
(Positive) implicative SBE-filters of SBE-algebras

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ABSTRACT. This study presents a comprehensive exploration of Sheffer stroke BE-algebras (SBE-algebras), delving into fundamental concepts, definitions, and associated results. We introduce the concept of transitive SBE-algebras, examining their properties and characterizing SBE-filters. Additionally, implicative SBE-filters are introduced, demonstrating their equivalence to SBE-filters and exploring their fundamental properties on transitive SBE-algebras. Moreover, the concept of positive implicative SBE-filters is introduced and its equivalence with both SBE-filters and implicative SBE-filters is established on transitive SBE-algebras.

Keywords: Sheffer stroke BE-algebras (SBE-algebras), SBE-filters, implicative SBE-filters.

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1. INTRODUCTION

H. M. Sheffer introduced the concept of the Sheffer operation, also referred to as the Sheffer stroke, in 1913 [4]. He illustrated in that all Boolean functions could be derived from a single binary operation as term operations [18]. Furthermore, it has been demonstrated that this principle extends not only to ortholattices but also to lattices with antitone involution.

The Sheffer stroke, also referred to as the NAND operator, signifies the negation of a conjunction and stands out as one of the two operators capable of independently forming a logical formal system, without requiring additional logical operators. There is considerable interest in uncovering straightforward axiom systems for diverse algebras and logics, with simplicity measured by the minimal number of axioms within a given system. In line with this pursuit, Oner and Senturk introduced Sheffer stroke basic algebras as an innovative approach to consolidating basic algebras into a unified single operation [11]. These pioneering advancements lay the groundwork for refined reasoning mechanisms and establish a fundamental algebraic framework applicable across various domains, offering avenues for reductions in a wide array of algebraic structures such as BCK-algebras, MV-algebras, Wajsberg algebras, MTL-algebras and etc [15, 16]. The P-Sheffer stroke defined on a partially ordered set serves as a foundation for generating various fuzzy logic operations and implications, offering practical applications in fuzzy reasoning and real-life decision-making problems [10].

The concept of BE-algebra was originally introduced by H.S. Kim and Y.H. Kim as a generalization of a dual BCK-algebra. They provided definitions for a filter and an upper set on this algebraic structure [8]. Ahn and So subsequently defined ideals on BE-algebras and offered various characterizations of these ideals [2, 3]. Additionally, Ahn and So extended the concept of upper sets in BE-algebras and examined certain properties of the characterizations of generalized upper sets concerning the structure of ideals in transitive and self-distributive BE-algebras. More recently, in 2012, Ahn, Kim, and Ko introduced the concept of a terminal section of BE-algebras. They provided several characterizations of commutative BE-algebras in relation to lattice-ordered relations and terminal sections [1]. Rezaei and Saeid introduced a definition of regular congruence relation to construct quotient BE-algebras from self-distributive BE-algebras and also introduced commutative ideals in BE-algebras with several properties [13, 14]. Furthermore, Rao examined prime filters of commutative BE-algebras [12], and Ciungu examined commutative pseudo BE-algebras [5]. N. Chunseei P. Julatha and A. lampan study fuzzy set approaches to ideal on Sheffer stroke BE-algebras

[7]. In alignment with these studies, investigations into congruences and filters extend to Sheffer stroke reducts algebras, as discussed in [17].

In this paper, we delve into the properties of transitive SBE-algebras and provide characterizations of SBE-filters, highlighting their essential properties and noteworthy findings. Additionally, we introduce implicative SBE-filters and explore their fundamental properties and associated results. By establishing connections among these filters, we construct bridges that contribute to theoretical advancements. The empirical contributions of this paper include introducing the concept of (positive) implicative SBE-filters, which will benefit researchers across various scientific domains requiring the concept of filters. Moreover, we present a theoretical framework for (positive) implicative filters, facilitating their adaptation to various algebraic structures. In Section 2, we offer a thorough review encompassing essential concepts, fundamental definitions, lemmas, and associated results concerning Sheffer stroke BE-algebras. Additionally, we revisit the concept of filters within the context of this study. In Section 3, we introduce the concept of transitive SBE-algebra. We explore various properties of this algebraic structure and provide a characterization of SBE-filters. We present several equivalent conditions for these filters and define the congruence concept, demonstrating that $[1_\theta]$ is an SBE-filter. Additionally, we investigate fundamental relationships between filters and congruences. Moreover, we illustrate that SBE-filters coincide with a congruence kernel. In Section 4, we present the concept of implicative SBE-filters. We demonstrate that each implicative SBE-filter is indeed an SBE-filter. Additionally, we examine various properties of implicative SBE-filters on transitive SBE-algebras. We illustrate that SBE-filters coincide with implicative SBE-filters on transitive SBE-algebras. Furthermore, we investigate fundamental relationships between SBE-filters and implicative SBE-filters. In Section 5, we introduce the concept of positive implicative SBE-filter. We demonstrate that a positive implicative SBE-filter is indeed an SBE-filter. Additionally, we provide a condition for proving the other direction of the previous assumption. Finally, we establish that SBE-filter, implicative SBE-filter, and positive implicative SBE-filter are equivalent to each other on transitive SBE-algebras.

2. PRELIMINARIES

We recall some basic definitions and results that are necessary in the sequel.

Definition 1 ([9]). Let $S = \langle S, \circ \rangle$ be a groupoid. The operation $\circ$ on S is said to be a Sheffer operation (or Sheffer stroke) if it satisfies the

following conditions:

- (S1) $p \circ q = q \circ p$,
 - (S2) $(p \circ p) \circ (p \circ q) = p$,
 - (S3) $p \circ ((q \circ r) \circ (q \circ r)) = ((p \circ q) \circ (p \circ q)) \circ r$,
 - (S4) $(p \circ ((p \circ p) \circ (q \circ q))) \circ (p \circ ((p \circ p) \circ (q \circ q))) = p$.
- for all $p, q, r \in S$.

Definition 2 ([9]). A Sheffer stroke BE-algebra (shortly, SBE-algebra) is a structure $(S; \circ, 1)$ of type $(2, 0)$ such that 1 is the constant in S , $\circ$ is a Sheffer operation on S and the following axioms are satisfied for all $p, q, r \in S$:

- (SBE-1) $p \circ (p \circ p) = 1$,
- (SBE-2) $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) = q \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))$.

Proposition 3 ([9]). *Every SBE-algebra $(S, \circ, 1)$ satisfies the following items.*

$$(\forall p \in S) (p \circ (1 \circ 1) = 1). \quad (2.1)$$

$$(\forall p \in S) (1 \circ (p \circ p) = p). \quad (2.2)$$

$$(\forall p, q \in S) (p \circ ((q \circ (p \circ p)) \circ (q \circ (p \circ p))) = 1). \quad (2.3)$$

$$(\forall p, q \in S) (p \circ (((p \circ (q \circ q)) \circ (q \circ q)) \circ ((p \circ (q \circ q)) \circ (q \circ q))) = 1). \quad (2.4)$$

$$(\forall p \in S) ((p \circ 1) \circ (p \circ 1) = p). \quad (2.5)$$

$$(\forall p, q \in S) (((p \circ q) \circ (p \circ q)) \circ (p \circ p) = 1 \text{ and } ((p \circ q) \circ (p \circ q)) \circ (q \circ q) = 1). \quad (2.6)$$

$$(\forall p, q \in S) (p \circ ((p \circ q) \circ (p \circ q))) = p \circ q = ((p \circ q) \circ (p \circ q)) \circ q. \quad (2.7)$$

Definition 4 ([9]). An SBE-algebra $(S, \circ, 1)$ is said to be

- *commutative* if it satisfies:

$$(\forall p, q, r \in S) ((p \circ (q \circ q)) \circ (q \circ q) = (q \circ (p \circ p)) \circ (p \circ p)). \quad (2.8)$$

- *self-distributive* if it satisfies:

$$(\forall p, q, r \in S) (p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) = (p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))). \quad (2.9)$$

In an SBE-algebra $(S, \circ, 1)$, a binary relation “ $\preceq$ ” is defined by

$$(\forall p, q \in S) (p \preceq q \Leftrightarrow p \circ (q \circ q) = 1). \quad (2.10)$$

The relation is not a partial order on S since it is only reflexive by (SBE-1).

