:=\tilde{a}^{ij}\kappa_i\kappa_j, f:=\tilde{b}^i\kappa_i,\ f_1:=\tilde{b}^i y^j\kappa_{ij},\ f_2:=\tilde{b}^i\tilde{b}^j\kappa_{ij}$. Here $c_1,c_2,c_3,c_4,\ c_5,c_7$ are the functions depending on the variable s and are independent of $\tilde{\alpha},\kappa_0,\kappa_{00},f,f_1,f_2,\tilde{a}^{ij}\kappa_{ij}$.

In this time, we suppose that $F=\alpha\exp(s)+\varepsilon\beta$ be a weak Einstein metric of conformally flat type and show that F is Ricci-flat.

Lemma 3.6. *Suppose that $F=\alpha\exp(s)+\varepsilon\beta$, $s:=\beta/\alpha$, be a weak Einstein metric of conformally flat type on a manifold M of dimension $n\geq 3$, with $\varepsilon\neq 0$ a real constant. Then F is Ricci flat.*

Proof. Since F is a weak Einstein (α,β)-metric of conformally flat type, equations (1.2) and (3.34) imply that

$$\begin{aligned}(n-1)\Big(\frac{3\theta}{F}+\sigma\Big)F^2=&c_1\|\nabla\kappa\|^2_{\tilde{\alpha}}\tilde{\alpha}^2+c_2\kappa_0^2+c_3\kappa_0 f\tilde{\alpha}+c_4 f^2\tilde{\alpha}^2\\&+c_5 f_1\tilde{\alpha}+c_6\tilde{\alpha}^2+c_7\kappa_{00}.\end{aligned}\tag{3.35}$$

We use the same coordinate transformation introduced in Theorem 3.4, for $\tilde{\alpha}$ and $\tilde{\beta}$. In fact, we choose at any point x an $\tilde{\alpha}$-orthonormal basis such that $\tilde{\alpha}=\sqrt{\sum_{i=1}^n(y^i)^2}$ and $\tilde{\beta}=\tilde{b}y^1$, where $\tilde{b}:=\|\tilde{\beta}_x\|_{\tilde{\alpha}}$. We then introduce the coordinate change

$$\psi:(s,v^A)\longrightarrow(y^i),$$

in T_xM, which

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = v^A, \quad A := 2, \dots, n,$$

where $\bar{\alpha} = \sqrt{\sum_{i=2}^{n}(v^A)^2}$. Under this transformation, we obtain

$$\tilde{\alpha} = \frac{\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}, \qquad \tilde{\beta} = \frac{\tilde{b}s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}.$$

Thus

$$\begin{aligned} F &= \exp(\kappa)\tilde{\alpha}\phi(s) = \frac{\exp(k)\tilde{b}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}\phi(s), \\ f &= \kappa_1\tilde{b}, \qquad f_1 = \frac{\tilde{b}s\kappa_{11}}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha} + \tilde{b}\bar{\kappa}_{10}, \\ f_2 &= \tilde{b}^2\kappa_{11}, \qquad \theta = \frac{t_1 s}{\sqrt{\tilde{b}^2 - s^2}}\bar{\alpha}^2 + \bar{t}_0, \end{aligned} \tag{3.36}$$

where $\bar{t}_0 := \Sigma_{A=2}^{n} t_A u^A$.

Using (3.36), the Eq. (3.35) is equivalent with two following equations.

$$\Big[c_1\tilde{b}^2\|\nabla\kappa\|_{\tilde{\alpha}}^2 + c_2\kappa_1^2 s^2 + c_3\tilde{b}^2\kappa_1^2 s + c_4\kappa_1^2\tilde{b}^4 + c_5\kappa_{11}\tilde{b}^2 s + (c_{61}\delta^{ij}\kappa_{ij} + c_{62}\kappa_{11}\tilde{b}^2)\tilde{b}^2 + c_7\kappa_{11}s^2\Big]\frac{\bar{\alpha}^2}{\tilde{b}^2 - s^2} + c_2\bar{\kappa}_0^2 + c_7\bar{\kappa}_{00} = \frac{(n-1)e^{\kappa}(3t_1\tilde{b}s + \sigma e^{\kappa}\tilde{b}^2\phi)}{\tilde{b}^2 - s^2}\bar{\alpha}^2\phi, \tag{3.37}$$

$$(2c_2 s + c_3\tilde{b}^2)\kappa_1\kappa_A + (c_5\tilde{b}^2 + 2c_7 s)\kappa_{1A} = 3(n-1)e^{\kappa}t_A\tilde{b}\phi. \tag{3.38}$$

