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On r-small intersection graph of modules

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Abstract. In this paper, we introduce a new algebraic graph associated with modules, called the r-small intersection graph of an A-module T, denoted by $\Gamma_r(T)$. The vertices of $\Gamma_r(T)$ are all nonzero submodules of T, and two distinct vertices are adjacent if and only if their intersection is an r-small submodule of T. We investigate how algebraic properties of modules influence graph-theoretic properties of $\Gamma_r(T)$. In particular, we study connectedness, diameter, girth, planarity and completeness of $\Gamma_r(T)$. Among other results, we show that for certain classes of modules, the graph $\Gamma_r(T)$ has diameter at most 2, and we characterize when $\Gamma_r(T)$ is complete or a tree.

Keywords: r-small, Intersection graph, Small submodule.

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1. Introduction

The study of graphs associated with algebraic structures has attracted considerable attention over the past decades. The intersection graphs were first introduced by Chakrabarty[6] in the context of ideals of rings

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and later developed by several authors in module theory and related areas. S. H. Jafari and N. Jafari Rad [7] investigated planarity properties of intersection graphs of ideals and explored domination in intersection graphs for both rings and modules during 2010 and 2011. Other contributions regarding intersection graphs of ideals can be found in [1, 8]. Akbari et al. [2] introduced the intersection graph of submodules of a module. Following the introduction of the small intersection graph by Mahdavi and Talebi[10], which captures interactions among submodules via small intersections, we extend this framework by defining the r-small intersection graph. This new graph provides a broader setting for studying the interplay between module-theoretic notions and graph-theoretic properties. The primary aim is to investigate how algebraic characteristics of a module, such as radicals, minimal and maximal submodules and hollow or semisimple structures, correspond to combinatorial properties of the associated graph.

Throughout the paper, A denotes an associative ring with identity and all modules are unitary right A-modules. A module T is *simple* if it has no nontrivial submodules other than 0 and itself. The *socle* of T, denoted $\mathrm{Soc}(T)$, is defined as the sum of all simple submodules of T. Let T be an A-module and let $E \leq T$. The submodule E is called *essential* in T, denoted by $E \leq_e T$, if for every nonzero submodule $X \leq T$, we have $E \cap X \neq 0$. T is uniform if and only if every nonzero submodule of T is essential in T. A submodule $a \leq T$ is called *small* in T, written $a \ll T$, if for every submodule $b \leq T$, the equality $a + b = T$ implies $b = T$. This notion of smallness is fundamental in understanding minimality conditions and adjacency relations among submodules. The *radical* of T, denoted $\mathrm{Rad}(T)$, is the intersection of all maximal submodules of T. Equivalently, Rad(T) is the sum of superfluous submodule of T. Nebiyev [12] introduced the notion of *r-small* submodules. A submodule K is *r-small* in W if $K \ll Rad(W)$, written $K \ll_r W$. Notably, any finitely generated or semisimple submodule contained in $\mathrm{Rad}(T)$ is r-small, which provides a rich source of vertices for the adjacency graph. We denote by $D(T)$ the set of all nonzero r-small submodules of T. Another key notion in this context is that of *hollow modules.* A module T is called hollow if every proper submodule of T is small. Let T be a module and $U, V \leq T$. If $T = U + V$ and $U \cap V \ll_r V$, then V is called an *r-supplement* of U in T. Furthermore, if every submodule of T has an r-supplement in T, then T is called an *r-supplemented module.* A graph G is defined as the pair $(V(G), E(G)$, where $(V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. For two distinct vertices a and b, we write $a - b$ to denote that a and b are adjacent. For distinct vertices x and y, an *x–y path* is a path starting at x and ending at y. The *distance* between x and y, denoted $d(x, y)$, is the length of the shortest x–y path; if no such path exists, we set $d(x, y) = \infty$. The *diameter* of G, written $\mathrm{diam}(G)$, is the supremum of the distances between all pairs of distinct vertices. A *cycle* in G is a path of length at least 3 that starts and ends at the same vertex, traversing distinct vertices. The *girth* of G is the length of its shortest cycle; if G contains no cycles, its girth is infinite. The graph G is *connected* if there exists a path between every pair of vertices. A *tree* is a connected graph without cycles, and a *star graph* is a tree in which a single central vertex is adjacent to all other vertices. A *complete graph* on n vertices is denoted by K_n and a *complete bipartite graph* with part sizes m and n is denoted by $K_{m,n}$. Finally, G is called *planar* if it can be drawn in the plane without any edges crossing. Any undefined terminology can be found in [5, 4].