Proposition 5 ([9]). *Every SBE-algebra $(S, \circ, 1)$ satisfies the following items.*

$$(\forall p, q \in S) (\text{ If } p \preceq q, \text{ then } q \circ q \preceq p \circ p). \quad (2.11)$$

$$(\forall p, q \in S) (p \preceq (q \circ (p \circ p))). \quad (2.12)$$

$$(\forall p \in S) (1 \preceq p \Rightarrow p = 1). \quad (2.13)$$

$$(\forall p, q \in S) (p \preceq (q \circ (p \circ p)) \circ (p \circ p)). \quad (2.14)$$

$$(\forall p, q \in S) (p \preceq (p \circ (q \circ q)) \circ (q \circ q)). \quad (2.15)$$

$$(\forall p, q \in S) (\text{ If } S \text{ is self-distributive, then } p \preceq q \text{ implies } q \circ r \preceq p \circ r). \quad (2.16)$$

$$(\forall p, q \in S) (\text{ If } S \text{ is self-distributive, then } q \circ (r \circ r) \preceq (r \circ (p \circ p)) \circ ((q \circ (p \circ p)) \circ (q \circ (p \circ p)))). \quad (2.17)$$

Definition 6 ([9]). A non-empty subset F of an SBE-algebra $(S, \circ, 1)$ is called an *SBE-filter* of $(S, \circ, 1)$ if it satisfies:

$$1 \in F, \quad (2.18)$$

$$(\forall p, q \in S) (p \in F, p \circ (q \circ q) \in F \Rightarrow q \in F). \quad (2.19)$$

Lemma 7 ([9]). *Let S be an SBE-algebra. Then a non-empty subset $F \subseteq S$ is an SBE-filter of S if and only if for all $p, q \in S$*
(i) $p \in F$ and $q \in F$ imply $(p \circ q) \circ (p \circ q) \in F$
(ii) $p \in F$ and $p \preceq q$ imply $q \in F$.

3. ON SBE-FILTERS

In this section, we study some properties of transitive SBE-algebra and we characterize SBE-filters. We prove that SBE-filters and congruence kernels coincide in a transitive SBE-algebra.

Theorem 8. *Let $\circ$ be a Sheffer operation on a non-empty set S with constant 1 satisfying (SBE-1). Then the following are equivalent.*

$$(SBE-2) \ p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) = q \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))).$$

$$(SBE-3) \ p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) = p \circ ((q \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ (q \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))))).$$

Proof. (SBE-2) $\Rightarrow$ (SBE-3): Assume (SBE-2). Let $p, q, r \in S$ and $A = p \circ (r \circ r)$. Then

$$\begin{aligned}
 & p \circ [\{q \circ (A \circ A)\} \circ \{q \circ (A \circ A)\}] \\
 \stackrel{(S3)}{=} & [(p \circ q) \circ (p \circ q)] \circ (A \circ A) \\
 \stackrel{(S1)}{=} & [(q \circ p) \circ (q \circ p)] \circ (A \circ A) \\
 \stackrel{(S3)}{=} & q \circ [\{p \circ (A \circ A)\} \circ \{p \circ (A \circ A)\}] \\
 \stackrel{(S3)}{=} & q \circ [\{(p \circ p) \circ (p \circ p)\} \circ \{(p \circ p) \circ (p \circ p)\}] \circ (r \circ r) \\
 \stackrel{(S2)}{=} & q \circ [(p \circ (r \circ r)) \circ (p \circ (r \circ r))] \\
 \stackrel{(SBE-2)}{=} & p \circ [(q \circ (r \circ r)) \circ (q \circ (r \circ r))]
 \end{aligned}$$

(SBE-3) $\Rightarrow$ (SBE-2): Assume (SBE-3). Let $p, q, r \in S$ and $A = p \circ (r \circ r)$. Then

$$\begin{aligned}
 & p \circ [(q \circ (r \circ r)) \circ (q \circ (r \circ r))] \\
 \stackrel{(SBE-3)}{=} & p \circ [\{q \circ (A \circ A)\} \circ \{q \circ (A \circ A)\}] \\
 \stackrel{(S3)}{=} & p \circ [(((q \circ p) \circ (q \circ p)) \circ (r \circ r)) \circ (((q \circ p) \circ (q \circ p)) \circ (r \circ r))] \\
 \stackrel{(S3)}{=} & [(p \circ \{(q \circ p) \circ (q \circ p)\}) \circ (p \circ \{(q \circ p) \circ (q \circ p)\})] \circ (r \circ r) \\
 \stackrel{(S3)}{=} & [\{(p \circ q) \circ (p \circ q)\} \circ p] \circ \{[(p \circ q) \circ (p \circ q)] \circ p\} \circ (r \circ r) \\
 \stackrel{(S1)}{=} & [\{(q \circ p) \circ (q \circ p)\} \circ p] \circ \{[(q \circ p) \circ (q \circ p)] \circ p\} \circ (r \circ r) \\
 \stackrel{(S3)}{=} & [q \circ [(p \circ p) \circ (p \circ p)]] \circ \{q \circ ((p \circ p) \circ (p \circ p))\} \circ (r \circ r) \\
 \stackrel{(S2)}{=} & [(q \circ p) \circ (q \circ p)] \circ (r \circ r) \\
 \stackrel{(S3)}{=} & q \circ [(p \circ (r \circ r)) \circ (p \circ (r \circ r))]
 \end{aligned}$$

□

Theorem 9. Let S be an SBE-algebra. Let F be a non-empty subset of S satisfying the following conditions:

$$(\forall p, q \in S)(p \in S \text{ and } q \in F \text{ imply } p \circ (q \circ q) \in F), \quad (3.1)$$

$$(\forall p, a, b \in S)(p \in S, a, b \in F \text{ imply } (a \circ ((b \circ (p \circ p)) \circ (b \circ (p \circ p)))) \circ (p \circ p) \in F). \quad (3.2)$$

Then F is an SBE-filter of S .