Let $A_1 := (s-1)$, $\;A_2 := (\tilde{b}^2 - s^2 - s + 1)$. Putting $\phi(s) = \exp(s) + \varepsilon s$ and multiplying (3.37) by $e^{2s}A_1^3A_2^4\phi^2(\tilde{b}^2 - s^2)$ and using Maple program we get

$$\mathcal{D}_0 + \mathcal{D}_1e^s + \mathcal{D}_2e^{2s} + \mathcal{D}_3e^{3s} + \mathcal{D}_4e^{4s} + \mathcal{D}_5e^{5s} + \mathcal{D}_6e^{6s} = 0, \tag{3.39}$$

where $\mathcal{D}_i$, $(0 \le i \le 6)$ are polynomials of s, specially we have

$$\mathcal{D}_5 := (n-1)s\tilde{b}(4e^{2\kappa}\varepsilon\tilde{b}\sigma + 3e^{\kappa}t_1)A_1^3A_2^4\bar{\alpha}^2, \tag{3.40}$$

$$\mathcal{D}_6 := (n-1)\tilde{b}^2(e^{2\kappa}\sigma)A_1^3A_2^4\bar{\alpha}^2. \tag{3.41}$$

Thus $\mathcal{D}_i = 0$, $(0 \le i \le 6)$, and therefore from (3.40) and (3.41) we conclude that $\sigma = t_1 = 0$.

Similarly, multiplying (3.38) with $e^sA_1A_2^4\phi^2$ and using Maple program we get

$$\mathcal{F}_0 + \mathcal{F}_1e^s + \mathcal{F}_2e^{2s} + \mathcal{F}_3e^{3s} + \mathcal{F}_4e^{4s} = 0, \tag{3.42}$$

where $\mathcal{F}_i,\ (0 \leq i \leq 4)$ are polynomials of s. Therefore $\mathcal{F}_i = 0,\ (0 \leq i \leq 4)$ and since

$$\mathcal{F}_4 := -6(n-1)\tilde{b}A_1A_2^4 e^{\kappa} t_A, \tag{3.43}$$

thus we get $t_A = 0$. From $t_1 = t_A = 0$, it follows that $\theta = 0$ and therefore $\mathbf{Ric} = 0$. □

Now, we consider weak Einstein G.R.C. of exponential metrics of conformally flat type and prove the subsequent Theorem.

Theorem 3.7. *Suppose that $F = \alpha \exp(s) + \varepsilon\beta$, $s := \beta/\alpha$ be a weak Einstein (α, β)-metric of the conformally flat type defined on a manifold M of the dimension $n \geq 3$, with $\varepsilon \neq 0$ a real constant. Then F is either a Riemannian or a locally Minkowskian metric.*

Proof. We assume that $\tilde{b} \neq 0$. Since $t_1 = \sigma = 0$, Eq. (3.37) is reduced to

$$\Big[c_1\tilde{b}^2\|\nabla\kappa\|^2_{\bar{\alpha}} + c_2\kappa_1^2 s^2 + c_3\tilde{b}^2\kappa_1^2 s + c_4\kappa_1^2\tilde{b}^4 + c_5\kappa_{11}\tilde{b}^2 s + (c_{61}\delta^{ij}\kappa_{ij} + c_{62}\kappa_{11}\tilde{b}^2)\tilde{b}^2 + c_7\kappa_{11}s^2\Big]\frac{\bar{\alpha}^2}{\tilde{b}^2 - s^2} + c_2\bar{\kappa}_0^2 + c_7\bar{\kappa}_{00} = 0, \tag{3.44}$$

Multiplying (3.44) by $e^{2s}A_1^3A_2^4\phi^2(\tilde{b}^2 - s^2)$ and using Maple program we have

$$E_0 + E_1e^s + E_2e^{2s} + E_3e^{3s} + E_4e^{4s} = 0, \tag{3.45}$$

where $E_i,\ (0 \leq i \leq 4)$ are functions of s which are independent of e^s. Thus $E_i = 0,\ (0 \leq i \leq 4)$.

One can see that

$$E_0 := \mathcal{N}_{14}s^{14} + \mathcal{N}_{13}s^{13} + \cdots + \mathcal{N}_1 s + \mathcal{N}_0, \tag{3.46}$$

where $\mathcal{N}_i$ $(0 \leq i \leq 14)$, are functions independent from s and

$$\mathcal{N}_{14} : = \varepsilon^4\tilde{b}^2[(\kappa_1^2 - \|\nabla\kappa\|^2_{\bar{\alpha}})\bar{\alpha}^2 + \bar{\kappa}_0^2],$$
$$\mathcal{N}_0 : = 6\varepsilon^4(n-1)\tilde{b}^6(\tilde{b}^2+1)^2\bar{\kappa}_0^2.$$

Thus $\mathcal{N}_i = 0, (0 \leq i \leq 14)$. From $\mathcal{N}_0 = 0$, we get

$$\kappa_A = 0, \tag{3.47}$$

thus

$$\|\nabla\kappa\|^2_{\bar{\alpha}} = 0. \tag{3.48}$$

Substituting (3.47) and (3.48) in $\mathcal{N}_{14} = 0$, we obtain $\kappa_1 = 0$. From $\kappa_1 = \kappa_A = 0$, we have that κ is constant. Consequently, the metric F is locally Minkowskian. □

4. Conclusion

In This paper, we study conformally flat G.R.C. of exponential metrics that have either relatively isotropic mean Landsberg curvature or weak Einstein metrics. We prove that such metrics are either locally Minkowskian or Riemannian metrics. Thus, we extended well-known results about some (α, β)-metrics such as Randers, Kropina, Matsumoto and exponential metrics of conformally flat type to a transformation of exponential metrics.

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