2. Preliminary Notes

We start this section by presenting our main definition and some fundamental examples. We use the results of this section to prove our main results in the next section.

Definition 2.1. Let T be an A-module. The graph $\Gamma_r(T)$ is the simple undirected graph whose vertices are all *nonzero proper submodules* of T. Two distinct vertices a_1 and a_2 are adjacent if and only if $a_1 \cap a_2 \ll_r T$.

It is clear that if a module T has a submodule N such that both N and T/N are simple, then $\Gamma_r(T)$ is isomorphic to K_1. So we examine modules with at least two nontrivial submodules.

Example 2.2. (1) Let $T = \mathbb{Z}_{24}$ be a $\mathbb{Z}$-module. The nonzero proper submodules of T are $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$, $\langle 6\rangle$, $\langle 8\rangle$, $\langle 12\rangle$. The radical of T is $\mathrm{Rad}(T) = \langle 6\rangle$.

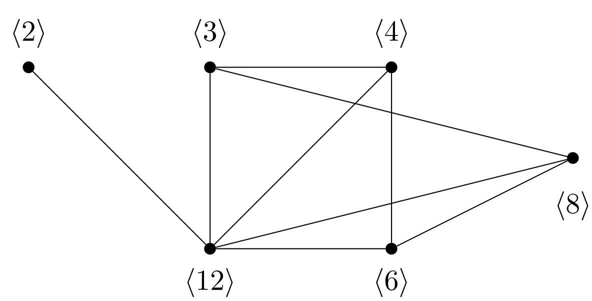

Figure 1. $\Gamma_r(\mathbb{Z}_{24})$

(2) Let $T = \mathbb{Z}_{12}$ as a $\mathbb{Z}$-module. The nonzero proper submodules of T are $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$ and $\langle 6\rangle$. Since $\mathrm{Rad}(T) = \langle 6\rangle$ and $\langle 6\rangle$ has no nonzero

proper submodules, no two distinct vertices have an r-small intersection. Therefore, $\Gamma_r(T)$ is a totally disconnected graph.

Proposition 2.3. *Let T be an A-module. Then $\Gamma_r(T)$ is complete if one of the following holds:*

(1) *$T = T_1 \oplus T_2$, where T_1 and T_2 are simple A-modules.*
(2) *T is hollow and Rad(T)=T.*

Proof. (1) Assume $T = T_1 \oplus T_2$ with T_1 and T_2 simple. Then the only nonzero proper submodules of T are T_1 and T_2. Moreover, $T_1 \cap T_2 = 0$. Since $T = T_1 \oplus T_2$ is semisimple, we have $\mathrm{Rad}(T) = 0$ and we have $T_1 \cap T_2 \ll_r T$. Thus $\Gamma_r(T)$ is complete.

(2) If T is hollow and $\mathrm{Rad}(T) = T$, then every submodule of T is small in T. Since $\mathrm{Rad}(T) = T$, we have $a_1 \cap a_2 \ll T = \mathrm{Rad}(T)$. Hence, $a_1 \cap a_2 \ll_r T$. Therefore, every pair of vertices is adjacent, and the graph is complete. □

Theorem 2.4. *Let T be a module such that every submodule of* $\mathrm{Rad}(T)$ *is r-small. If $\Gamma_r(T)$ is connected, then* $\mathrm{diam}(\Gamma_r(T)) \leq 2$.