Proof. Let F be a non-empty subset of S satisfying (3.1) and (3.2). Then $1 \in F$. Hence (2.18) holds. Let $p \in F$ and $p \circ (q \circ q) \in F$. Then

$$q \stackrel{\text{Proposition 3(3)}}{=} 1 \circ (q \circ q) \stackrel{(SBE-1)}{=} [(p \circ (q \circ q)) \circ \{(p \circ (q \circ q)) \circ (p \circ (q \circ q))\}] \circ (q \circ q) \in F$$

and hence (2.19) holds. Therefore F is an SBE-filter of S . $\square$

Theorem 10. *If S is an SBE-algebra and F is an SBE-filter of S , then F satisfies (3.1) and (3.2).*

Proof. Let F be an SBE-filter of S . Let $a \in F$ and $p \in S$. Then, by Proposition 3(3),

$$a \circ ((p \circ (a \circ a)) \circ (p \circ (a \circ a))) = 1.$$

Since $1 \in F$ and $a \in F$, it follows from the filter condition (2.19) that

$$p \circ (a \circ a) \in F.$$

Hence, F satisfies (3.1). Let $a, b \in F$ and $p \in S$. Set

$$A = (b \circ (p \circ p)) \circ (b \circ (p \circ p)).$$

Then, by Proposition 5(2.15), we have

$$a \circ (((a \circ A) \circ A) \circ ((a \circ A) \circ A)) = 1.$$

Since $1 \in F$ and $a \in F$, applying the filter condition (2.19), we obtain

$$(a \circ A) \circ A \in F.$$

Now using (SBE-3), we get

$$b \circ (((a \circ A) \circ (p \circ p)) \circ ((a \circ A) \circ (p \circ p))) = b \circ (((a \circ A) \circ A) \circ ((a \circ A) \circ A)).$$

By (3.1) and $(a \circ A) \circ A \in F$, we have $b \circ (((a \circ A) \circ A) \circ ((a \circ A) \circ A)) \in F$. Hence

$$b \circ (((a \circ A) \circ (p \circ p)) \circ ((a \circ A) \circ (p \circ p))) \in F.$$

Since $b \in F$, again applying (2.19), we conclude that

$$(a \circ A) \circ (p \circ p) \in F.$$

Substituting back A , we obtain

$$(a \circ ((b \circ (p \circ p)) \circ (b \circ (p \circ p)))) \circ (p \circ p) \in F.$$

Hence, F satisfies (3.2). Therefore, F satisfies both (3.1) and (3.2). $\square$

Theorem 11. *Let F be a non-empty subset of S . Then F is an SBE-filter of S if and only if for every $a, b \in F$ and $p \in S$, $a \circ ((b \circ (p \circ p)) \circ (b \circ (p \circ p))) = 1$ implies $p \in F$.*

Proof. Suppose F is an SBE-filter of S and $a, b \in F, p \in S$ such that $a \circ ((b \circ (p \circ p)) \circ (b \circ (p \circ p))) = 1$. By (2.18), we have $a \circ ((b \circ (p \circ p)) \circ (b \circ (p \circ p))) \in F$. Then, by (2.19), we obtain $p \in F$. Conversely, assume that the condition holds. Let $a \in F$. Then $a \circ ((a \circ (1 \circ 1)) \circ (a \circ (1 \circ 1))) = 1$ implies $1 \in F$. Suppose $p \circ (q \circ q) \in F$ and $p \in F$. Then $p \circ [(p \circ (q \circ q)) \circ (q \circ q)] \circ ((p \circ (q \circ q)) \circ (q \circ q)) = 1$ implies $q \in F$. Hence F is an SBE-filter of S . $\square$

Lemma 12. *Let F be an SBE-filter of S . Then $(a \circ (p \circ p)) \circ (p \circ p) \in F$ for all $a \in F$ and $p \in S$.*

Proof. Since F is an SBE-filter of S , $1 \in F$ and F satisfies (3.2) by Theorem 10. If $b = 1$ in (3.2) then, by Proposition 3(2), $(a \circ (p \circ p)) \circ (p \circ p) \in F$ for all $a \in F$ and $p \in S$. $\square$

Proposition 13. *A non-empty subset F of an SBE-algebra S is an SBE-filter of S if and only if it satisfies (i) $1 \in F$ (ii) $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F, q \in F$ implies $p \circ (r \circ r) \in F$ for all $p, q, r \in S$.*

Proof. Suppose F is an SBE-filter of S . Then $1 \in F$. Let $p, q \in S$ be such that $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F$ and $q \in F$. Then, by (SBE-2), $q \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) = p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F$. Since $q \in F$ and F is an SBE-filter of S , we have $(p \circ (r \circ r)) \circ (p \circ (r \circ r)) \in F$. Now, $1 \in F$ and $1 \circ (p \circ (r \circ r)) = 1 \circ (((p \circ (r \circ r)) \circ (p \circ (r \circ r))) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) = (p \circ (r \circ r)) \circ (p \circ (r \circ r)) \in F$ implies that $p \circ (r \circ r) \in F$. Conversely, assume that the conditions hold. Let $p, q \in S$ be such that $p \circ (q \circ q) \in F$ and $p \in F$. Then $1 \circ ((p \circ (q \circ q)) \circ (p \circ (q \circ q))) = p \circ (q \circ q) \in F$ and $p \in F$ and hence, by (ii), $q = 1 \circ (q \circ q) \in F$. $\square$

Theorem 14. *Let $(S, \circ, 1)$ be an SBE-algebra. Then, for all $p, q, r \in S$, the following are equivalent.*

- (1) $p \circ (q \circ q) \preceq (r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))$,
- (2) $p \circ (q \circ q) \preceq (q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))$.

Proof. Let $p, q, r \in S$.

(1) $\Rightarrow$ (2): Assume (1). Then, by (SBE-2) and (1), $p \circ (q \circ q) \circ (((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ ((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) = q \circ (r \circ r) \circ (((p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ ((p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) = 1$.

(2) $\Rightarrow$ (1): Assume (2). Then, by (SBE-2) and (2), $p \circ (q \circ q) \circ (((r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))) \circ ((r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))))) = r \circ (p \circ p) \circ (((p \circ (q \circ q)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))) \circ ((p \circ (q \circ q)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))))) = 1$. $\square$

Definition 15. an SBE-algebra $(S, \circ, 1)$ is said to be transitive if it satisfies

$$p \circ (q \circ q) \preceq (q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))$$

for all $p, q, r \in S$.

Example 1. Let $S = \{0, 1, 2, 3, 4, 5\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5
0	3	3	3	1	1	1
1	3	5	4	0	2	1
2	3	4	4	1	1	1
3	1	0	1	0	2	1
4	1	2	1	2	2	1
5	1	1	1	1	1	1

Then $(S, \circ, 1)$ is a transitive SBE-algebra.

Theorem 16. In a transitive SBE-algebra $(S, \circ, 1)$, for all $p, q, r \in S$, the following conditions hold:

- (1) $p \preceq q$ implies $r \circ (p \circ p) \preceq r \circ (q \circ q)$.
- (2) $p \circ (q \circ q) \preceq (q \circ (r \circ r)) \circ (p \circ (r \circ r))$.
- (3) $p \preceq q$ implies $q \circ (r \circ r) \preceq p \circ (r \circ r)$.
- (4) $((p \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) \preceq p \circ (r \circ r)$.
- (5) $p \preceq q$ and $q \preceq r$ imply $p \preceq r$.

Proof. Let $(S, \circ, 1)$ be a transitive SBE-algebra and $p, q, r \in S$.

