

Inextensible Flows of Curves in Lie Groups

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ABSTRACT In this paper, we study inextensible flows of curves in three dimensional Lie groups with a bi-invariant metric. The necessary and sufficient conditions for inextensible curve flow are expressed as a partial differential equation involving the curvatures. Also, we give some results for special cases of Lie groups.

Keywords: Inextensible flows; Lie groups

1. INTRODUCTION

Flows of curves are used in engineering applications such as modeling ship hulls, buildings, airplane wings, garments, ducts, automobile parts. Moreover Chirikjian and Burdick explain the kinematics of hyperredundant (or “serpentine”) robot as the flow of plane curve in [1].

Inextensible flows of curves and developable surfaces in Euclidean 3-space have been studied by Kwon and Park in [8]. The flow of a curve is said to be inextensible if, its arc-length is preserved. Inextensible flows of curves have great importance in computer vision and computer animation in addition to the structural mechanics (see [4],[7], [9]).

Inextensible flows of curves have been studied in many studies. For example, Yıldız et al. [12] have investigated inextensible flows of curves

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according to darboux frame in Euclidean 3-space. Recently, Gürbüz [6] have introduced inextensible flows of spacelike, timelike and null curves in Minkowski space. Quite recently, Öğrenmiş et al. [11] have examined inextensible curves in Galilean space and etc.

In the literature, there are many papers about curves in Lie groups. For example, Çiftçi [3] has defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem and given a relation between the geodesics of the so-called cylinders and general helices. Also, Okuyucu et al. [10] have studied slant helices in Lie groups. Further, AW(k)-type general helices in Lie groups have been given by Yoon in [13].

In the present paper, we study inextensible flows of curves in three dimensional Lie groups with a bi-invariant metric and we obtain some results for special cases of three dimensional Lie groups. So we hope that this study contributes to the curves theory in Lie groups.

2. PRELIMINARIES

Let G be a Lie group with a bi-invariant metric \langle, \rangle and D be the Levi-Civita connection of Lie group G . If \mathfrak{g} denotes the Lie algebra of G then we know that \mathfrak{g} is isomorphic to T_eG , where e is neutral element of G . If \langle, \rangle is a bi-invariant metric on G then we will have that

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$$

and

$$D_X Y = \frac{1}{2} [X, Y]$$

for all X, Y and $Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc-lengthed curve and $\{X_1, X_2, \dots, X_n\}$ be an orthonormal basis of \mathfrak{g} . In this case, we write that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^n w_i X_i$ and $Z = \sum_{i=1}^n z_i X_i$, where $w_i : I \rightarrow \mathbb{R}$ and $z_i : I \rightarrow \mathbb{R}$ are smooth functions for $1 \leq i \leq n$. Also the Lie bracket of two vector fields W and Z is given by

$$[W, Z] = \sum_{i=1}^n w_i z_i [X_i, X_j]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha'} W$ is given by

$$D_{\alpha'} W = \dot{W} + \frac{1}{2} [T, W], \quad (2.1)$$

where $T = \alpha'$ and $\dot{W} = \sum_{i=1}^n \dot{w}_i X_i$ or $\dot{W} = \sum_{i=1}^n \frac{dw}{dt} X_i$. Note that if W is the restriction of a left-invariant vector field to the curve α , then $\dot{W} = 0$ (see [2] for details).

Let G be a three dimensional Lie group and $(T, N, B, \varkappa, \tau)$ denote the Frenet apparatus of the curve α , then we calculate $\varkappa = \|\dot{T}\|$.

Definition 2.1. ([3]) Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc-length parameterized curve with the Frenet apparatus $(T, N, B, \varkappa, \tau)$. Then

$$\tau_G = \frac{1}{2} \langle [T, N], B \rangle$$

or

$$\tau_G = \frac{1}{2\varkappa^2\tau} \langle \ddot{T}, [T, T] \rangle + \frac{1}{4\varkappa^2\tau} \|[T, T]\|^2.$$

Proposition 2.2. ([10]) Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be an arc-length parameterized curve with the Frenet apparatus $\{T, N, B\}$. Then the following equalities

$$\begin{aligned} [T, N] &= \langle [T, N], B \rangle B = 2\tau_G B \\ [T, B] &= \langle [T, B], N \rangle N = -2\tau_G N \end{aligned}$$

hold.