Proof. Let a_1 and a_2 be two distinct vertices of the connected graph $\Gamma_r(T)$. If a_1 and a_2 are adjacent, then $d(a_1, a_2) = 1$ and there is nothing to prove. Assume that a_1 and a_2 are not adjacent. Since $\Gamma_r(T)$ is connected, there exists a shortest path $a_1 = b_0 - b_1 - b_2 - \cdots - b_k = a_2$. Because the path is shortest, we have $b_0 \cap b_2 \not\ll_r T$, for otherwise b_0 and b_2 would be adjacent, contradicting the minimality of the path. On the other hand, since b_0 is adjacent to b_1 and b_1 is adjacent to b_2, we have $b_0 \cap b_1 \ll_r T$, $b_1 \cap b_2 \ll_r T$. Hence $(b_0 \cap b_2) \subseteq (b_0 \cap b_1) + (b_1 \cap b_2) \subseteq Rad(T)$. By assumption, every submodule contained in the radical is r-small. Thus $b_0 \cap b_2 \ll_r T$, which contradicts the fact that b_0 and b_2 are not adjacent. Therefore k can not exceed 2; hence $k = 2$. So $d(a_1, a_2) = 2$. Since a_1 and a_2 were arbitrary vertices, the diameter of $\Gamma_r(T)$ is at most 2. □

3. Main Results

In this section, we investigate several graph-theoretic properties of the r-small intersection graph $\Gamma_r(T)$ and relate them to algebraic properties of the module T. Throughout this section, vertices of $\Gamma_r(T)$ are assumed to be nonzero proper submodules of T.

3.1. Diameter and girth of $\Gamma_r(T)$.

Proposition 3.1. *Let A be an integral domain with $0 \neq \mathrm{Rad}(A)$ and T be a finitely generated torsion-free A-module. Assume that every nonzero submodule of* $\mathrm{Rad}(T)$ *is r-small in T. If T has a proper essential submodule, then the graph $\Gamma_r(T)$ is connected. Moreover,* $\mathrm{diam}(\Gamma_r(T)) \leq 2$.

Proof. Let E be a proper essential submodule of T. Since A is an integral domain and T is torsion-free, multiplication by any nonzero element of A is injective on T. Choose $0 \neq r \in \mathrm{Rad}(A)$. Then $rT \subseteq \mathrm{Rad}(T)$ and hence $rE \neq 0$. Let K and N be arbitrary vertices of $\Gamma_r(T)$. Since E is essential in T, we have $K \cap E \neq 0$ and $N \cap E \neq 0$. Multiplying by r, we obtain $r(K \cap E) \subseteq K \cap rE$, $r(N \cap E) \subseteq N \cap rE$, and both submodules are nonzero and contained in $\mathrm{Rad}(T)$. By assumption, they are r-small in T. Hence K is adjacent to rE and N is adjacent to rE. Therefore, there exists a path $K - rE - N$ of length at most 2. Thus $\Gamma_r(T)$ is connected and has diameter at most 2. □

Proposition 3.2. *Let T be an A-module such that* $\mathrm{Rad}(T)$ *is hollow and every submodule of* $\mathrm{Rad}(T)$ *is r-small. If $\Gamma_r(T)$ contains a cycle, then* $\mathrm{girth}(\Gamma_r(T)) = 3$.

Proof. Assume that $\Gamma_r(T)$ contains a cycle and let $C = A_1 - A_2 - \cdots - A_n - A_1$ be a cycle of minimal length $n \geq 3$. Suppose that $n > 3$. Choose two nonconsecutive vertices A_i and A_j in C. Then $A_i \cap A_j \subseteq \mathrm{Rad}(T)$. By assumption, every submodule of $\mathrm{Rad}(T)$ is r-small, so we have $A_i \cap A_j \ll_r T$, which implies that A_i and A_j are adjacent. This yields a cycle shorter than C, contradicting the minimality of C. Hence $n = 3$ and the girth is 3. □

Proposition 3.3. *Let T be a finitely generated hollow A-module such that* $0 \neq \mathrm{Rad}(T)$, $\mathrm{Soc}(T) \subseteq \mathrm{Rad}(T)$, *and every nonzero submodule of* $\mathrm{Rad}(T)$ *is r-small in T. Then:*

(1) $\mathrm{Soc}(T)$ *is adjacent to every vertex of* $\Gamma_r(T)$.
(2) $d\big(\mathrm{Soc}(T), \mathrm{Rad}(T)\big) \leq 2$.