- (1). Suppose $p \preceq q$. Then $p \circ (q \circ q) = 1$. Hence, by (2.1) and Theorem 14, $(r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q))) = 1 \circ (((r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))) \circ ((r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))))) = (p \circ (q \circ q)) \circ (((r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))) \circ ((r \circ (p \circ p)) \circ ((r \circ (q \circ q)) \circ (r \circ (q \circ q)))))) = 1$.
- (2). $(p \circ (q \circ q)) \circ (((q \circ (r \circ r)) \circ (p \circ (r \circ r))) \circ ((q \circ (r \circ r)) \circ (p \circ (r \circ r)))) = 1$ by Theorem 14.
- (3). Suppose $p \preceq q$. Then $p \circ (q \circ q) = 1$. Hence, by (2.1) and Theorem 14 $(q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) = 1 \circ (((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ ((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) = (p \circ (q \circ q)) \circ (((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ ((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) = 1$.
- (4). $p \circ (((p \circ (q \circ q)) \circ (q \circ q)) \circ ((p \circ (q \circ q)) \circ (q \circ q))) = (p \circ (q \circ q)) \circ ((p \circ (q \circ q)) \circ (p \circ (q \circ q))) = 1$. Hence $p \preceq ((p \circ (q \circ q)) \circ (q \circ q))$. Therefore, by (3), $((p \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) \preceq p \circ (r \circ r)$.
- (5). Suppose $p \preceq q$ and $q \preceq r$. Then $p \circ (q \circ q) = 1$ and $q \circ (r \circ r) = 1$. Hence, by (2.1) and Theorem 14, $p \circ (r \circ r) = 1 \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) = (q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) = 1 \circ (((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ ((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) = (p \circ (q \circ q)) \circ (((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ ((q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) = 1$. Hence $p \preceq r$. $\square$

Let θ be a binary relation on an SBE-algebra $(S, \circ, 1)$. We denote $\{p \in S \mid (p, 1) \in \theta\}$ by $[1]_\theta$. If θ is a congruence relation on S then $[1]_\theta$ is called a congruence kernel.

Lemma 17. *If θ is a congruence relation on S , then kernel $[1]_\theta$ is an SBE-filter of S .*

Proof. Clearly $1 \in [1]_\theta$. Suppose $p \in [1]_\theta$ and $p \circ (q \circ q) \in [1]_\theta$. Then $(p, 1) \in \theta$, $(p \circ (q \circ q), 1) \in \theta$ and hence $(p \circ (q \circ q), q) = (p \circ (q \circ q), 1 \circ (q \circ q)) \in \theta$. By symmetry of θ , $(q, p \circ (q \circ q)) \in \theta$. Therefore, by transitivity of θ , we obtain $(q, 1) \in \theta$ proving $q \in [1]_\theta$. $\square$

The converse is not true in general. We demonstrate this with the following counterexample.

Example 2. *Consider the SBE-algebra S in Example 1. Define the equivalence relation:*

$$\varrho' = \{(p, p) \mid p \in S\} \cup \{(0, 1), (1, 0), (0, 2), (2, 0), (1, 2), (2, 1)\}$$

with equivalence classes $[0] = [1] = [2] = \{0, 1, 2\}$ and $[3] = \{3\}$, $[4] = \{4\}$, $[5] = \{5\}$. $[1]_{\varrho'} = \{0, 1, 2\}$ is an SBE-filter of S . But ϱ' is not congruence relation on S since the substitution property fails. Take $(0, 1) \in \varrho'$ and $(0, 1) \in \varrho'$. Then,

$$0 \circ 0 = 3 \quad \text{and} \quad 1 \circ 1 = 5.$$

But $(3, 5) \notin \varrho'$. Therefore, $[1]_{\varrho'}$ is an SBE-filter of S but ϱ' is not a congruence relation on S

Example 3. *Consider the SBE-algebra S in Example 1. We can observe that the equivalence relation*

$$\theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (0, 1), (1, 0), (0, 2), (2, 0), (1, 2), (2, 1), \\ (3, 4), (4, 3), (3, 5), (5, 3), (4, 5), (5, 4)\}$$

is a congruence relation on S and the kernel $[1]_\theta = \{0, 1, 2\}$ is an SBE-filter of S .

Theorem 18. *Let $(S, \circ, 1)$ be a transitive SBE-algebra. Then every SBE-filter F of S is a kernel of a congruence θ_F given by*

$$(p, q) \in \theta_F \text{ if and only if } p \circ (q \circ q) \in F \text{ and } q \circ p \in F.$$

Moreover, θ_F is the greatest congruence on S having the kernel F .

Proof. Let F be an SBE-filter of S . Since $1 \in F$, we have θ_F is reflexive. Clearly θ_F is symmetric. We prove transitivity of θ_F . Let $(p, q) \in \theta_F$ and $(q, r) \in \theta_F$. Then $p \circ (q \circ q), q \circ (p \circ p), q \circ (r \circ r), r \circ (q \circ q) \in F$. Since S is transitive, we have $(q \circ (r \circ r)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) \in F$. Therefore $p \circ (r \circ r) \in F$. Similarly, we can prove that $r \circ (p \circ p) \in F$.

Thus $(p, r) \in \theta_F$. Now, we prove the substitution property of θ_F . Let $(p, q) \in \theta_F$ and $(u, v) \in \theta_F$. Then $p \circ (q \circ q), q \circ (p \circ p), u \circ (v \circ v), v \circ (u \circ u) \in F$ and hence $(v \circ v) \circ ((u \circ u) \circ (u \circ u)) = (v \circ v) \circ u \in F$. Since S is transitive and F is SBE-filter of S , we have $(p \circ ((v \circ v) \circ (v \circ v))) \circ ((p \circ ((u \circ u) \circ (u \circ u))) \circ (p \circ ((u \circ u) \circ (u \circ u)))) = (p \circ v) \circ ((p \circ u) \circ (p \circ u)) \in F$. Similarly, we can prove that $(p \circ u) \circ ((p \circ v) \circ (p \circ v)) \in F$. Hence $(p \circ u, p \circ v) \in \theta_F$. Also $(q \circ q) \circ ((p \circ p) \circ (p \circ p)) = (q \circ q) \circ p \in F$. Since S is transitive and F is SBE-filter of S , we have $(v \circ ((q \circ q) \circ (q \circ q))) \circ ((v \circ ((p \circ p) \circ (p \circ p))) \circ (v \circ ((p \circ p) \circ (p \circ p)))) = (v \circ q) \circ ((v \circ p) \circ (v \circ p)) \in F$. Similarly, we can prove that $(v \circ p) \circ ((v \circ q) \circ (v \circ q)) \in F$. Hence $(q \circ v, p \circ v) \in \theta_F$. By transitivity of θ_F , we conclude $(p \circ u, q \circ v) \in \theta_F$. Thus θ_F is a congruence relation on S .

If $p \in F$ then $1 \circ (p \circ p) = p \in F$ and $p \circ (1 \circ 1) = 1 \in F$. Therefore $(p, 1) \in \theta_F$, i.e., $p \in [1]_{\theta_F}$. Conversely, let $p \in [1]_{\theta_F}$. Then $(p, 1) \in \theta_F$ and hence $p = 1 \circ (p \circ p) \in F$ which shows that $F = [1]_{\theta_F}$. Thus F is the kernel of the congruence θ_F . Finally, if ψ is a congruence relation on S such that $[1]_{\psi} = F$ then, for $(p, q) \in \psi$, we have $(p \circ (q \circ q), 1) = (p \circ (q \circ q), q \circ (q \circ q)) \in \psi$ and $(q \circ (p \circ p), 1) = (q \circ (p \circ p), q \circ (q \circ q)) \in \psi$. Thus $p \circ (q \circ q) \in F$ and $q \circ (p \circ p) \in F$ which gives $(p, q) \in \theta_F$. Hence $\psi \subseteq \theta_F$ i.e., θ_F is the greatest congruence relation of S having the kernel F . $\square$

The following example shows that SBE-filters need not be congruence kernels:

Example 4. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5	6	7	8	9	10	11
0	5	5	8	1	8	1	1	1	1	8	5	1
1	5	6	3	2	7	0	1	4	9	8	11	10
2	8	3	3	1	3	10	1	1	10	8	8	10
3	1	2	1	2	1	1	1	4	2	1	2	1
4	8	7	3	1	7	10	1	1	10	8	8	10
5	1	0	10	1	10	0	1	1	0	1	1	10
6	1	1	1	1	1	1	1	1	1	1	1	1
7	1	4	1	4	1	1	1	4	4	1	4	1
8	1	9	10	2	10	0	1	4	9	1	2	10
9	8	8	8	1	8	1	1	1	1	8	8	1
10	5	11	8	2	8	1	1	4	2	8	11	1
11	1	10	10	1	10	10	1	1	10	1	1	10

Then $(S, \circ, 1)$ is a SBE-algebra. But it is not a transitive since $(0 \circ (2 \circ 2)) \circ \{(3 \circ (0 \circ 0)) \circ ((3 \circ (2 \circ 2)) \circ (3 \circ (2 \circ 2)))\} \circ [(3 \circ (0 \circ 0)) \circ ((3 \circ (2 \circ 2)) \circ (3 \circ (2 \circ 2)))] \circ \{(1 \circ (2 \circ 2)) \circ (1 \circ (2 \circ 2))\} = 1 \circ (2 \circ 2) = 1 \circ 3 = 2 \neq 1$.

Clearly, $F = \{1, 2, 4, 5, 8, 11\}$ is an SBE-filter of S . Let $(5, 4) \in \theta$ for some congruence relation θ on S . As θ is reflexive, we have $(0, 0) \in \theta$. Using compatibility of congruence, $(1, 8) = (0 \circ 5, 0 \circ 4) \in \theta$ and hence $8 \in [1]_\theta \neq \{1, 2, 4, 5, 8, 11\} = F$.

We conclude this section with the following

Theorem 19. *Let $(S, \circ, 1)$ be a transitive SBE-algebra. Then SBE-filters of S coincide with congruence kernels.*

4. IMPLICATIVE SBE-FILTERS

In this section, we define the notion of an implicative SBE-filter of an SBE-algebra and study its properties. We establish the relationship between SBE-filter and implicative SBE-filter. We provide conditions for the trivial SBE-filter to be an implicative SBE-filter. We show that every SBE-filter is an implicative SBE-filter in a transitive SBE-algebra.

Definition 20. A subset F of an SBE-algebra S is called an *implicative SBE-filter* of S if it satisfies (2.18) and for all $p, q, r \in S$, if

$$p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F, \quad p \circ (q \circ q) \in F \quad \text{then} \quad p \circ (r \circ r) \in F. \quad (4.1)$$

Example 5. Let $S = \{0, 1, 2, 3, 4, 5\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5
0	3	3	3	1	1	1
1	3	4	5	0	1	2
2	3	5	5	1	1	1
3	1	0	1	0	1	2
4	1	1	1	1	1	1
5	1	2	1	2	1	2

Then $(S, \circ, 1)$ is a SBE-algebra. Then $F = \{0, 1, 2\}$ is an implicative SBE-filter of S .

We establish the relationship between implicative SBE-filters and SBE-filters.

Theorem 21. *In an SBE-algebra, every implicative SBE-filter is an SBE-filter.*

Proof. Let F be an implicative SBE-filter of an SBE-algebra S . Let $p, q \in S$ be such that $p \circ (q \circ q) \in F$ and $p \in F$. If we substitute p, q and r with $1, p$ and q respectively in (4.1) and use (2.2), then $1 \circ ((p \circ (q \circ q)) \circ (p \circ (q \circ q))) = p \circ (q \circ q) \in F$ and $1 \circ (p \circ p) = p \in F$. It follows from (4.1) that $q = 1 \circ (q \circ q) \in F$. Hence F is an SBE-filter of S . $\square$

The following example shows that the converse of Theorem 21 is not true in general.

Example 6. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	3	3	1	1	3	3	1	1	1	1	3	1	1	3
1	3	6	7	0	8	9	1	2	4	5	11	10	13	12
2	1	7	7	1	7	7	1	1	1	1	7	1	1	7
3	1	0	1	0	10	10	1	13	4	5	4	10	13	1
4	3	8	7	10	8	8	1	10	1	1	12	10	10	12
5	3	9	7	10	8	9	1	10	1	1	12	10	10	12
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	2	1	13	10	10	1	2	4	5	4	10	13	1
8	1	4	1	4	1	1	1	4	4	5	4	1	4	1
9	1	5	1	5	1	1	1	5	5	5	5	1	5	1
10	3	11	7	4	12	12	1	4	4	5	11	1	4	12
11	1	10	1	10	10	10	1	10	1	1	1	10	10	1
12	1	13	1	13	10	10	1	13	4	5	4	10	13	1
13	3	12	7	1	12	12	1	1	1	1	12	1	1	12

Then $(S, \circ, 1)$ is a SBE-algebra. Then $F = \{1, 4, 5\}$ is an SBE-filter of S . But F is not an implicative SBE-filter of S since $0 \circ [(2 \circ (3 \circ 3)) \circ (2 \circ (3 \circ 3))] = 0 \circ [(2 \circ 0) \circ (2 \circ 0)] = 0 \circ (1 \circ 1) = 0 \circ 6 = 1 \in F$ and $0 \circ (2 \circ 2) = 0 \circ 7 = 1 \in F$ but $0 \circ (3 \circ 3) = 0 \circ 0 = 3 \notin F$.

In an SBE-algebra S , consider the following condition:

$$(\forall p, q, r \in S) (p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F \Rightarrow (p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) \in F). \quad (4.2)$$

The following example shows that an SBE-filter of S may not satisfy the condition (4.2).

Example 7. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	5	5	8	10	5	1	1	1	10	3	12	10	10	12
1	5	6	8	9	7	0	1	4	2	3	11	10	13	12
2	8	8	8	1	8	1	1	1	1	1	8	1	1	8
3	10	9	1	9	10	0	1	4	13	1	0	10	13	1
4	5	7	8	10	7	1	1	1	10	3	12	10	10	12
5	1	0	1	0	1	0	1	4	0	1	0	1	0	1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	4	1	4	1	4	1	4	4	1	4	1	4	1
8	10	2	1	13	10	0	1	4	2	1	0	10	13	1
9	3	3	1	1	3	1	1	1	1	3	3	1	1	3
10	12	11	8	0	12	0	1	4	0	3	11	1	0	12
11	10	10	1	10	10	1	1	1	10	1	1	10	10	1
12	10	13	1	13	10	0	1	4	13	1	0	10	13	1
13	12	12	8	1	12	1	1	1	1	3	12	1	1	12

Then $(S, \circ, 1)$ is a SBE-algebra. Then $F = \{0, 1, 4\}$ is an SBE-filter of S . But F does not satisfy the condition (4.2) since

$$0 \circ ((2 \circ (3 \circ 3)) \circ (2 \circ (3 \circ 3))) = 0 \circ ((2 \circ 9) \circ (2 \circ 9)) = 0 \circ (1 \circ 1) = 0 \circ 6 = 1 \in F$$

$$\text{but } (0 \circ (2 \circ 2)) \circ ((0 \circ (3 \circ 3)) \circ (0 \circ (3 \circ 3))) = (0 \circ 8) \circ ((0 \circ 9) \circ (0 \circ 9)) = 10 \circ (3 \circ 3) = 10 \circ 9 = 3 \notin F.$$

We explore the conditions for an SBE-filter to be an implicative SBE-filter.