Proof. (1) Let X be a vertex of $\Gamma_r(T)$. Since T is hollow, $X \ll T$. Moreover, $X \cap \mathrm{Soc}(T) \subseteq \mathrm{Rad}(T)$. By hypothesis, $X \cap \mathrm{Soc}(T)$ is r-small in T. Hence X is adjacent to $\mathrm{Soc}(T)$.

(2) If $\mathrm{Rad}(T) \neq T$, then $\mathrm{Soc}(T) \cap \mathrm{Rad}(T) \ll_r T$, and thus they are adjacent. Otherwise, choose any vertex X of $\Gamma_r(T)$. Then $\mathrm{Soc}(T) \sim X$ and $X \sim \mathrm{Rad}(T)$, yielding a path of length 2. □

3.2. **Connectedness.**

Theorem 3.4. *Let $h : T_1 \to T_2$ be an A-monomorphism such that* $h(\mathrm{Rad}(T_1)) \subseteq \mathrm{Rad}(T_2)$. *Assume that every nonzero r-small submodule of* $\mathrm{Rad}(T_1)$ *maps under h to an r-small submodule of* $\mathrm{Rad}(T_2)$. *If $a_1 \sim a_2$ in $\Gamma_r(T_1)$, then $h(a_1) \sim h(a_2)$ in $\Gamma_r(T_2)$.*

Proof. Since h is injective, for any submodules $a_1, a_2 \leq T_1$ we have $h(a_1 \cap a_2) = h(a_1) \cap h(a_2)$. If $a_1 \sim a_2$ in $\Gamma_r(T_1)$, then by definition $a_1 \cap a_2 \ll_r T_1 \implies a_1 \cap a_2 \ll \mathrm{Rad}(T_1)$. Applying h, we obtain $h(a_1 \cap a_2) \subseteq$

$h(\mathrm{Rad}(T_1)) \subseteq \mathrm{Rad}(T_2)$ and by the assumption on r-small submodules, $h(a_1 \cap a_2)$ is r-small in T_2. Hence $h(a_1) \cap h(a_2) = h(a_1 \cap a_2) \ll_r T_2$, so by definition of $\Gamma_r(T_2)$, we have $h(a_1) \sim h(a_2)$. □

Example 3.5. Let $A = \mathbb{Z}$ and consider the modules $T_1 = \mathbb{Z}/4 \oplus \mathbb{Z}/2$, $T_2 = \mathbb{Z}/8 \oplus \mathbb{Z}/4$. Then $Rad(\Gamma_r(\mathbb{Z}/4 \oplus \mathbb{Z}/2)) = 2\mathbb{Z}/4 \oplus \{0\} = \{(0,0),(2,0)\}$, $\mathrm{Rad}(\mathbb{Z}/8 \oplus \mathbb{Z}/4) = 2\mathbb{Z}/8 \oplus 2\mathbb{Z}/4$. Consider the nonzero submodules $A = \langle(1,0)\rangle$, $B = \langle(2,0)\rangle$, $C = \langle(0,1)\rangle$, $D = \langle(1,1)\rangle$. $\mathrm{Rad}(\mathbb{Z}/4 \oplus \mathbb{Z}/2) = B$. Consider $h : \Gamma_r(\mathbb{Z}/4 \oplus \mathbb{Z}/2) \to \mathbb{Z}/8 \oplus \mathbb{Z}/4$ by $h(a,b) = (2a, 2b)$ which is injective.

$$
\begin{aligned}
&A \cap B = B \ll \mathrm{Rad}(T_1) \Rightarrow A \cap B \ll_r T_1 \Rightarrow A \sim B, \\
&A \cap C = 0 \ll \mathrm{Rad}(T_1) \Rightarrow A \cap C \ll_r T_1 \Rightarrow A \sim C, \qquad h(A) = \langle(2,0)\rangle, \\
&A \cap D = B \ll \mathrm{Rad}(T_1) \Rightarrow A \cap D \ll_r T_1 \Rightarrow A \sim D, \quad h(B) = \langle(4,0)\rangle, \\
&B \cap C = 0 \ll \mathrm{Rad}(T_1) \Rightarrow B \cap C \ll_r T_1 \Rightarrow B \sim C, \qquad h(C) = \langle(0,2)\rangle, \\
&B \cap D = B \ll \mathrm{Rad}(T_1) \Rightarrow B \cap D \ll_r T_1 \Rightarrow B \sim D, \quad h(D) = \langle(2,2)\rangle. \\
&C \cap D = 0 \ll \mathrm{Rad}(T_1) \Rightarrow C \cap D \ll_r T_1 \Rightarrow C \sim D.
\end{aligned}
$$

All submodules are pairwise adjacent, so $\Gamma_r(\mathbb{Z}/4 \oplus \mathbb{Z}/2)$ is a complete graph on 4 vertices and $\Gamma_r(\mathbb{Z}/8 \oplus \mathbb{Z}/4)$ is also a complete graph on 4 vertices. This illustrates that the monomorphism h preserves adjacency between submodules.

(A) $\Gamma_r(\mathbb{Z}/4 \oplus \mathbb{Z}/2)$ (B) $\Gamma_r(\mathbb{Z}/8 \oplus \mathbb{Z}/4)$

Figure 2

Proposition 3.6. *Let $h : T_1 \to T_2$ be an isomorphism. Then* $\Gamma_r(T_1) \cong \Gamma_r(T_2)$.

Proof. Since $\Gamma_r(T_1)$ is complete, for any two distinct nonzero submodules $a_1, a_2 \leq T_1$, we have $a_1 \cap a_2 \ll \mathrm{Rad}(T_1)$, i.e., $a_1 \sim a_2$ in $\Gamma_r(T_1)$. By Theorem 3.4, monomorphism preserves adjacency, so $a_1 \sim a_2$ implies $h(a_1) \sim h(a_2)$ in $\Gamma_r(T_2)$. Since $T_2 = h(T_1)$, every nonzero submodule of T_2 is of the form $h(a)$ for some nonzero $a \leq T_1$. Therefore, every pair of distinct nonzero submodules of T_2 is adjacent, and hence $\Gamma_r(T_2)$ is complete. $\square$

Theorem 3.7. *Let T be a Noetherian A-module. If $\Gamma_r(T)$ is complete, then T has a unique maximal submodule.*

Proof. Assume that T has two distinct maximal submodules M_1 and M_2. Then $M_1 \cap M_2 \subseteq \mathrm{Rad}(T)$. Since $M_1 \neq M_2$, we have $M_1 \cap M_2 \not\ll_r T$. Hence M_1 and M_2 are not adjacent in $\Gamma_r(T)$, contradicting completeness. Therefore, T has a unique maximal submodule. $\square$

Theorem 3.8. *Let T be a left Artinian A-module. If T has a unique minimal submodule, then $\Gamma_r(T)$ is complete.*

Proof. Let M be the unique minimal submodule of T. Since T is Artinian, $\mathrm{Soc}(T)$ is nonzero and equals M. Let A, B be distinct nonzero submodules of T. Then $A \cap B$ contains M or is zero. In either case, $A \cap B \subseteq \mathrm{Rad}(T)$. Since $M \subseteq \mathrm{Rad}(T)$ and every submodule of $\mathrm{Rad}(T)$ is r-small, we have $A \cap B \ll_r T$. Hence A and B are adjacent. Thus $\Gamma_r(T)$ is complete. $\square$

Proposition 3.9. *Let T be an A-module such that* $\mathrm{Rad}(T)$ *is uniform. If $N \in D(T)$, then N is an isolated vertex in $\Gamma_r(T)$.*

Proof. Let $N \in D(T)$, so $N \ll_r T$. Suppose, on the contrary, that N is adjacent to some vertex M in $\Gamma_r(T)$. Then $N \cap M \ll_r T \Rightarrow N \cap M \ll \mathrm{Rad}(T)$. Since $\mathrm{Rad}(T)$ is uniform, every nonzero submodule of $\mathrm{Rad}(T)$ is essential in $\mathrm{Rad}(T)$. However, an essential submodule cannot be small. Therefore, we must have $N \cap M = 0$. Thus N has no adjacent vertices in $\Gamma_r(T)$ and hence $\deg(N) = 0$. $\square$