Theorem 22. If an SBE-filter F of an SBE-algebra S satisfies the condition (4.2), then F is an implicative SBE-filter of S .

Proof. Let F be an SBE-filter of an SBE-algebra S which satisfies the condition (4.2). Let $p, q, r \in S$ be such that $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F$ and $p \circ (q \circ q) \in F$. By the condition (4.2), we have $(p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) \in F$ and $p \circ (q \circ q) \in F$. Since F is an SBE-filter of S , it follows from (2.19) that $(p \circ (r \circ r)) \circ (p \circ (r \circ r)) \in F$. Hence $p \circ (r \circ r) \in F$. Therefore F is an implicative SBE-filter of S . $\square$

Lemma 23. Let $(S, \circ, 1)$ be a transitive SBE-algebra. Then, for any $p, q, r \in S$,

$$p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \preceq (p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))).$$

Proof. Let $p, q, r \in S$. Then, by (SBE-2), Theorem 14 and (2.7), $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) = q \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r))) \preceq (p \circ (q \circ q)) \circ ((p \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))) \circ (p \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))))) =$

$(p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))$. Hence $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \preceq (p \circ (q \circ q)) \circ ((p \circ (r \circ r)) \circ (p \circ (r \circ r)))$. $\square$

Theorem 24. *Let $(S, \circ, 1)$ be a transitive SBE-algebra. Then every SBE-filter of S is an implicative SBE-filter of S .*

Proof. Let F be an SBE-filter of a transitive SBE-algebra S and $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F, p \circ (q \circ q) \in F$. Then, by Lemma 23 and Lemma 7, we get $p \circ (r \circ r) \in F$. $\square$

Given a point α and a non-empty subset F of an SBE-algebra S , we consider a special set:

$$F_\alpha := \{p \in S \mid \alpha \circ (p \circ p) \in F\}. \quad (4.3)$$

If F is an SBE-filter of an SBE-algebra S , then $1, \alpha \in F_\alpha$ for all $\alpha \in S$.

We have the following questions.

Question 1. *If F is an SBE-filter of an SBE-algebra S , then is the set F_α in (4.3) an SBE-filter of S ?*

The following example gives a negative answer to the Question 1.

Example 8. *Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ be the SBE-algebra in Example 7. Take SBE-filter $F = \{0, 1, 4\}$. Then $F_{10} = \{0, 1, 2, 4, 9, 10, 13\}$ and it is not an SBE-filter of S since $9 \circ (8 \circ 8) = 9 \circ 2 = 1 \in F_{10}$ and $9 \in F_{10}$ but $8 \notin F_{10}$.*

We suggest conditions that will lead to a positive answer to the Question 1.

Theorem 25. *If F is an implicative SBE-filter of an SBE-algebra S , then the set F_α in (4.3) is an SBE-filter of S .*

Proof. Assume that F is an implicative SBE-filter of an SBE-algebra S . Let $p, q \in S$ be such that $p \circ (q \circ q) \in F_\alpha$ and $p \in F_\alpha$. Then $\alpha \circ ((p \circ (q \circ q)) \circ (p \circ (q \circ q))) \in F$ and $\alpha \circ (p \circ p) \in F$. It follows from (4.1) that $\alpha \circ (q \circ q) \in F$, that is, $q \in F_\alpha$. Hence F_α is an SBE-filter of S . $\square$

We suggest the conditions under which an SBE-filter can be an implicative SBE-filter.

Theorem 26. *For every subset F of an SBE-algebra S , if $1 \in F$ and the set F_α in (4.3) is an SBE-filter of S for every $\alpha \in S$, then F is an implicative SBE-filter of S .*

Proof. Suppose that $1 \in F$ and the set F_α in (4.3) is an SBE-filter of S for every $\alpha \in S$. Let $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F$ and $p \circ (q \circ q) \in F$. Then $q \circ (r \circ r) \in F_p$ and $q \in F_p$. Since F_p is an SBE-filter of S , we have

$r \in F_p$ and so $p \circ (r \circ r) \in F$. Hence F is an implicative SBE-filter of S . $\square$

Corollary 27. *Given a SBE-filter F of an SBE-algebra S , if the set F_α in (4.3) is an SBE-filter of S for every $\alpha \in S$, then F is an implicative SBE-filter of S .*

Theorem 28. *If F is an implicative SBE-filter of an SBE-algebra S , then the set F_α in (4.3) is the least SBE-filter of S containing F and α .*

Proof. Assume that F is an implicative SBE-filter of an SBE-algebra S and let $\alpha \in S$. Then F_α is an SBE-filter of S (see Theorem 25). It is obvious that F_α contains F and α . Let G be an SBE-filter of S containing F and α . If $p \in F_\alpha$, then $\alpha \circ (p \circ p) \in F \subseteq G$ and so $p \in G$. Hence $F_\alpha \subseteq G$ and F_α is the least SBE-filter of S containing F and α . $\square$

The following example shows that the trivial filter $\{1\}$ of an SBE-algebra S is not an implicative SBE-filter of S .

Example 9. *Let $S = \{0, 1, 2, 3, 4, 5\}$ be a set with binary operation $\circ$ given in the following table:*

$\circ$	0	1	2	3	4	5
0	3	3	1	1	1	1
1	3	4	5	0	1	2
2	1	5	5	1	1	1
3	1	0	1	0	1	1
4	1	1	1	1	1	1
5	1	2	1	1	1	2

Then $(S, \circ, 1)$ is a SBE-algebra. Then $F = \{1\}$ is an SBE-filter of S . But F is not an implicative SBE-filter of S since $0 \circ ((2 \circ (3 \circ 3)) \circ (2 \circ (3 \circ 3))) = 0 \circ ((2 \circ 0) \circ (2 \circ 0)) = 0 \circ (1 \circ 1) = 1 \in F$ and $0 \circ (2 \circ 2) = 0 \circ 5 = 1 \in F$ but $0 \circ (3 \circ 3) = 0 \circ 0 = 3 \notin F$.

Given an element a in an SBE-algebra S , the set $\vec{a} := \{p \in S \mid a \preceq p\}$ is called the *upset* of a .

In general, the set $\vec{a}$ is not an SBE-filter of S as seen in the following example.

Example 10. *From Example 9, we can observe that $\vec{2} := \{0, 1, 2, 3\}$ is not an SBE-filter of S since $3 \in \vec{2}$ and $3 \circ (4 \circ 4) = 3 \circ 1 = 0 \in \vec{2}$ but $4 \notin \vec{2}$.*

We provide conditions for the set $\vec{a}$ to be an SBE-filter.