Theorem 3.10. *Let T be an A-module. If $|D(T)| \geq 1$ and $\Gamma_r(T)$ is a tree such that for every pair of distinct non-central vertices $u, v \in D(T)$, $u \cap v \not\ll \mathrm{Rad}(T)$, then $\Gamma_r(T) \cong K_1$ or $\Gamma_r(T)$ is a star graph.*

Proof. Let $|D(T)| = 1$. Then $\Gamma_r(T)$ has a single vertex and no edges. Clearly, $\Gamma_r(T) \cong K_1$. If $|D(T)| > 1$. Since $\Gamma_r(T)$ is a tree, it is connected and contains no cycles. Let $v \in D(T)$ be a vertex adjacent to the maximal number of vertices and call it the *central vertex*. For any two non-central vertices $u, w \in D(T)$, if $u \cap w \ll \mathrm{Rad}(T)$, then u and w would be adjacent, creating a cycle $u - v - w - u$, which contradicts the tree property. Therefore, no two non-central vertices are adjacent. Consequently, all edges must be incident to the central vertex, and the tree is a *star graph*. $\square$

Proposition 3.11. *If T is an r-supplemented module, then $\Gamma_r(T)$ is connected.*

Proof. Let A and B be two arbitrary vertices of $\Gamma_r(T)$. Since T is r-supplemented, there exist submodules V and W of T such that $T = A + V$ with $A \cap V \ll_r T$ and $T = B + W$ with $B \cap W \ll_r T$. Hence A is adjacent to V and B is adjacent to W in $\Gamma_r(T)$. Now consider the submodules V and W. Since $V \cap W \subseteq (A \cap V) + (B \cap W)$ and both $A \cap V$ and $B \cap W$ are r-small in T. It follows that $V \cap W \ll_r T$. Thus V is adjacent to W in $\Gamma_r(T)$. Therefore, there exists a path $A - V - W - B$ in $\Gamma_r(T)$ connecting A and B. Since A and B were arbitrary vertices, the graph $\Gamma_r(T)$ is connected. □

Proposition 3.12. *Suppose that T is an r-supplemented module and every submodule of T has a unique r-supplement. Then $\Gamma_r(T)$ is a tree.*

Proof. Since T is r-supplemented, so $\Gamma_r(T)$ is connected. Assume that $\Gamma_r(T)$ contains a cycle $U_0 - U_1 - \cdots - U_n - U_0$. Each adjacency implies $U_i \cap U_{i+1} \ll_r T$. Thus U_{i+1} is an r-supplement of U_i. But uniqueness of r-supplements implies that no submodule can have two distinct r-supplements. The existence of a cycle contradicts this uniqueness. Therefore, $\Gamma_r(T)$ has no cycles and hence is a tree. □

3.3. **Planarity of $\Gamma_r(T)$.** To establish the forthcoming results, we recall a celebrated characterization of planar graphs due to Kuratowski.

Theorem 3.13. *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*[5, Theorem 10.30]

Proposition 3.14. *Let T be an A-module. Then the following conditions hold:*

(1) *If $|D(T)| = 1$ or $|D(T)| = 2$, and the intersection of every pair of non r-small submodules of T is a non r-small submodule, then $\Gamma_r(T)$ is planar.*
(2) *If $|D(T)| \geq 3$, then $\Gamma_r(T)$ is not planar.*

Proof. (1) Suppose that $|D(T)| = 1$. Then $\Gamma_r(T)$ consists of a single vertex and no edges, hence it is planar.

Now assume that $|D(T)| = 2$ and let $D(T) = \{X, Y\}$. Since X and Y are r-small submodules, at most one edge can occur between them. All remaining vertices of $\Gamma_r(T)$ correspond to non-r-small submodules. By assumption, the intersection of any two non-r-small submodules is not r-small; therefore, no two such vertices are adjacent. Consequently, $\Gamma_r(T)$ contains no subgraph isomorphic to K_5 or $K_{3,3}$. By Kuratowski's theorem, $\Gamma_r(T)$ is planar.