Theorem 29. *Given an element a in an SBE-algebra S , the following are equivalent.*

- (i) *The set $\vec{a} := \{p \in S \mid a \preceq p\}$ is an SBE-filter of S .*
- (ii) *S satisfies:*

$$(\forall p, q \in S) (a \preceq p \circ (q \circ q), a \preceq p \Rightarrow a \preceq q). \quad (4.4)$$

Proof. Assume that $\vec{a}$ is an SBE-filter of S . Let $p, q \in S$ be such that $a \preceq p \circ (q \circ q)$ and $a \preceq p$. Then $p \circ (q \circ q) \in \vec{a}$ and $p \in \vec{a}$. Since $\vec{a}$ is an SBE-filter of S , it follows that $q \in \vec{a}$, that is, $a \preceq q$. Suppose that S satisfies the condition (4.4). It is clear that $1 \in \vec{a}$. Let $p, q \in S$ be such that $p \circ (q \circ q) \in \vec{a}$ and $p \in \vec{a}$. Then $a \preceq p \circ (q \circ q)$ and $a \preceq p$ which imply from (4.4) that $a \preceq q$. Hence $q \in \vec{a}$, and therefore $\vec{a}$ is an SBE-filter of S . $\square$

Theorem 30. *In an SBE-algebra S , the following are equivalent.*

- (i) *The trivial SBE-filter $\{1\}$ is an implicative SBE-filter.*
- (ii) *For every $a \in S$, the set $\vec{a} := \{p \in S \mid a \preceq p\}$ is an SBE-filter of S .*

Proof. Assume the trivial SBE-filter $\{1\}$ is an implicative SBE-filter of S . It is clear that $1 \in \vec{a}$. Let $p, q \in S$ be such that $p \circ (q \circ q) \in \vec{a}$ and $p \in \vec{a}$. Then $a \preceq p \circ (q \circ q)$ and $a \preceq p$, that is, $a \circ ((p \circ (q \circ q)) \circ (p \circ (q \circ q))) = 1 \in \{1\}$ and $a \circ (p \circ p) = 1 \in \{1\}$. Since $\{1\}$ is an implicative SBE-filter of S , it follows from (4.1) that $a \circ (q \circ q) \in \{1\}$. Hence $q \in \vec{a}$. Therefore $\vec{a}$ is an SBE-filter of S .

Conversely, suppose that the set $\vec{a}$ is an SBE-filter of S for every $a \in S$. Let $p, q, r \in S$ be such that $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in \{1\}$ and $p \circ (q \circ q) \in \{1\}$. Then $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) = 1$ and $p \circ (q \circ q) = 1$, i.e., $p \preceq q \circ (r \circ r)$ and $p \preceq q$. Hence $q \circ (r \circ r) \in \vec{p}$ and $q \in \vec{p}$. Since $\vec{p}$ is an SBE-filter of S , we have $r \in \vec{p}$, that is, $p \preceq r$. Thus $p \circ (r \circ r) = 1 \in \{1\}$, and therefore $\{1\}$ is an implicative SBE-filter of S . $\square$

5. POSITIVE IMPLICATIVE SBE-FILTERS

In this section, we define the notion of a positive implicative SBE-filter of an SBE-algebra and study its properties. We provide equivalent conditions for an SBE-filter to be a positive implicative SBE-filter. We show that every SBE-filter is a positive implicative SBE-filter in a transitive SBE-algebra.

Definition 31. A subset F of an SBE-algebra S is called a *positive implicative SBE-filter* of S if it satisfies (2.18) and for all $p, q, r \in S$, if

$$p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F, p \in F \text{ then } ((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) \in F. \quad (5.1)$$

Example 11. From the Example 4, we can observe that $(S, \circ, 1)$ is SBE-algebra and $F = \{1, 2, 4\}$ is a positive implicative SBE-filter of S .

Theorem 32. Every positive implicative SBE-filter is an SBE-filter.

Proof. Let F be a positive implicative SBE-filter of an SBE-algebra S and let $p, q \in S$ be such that $p \circ (q \circ q) \in F$ and $p \in F$. Then $p \circ ((1 \circ (q \circ q)) \circ (1 \circ (q \circ q))) = p \circ (q \circ q) \in F$ and $p \in F$. It follows from (SBE-1), (2.1), (2.2) and (5.1) that $q = ((q \circ (1 \circ 1)) \circ (1 \circ 1)) \circ (q \circ q) \in F$. Thus F is an SBE-filter of S . $\square$

The following example shows that the converse of Theorem 32 is not true.

Example 12. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	5	5	1	5	5	1	1	1	1	1	5	1	1	5
1	5	6	9	7	8	0	1	3	4	2	11	10	13	12
2	1	9	9	10	10	13	1	3	4	1	3	10	13	1
3	5	7	10	7	7	10	1	1	1	2	12	10	10	12
4	5	8	10	7	8	10	1	1	1	2	12	10	10	12
5	1	0	13	10	10	0	1	3	4	1	3	10	13	1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	3	3	1	1	3	1	3	4	1	3	1	3	1
8	1	4	4	1	1	4	1	4	4	1	4	1	4	1
9	1	2	1	2	2	1	1	1	1	2	2	1	1	2
10	5	11	3	12	12	3	1	3	4	2	11	1	3	12
11	1	10	10	10	10	10	1	1	1	1	1	10	10	1
12	1	13	13	10	10	13	1	3	4	1	3	10	13	1
13	5	12	1	12	12	1	1	1	1	2	12	1	1	12

Then $(S, \circ, 1)$ is a SBE-algebra. Then $F = \{1, 3, 4\}$ is an SBE-filter of S . But F is not a positive implicative SBE-filter of S since $1 \circ ((0 \circ (2 \circ 2)) \circ (0 \circ (2 \circ 2))) = 1 \circ (1 \circ 1) = 1 \circ 6 = 1 \in F$ but $((2 \circ (0 \circ 0)) \circ (0 \circ 0)) \circ (2 \circ 2) = ((2 \circ 5) \circ 5) \circ 9 = (13 \circ 5) \circ 9 = 1 \circ 9 = 2 \notin F$.

We provide conditions for an SBE-filter to be a positive implicative SBE-filter.

Theorem 33. *Let F be an SBE-filter of an SBE-algebra S . Then F is a positive implicative SBE-filter of S if and only if it satisfies:*

$$(\forall p, q \in S) (p \circ (q \circ q) \in F \Rightarrow ((q \circ (p \circ p)) \circ (p \circ p)) \circ (q \circ q) \in F). \quad (5.2)$$

Proof. Assume that F is a positive implicative SBE-filter of S and $p \circ (q \circ q) \in F$ for all $p, q \in S$. Using (2.2) and (2.18), we have $1 \circ ((p \circ (q \circ q)) \circ (p \circ (q \circ q))) = p \circ (q \circ q) \in F$ and $1 \in F$. It follows from (5.1) that $((q \circ (p \circ p)) \circ (p \circ p)) \circ (q \circ q) \in F$.

Conversely, let F be an SBE-filter of S which satisfies the condition (5.2). Let $p, q, r \in S$ be such that $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in F$ and $p \in F$. Then $q \circ (r \circ r) \in F$ by (2.19), and so $((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) \in F$ by (5.2). Therefore F is a positive implicative SBE-filter of S . $\square$

Theorem 34. *Given an element c in an SBE-algebra S , the upset $\vec{c}$ of c is a positive implicative SBE-filter of S if and only if S satisfies:*

$$(\forall p, q \in S) (a \preceq p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))), a \preceq p \Rightarrow a \preceq ((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r)). \quad (5.3)$$

Proof. Assume that $\vec{a}$ is a positive implicative SBE-filter of S . Let $p, q, r \in S$ be such that $a \preceq p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r)))$ and $a \preceq p$. Then $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in \vec{a}$ and $p \in \vec{a}$. Since $\vec{a}$ is a positive implicative SBE-filter of S , it follows that $((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) \in \vec{a}$, that is, $a \preceq ((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r)$.

Conversely, suppose that S satisfies the condition (5.3). It is clear that $1 \in \vec{a}$. Let $p, q, r \in S$ be such that $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in \vec{a}$ and $p \in \vec{a}$. Then $a \preceq p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r)))$ and $a \preceq p$ which imply from (5.3) that $a \preceq ((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r)$. Hence $((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) \in \vec{a}$, and therefore $\vec{a}$ is a positive implicative SBE-filter of S . $\square$

In general the trivial SBE-filter $\{1\}$ is not a positive implicative SBE-filter as seen in the following example.

Example 13. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with binary operation $\circ$ given in the following table:

$\circ$	0	1	2	3	4	5	6	7
0	4	4	1	1	1	1	4	1
1	4	5	3	2	0	1	7	6
2	1	3	3	1	6	1	1	6
3	1	2	1	2	1	1	2	1
4	1	0	6	1	0	1	1	6
5	1	1	1	1	1	1	1	1
6	4	7	1	2	1	1	1	1
7	1	6	6	1	6	1	1	6

Then $(S, \circ, 1)$ is a SBE-algebra. Then $F = \{1\}$ is an SBE-filter of S . But F is not a positive implicative SBE-filter of S since $1 \circ ((0 \circ (2 \circ 2)) \circ (0 \circ (2 \circ 2))) = 1 \circ ((0 \circ 3) \circ (0 \circ 3)) = 1 \circ (1 \circ 1) = 1 \in F$ and $1 \in F$ but $((2 \circ (0 \circ 0)) \circ (0 \circ 0)) \circ (2 \circ 2) = ((2 \circ 4) \circ 4) \circ 3 = (6 \circ 4) \circ 3 = 1 \circ 3 = 2 \notin F$.

In the following theorem, we look for conditions under which the trivial SBE-filter is a positive implicative SBE-filter.

Theorem 35. In a commutative SBE-algebra, the trivial SBE-filter is a positive implicative SBE-filter.

Proof. Let S be a commutative SBE-algebra and let $p, q, r \in S$ be such that $p \circ ((q \circ (r \circ r)) \circ (q \circ (r \circ r))) \in \{1\}$ and $p \in \{1\}$. Then $q \circ (r \circ r) = 1$, and so $((r \circ (q \circ q)) \circ (q \circ q)) \circ (r \circ r) = ((q \circ (r \circ r)) \circ (r \circ r)) \circ (r \circ r) = (1 \circ (r \circ r)) \circ (r \circ r) = r \circ (r \circ r) = 1 \in \{1\}$ by (2.8), (SBE-1) and (2.1). Therefore $\{1\}$ is a positive implicative SBE-filter of S . $\square$

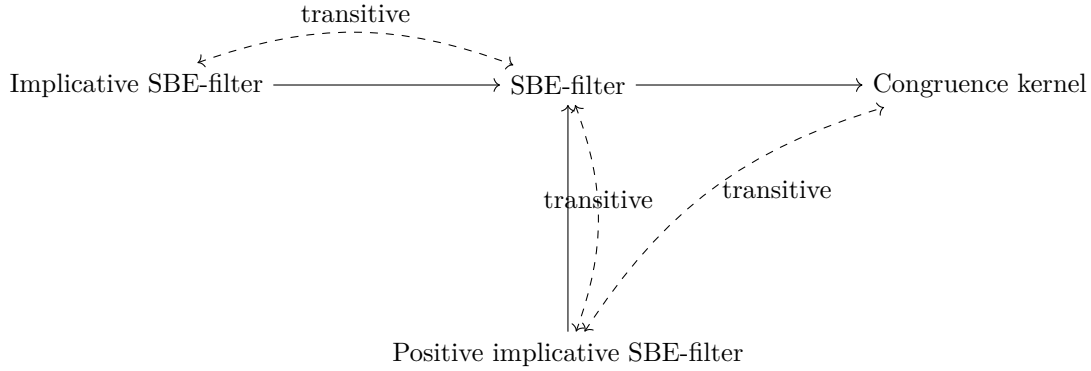
Theorem 36. Let $(S, \circ, 1)$ be a transitive SBE-algebra. Then every SBE-filter of S is a positive implicative SBE-filter of S .

Proof. Let F be an SBE-filter of a transitive SBE-algebra S and $q \circ (p \circ p) \in F$. Then $q \circ (p \circ p) \preceq ((p \circ (q \circ q)) \circ (q \circ q)) \circ ((p \circ (q \circ q)) \circ (p \circ p))$. Since $(p \circ (q \circ q)) \circ (p \circ p) = p$, we have $q \circ (p \circ p) \preceq ((p \circ (q \circ q)) \circ (q \circ q)) \circ (p \circ p)$. Since F is an SBE-filter of S , we have $((p \circ (q \circ q)) \circ (q \circ q)) \circ (p \circ p) \in F$. Hence, by Theorem 33, F is a positive implicative SBE-filter of S . $\square$

Theorem 37. Let $(S, \circ, 1)$ be a transitive SBE-algebra and F a non-empty subset of S . Then the following are equivalent.

- (i) F is an SBE-filter of S .
- (ii) F is an implicative SBE-filter of S .
- (iii) F is a positive implicative SBE-filter of S .

In the following diagram, we illustrate the relationships between different types of SBE-filters. The notation $A \rightarrow B$ means that A implies B .



6. CONCLUSION

In this study, we have investigated into the intricate properties of transitive SBE-algebras, shedding light on their essential characteristics and unveiling characterizations of SBE-filters. Furthermore, our exploration extends to the introduction of implicative SBE-filters, where we have meticulously examined their fundamental properties and associated results. Through establishing interconnections among these filters, we have erected theoretical bridges that advance our understanding of this algebraic domain. Our empirical contributions extend to introducing the concept of (positive) implicative SBE-filters, a novel addition that holds promise for researchers across diverse scientific disciplines requiring the utilization of filters. Moreover, we have presented a robust theoretical framework for (positive) implicative filters, facilitating their seamless adaptation to various algebraic structures.

In introducing the concept of transitive SBE-algebras, we have embarked on a comprehensive exploration of their properties, culminating in a detailed characterization of SBE-filters. Additionally, we have elucidated several equivalent conditions for these filters, alongside defining the congruence concept and showcasing the pivotal role of $[1_\theta]$ as an SBE-filter. Furthermore, our investigation has unveiled the inherent relationship between filters and congruences, emphasizing the correspondence between SBE-filters and congruence kernels. Moreover, we have introduced implicative SBE-filters and meticulously demonstrated their status as SBE-filters. Through an exhaustive examination of their properties on transitive SBE-algebras, we have illustrated the alignment between SBE-filters and implicative SBE-filters. Furthermore, our inquiry into the fundamental relationships between SBE-filters and implicative SBE-filters has enriched our understanding of their interconnectedness.

Finally, we have introduced the concept of positive implicative SBE-filter and demonstrated its equivalence to SBE-filters and implicative SBE-filters on transitive SBE-algebras. By providing a condition for proving the reciprocal of the previous assumption, we have further solidified the equivalence between these filter concepts, underscoring their significance in the realm of algebraic structures.

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