(2) Suppose that $|D(T)| \geq 3$ and choose three distinct vertices X,Y,Z $\in$ D(T). Since X, Y, Z are r-small, any two of them may be adjacent, and in particular, the subgraph induced by $\{X, Y, Z\}$ contains a triangle

whenever they are pairwise adjacent. Now consider three distinct non-r-small vertices U_1, U_2, U_3 of $\Gamma_r(T)$ (if such vertices exist). If each U_i is adjacent to at least two of X, Y, Z, then the subgraph induced by $\{X, Y, Z, U_1, U_2, U_3\}$ contains a subgraph isomorphic to $K_{3,3}$. Hence $\Gamma_r(T)$ contains a subgraph isomorphic to $K_{3,3}$ or K_5. By Kuratowski's theorem, $\Gamma_r(T)$ is not planar. Therefore, if $|D(T)| \geq 3$, the graph $\Gamma_r(T)$ is not planar. □

Two distinct vertices I and J of the graph $\Gamma_r(T)$ are said to be *orthogonal*, denoted by $I \perp J$, if $I \cap J \ll_r T$ and there is no vertex $Z \in \Gamma_r(T)$ such that $I \cap Z \ll_r T$ and $J \cap Z \ll_r T$. A graph $\Gamma_r(T)$ is called *complemented* if for each vertex v of $\Gamma_r(T)$, there exists a vertex w of $\Gamma_r(T)$ (called a *complement* of v) such that $v \perp w$. According to these definitions, we prove following result.

Proposition 3.15. *Let $T = G \oplus K$ be a finitely generated A-module such that $Rad(G) = G \neq (0)$ and $Rad(K) = K \neq (0)$. Then the following conditions hold:*

(1) *The vertices G and K are not orthogonal in the graph $\Gamma_r(T)$.*
(2) *The graph $\Gamma_r(T)$ is not complemented.*
(3) *The graph $\Gamma_r(T)$ is not planar.*

Proof. Since $T = G + K$, so by [4, Proposition 9.19], we have $Rad(T) = Rad(G) \oplus Rad(K)$. Therefore $Rad(T) = G + K = T$. Now by [12], a submodule N of T is r-small if and only if N is small. We can use this observation to prove this theorem as follows.

(1) Suppose that $T = G \oplus K$. Then $G \cap K = (0) \ll T$ and since T is a finitely generated R-module, $Rad(T) \ll T$. Hence by [4, Proposition 5.17], $G \cap Rad(T) \ll T$ and $K \cap Rad(T) \ll T$. Thus $\Gamma_r(T)$ has a triangle of the form $(G, Rad(T), K)$. Therefore, the vertices G and K are not orthogonal in the graph $\Gamma_r(T)$.

(2) It is clear by definition.

(3) Let $T = G \oplus K$. Then $Rad(T) = Rad(G) \oplus Rad(K)$ and $Rad(G) \cap Rad(K) = (0) \ll T$. Also, $Rad(G) \subseteq G$ and $G \cap Rad(K) \subseteq G \cap K = (0)$ and so the modularity condition implies that $G \cap Rad(T) = Rad(G)$ and similarly $K \cap Rad(T) = Rad(K)$. Moreover, $G \cap Rad(G) = Rad(G) \ll T$, $K \cap Rad(K) = Rad(K) \ll T$, $G \cap Rad(T) = Rad(G) \ll T$, $K \cap Rad(T) = Rad(K) \ll T$, $Rad(G) \cap Rad(T) = Rad(G) \ll T$, $Rad(K) \cap Rad(T) = Rad(K) \ll T$. Hence G, K, $Rad(G)$, $Rad(K)$ and $Rad(T)$ are adjacent vertices in the graph $\Gamma_r(T)$. Therefore, the set {G, K, Rad(G), Rad(K), Rad(T)} induces a complete subgraph K_5 in $\Gamma_r(T)$. Hence, by Theorem 3.13 $\Gamma_r(T)$ is not a planar graph. □

Proposition 3.16. *Let T be an A-module and $K \leq T$. If T is an r-supplemented module, then $\Gamma_r(T/K)$ is connected.*

Proof. Since T is r-supplemented, so by [13] T is r-supplemented. Hence T/K is an r-supplemented module. By Proposition 3.5, the graph $\Gamma_r(T/K)$ is connected. □

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