Remark 2.3. ([3] and [5]) Let G be a Lie group with a bi-invariant metric $\langle \cdot, \cdot \rangle$. Then the following equalities can be given in different Lie groups.

- i) If G is Abelian group then $\tau_G = 0$.
- ii) If G is SO^3 then $\tau_G = \frac{1}{2}$.
- iii) If G is SU^2 then $\tau_G = 1$.

Let $\alpha : I \subset \mathbb{R} \rightarrow G$ be a curve with arc-length parameter s . For Frenet vectors of the curve α , we have by (2.1) and Proposition 2.2 that

$$\begin{bmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dB}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & (\tau - \tau_G) \\ 0 & -(\tau - \tau_G) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (2.2)$$

where $\{T, N, B\}$ is Frenet frame, $\tau_G = \frac{1}{2} \langle [T, N], B \rangle$ and \varkappa, τ are curvatures of the curve α in three dimensional Lie group G , respectively.

3. INEXTENSIBLE FLOWS OF CURVES IN LIE GROUPS

Throughout this paper, unless otherwise stated, we assume that

$$\alpha : [0, l] \times [0, w] \longrightarrow G$$

is a one - parameter family of smooth curves in three dimensional Lie group G , where l is the arc-length of the initial curve. Let u be the curve parametrization variable, $0 \leq u \leq l$. If the speed of the curve α is given by $v = \left\| \frac{d\alpha}{du} \right\|$, then the arc-length of α is given as a function of u by

$$s(u) = \int_0^u \left\| \frac{\partial \alpha}{\partial u} \right\| du = \int_0^u v du.$$

The operator $\frac{\partial}{\partial s}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}. \quad (3.1)$$

In this case; the arc-length parameter is $ds = vdu$.

Let $\alpha(u, t) : [0, l] \times [0, w] \rightarrow G$ be a differentiable curve with the Frenet vector field $\{T, N, B\}$ in three dimensional Lie group G . Any flow of the curve can be expressed as follows

$$\frac{\partial \alpha}{\partial t} = fT + gN + hB,$$

where f, g and h are scalar speed functions of the curve α .

Let the arc-length variation be

$$s(u, t) = \int_0^u v du.$$

In three dimensional Lie group G , the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0, \quad (3.2)$$

where $u \in [0, l]$.

Definition 3.1. A curve evolution $\alpha(u, t)$ and its flow $\frac{\partial \alpha}{\partial t}$ in three dimensional Lie group G are said to be inextensible if

$$\frac{\partial}{\partial t} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0.$$

Before deriving the necessary and sufficient condition for inelastic curve flow, we need the following lemma.

Lemma 3.2. *Let $\alpha(u, t)$ be a differentiable curve with the Frenet vector field $\{T, N, B\}$ and the smooth flow of the curve $\alpha(u, t)$ be $\frac{\partial \alpha}{\partial t} = fT + gN + hB$ in three dimensional Lie group G . Then we have the following equality:*

$$\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} - gv\kappa. \quad (3.3)$$

Proof. Since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative and $v^2 = \langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \rangle$, we have showing (2.2) that

$$\begin{aligned}
 2v \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle \\
 &= 2 \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u} \left(\frac{\partial \alpha}{\partial t} \right) \right\rangle \\
 &= 2 \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u} (fT + gN + hB) \right\rangle \\
 &= 2 \left\langle vT, \frac{\partial f}{\partial u} T + fv\kappa N + \frac{\partial g}{\partial u} N + g(-v\kappa T + v(\tau - \tau_G)B) + \frac{\partial h}{\partial u} B \right. \\
 &\quad \left. + h(-v(\tau - \tau_G)N) \right\rangle = 2v \left(\frac{\partial f}{\partial u} - gv\kappa \right). (\langle \rangle)
 \end{aligned}$$

This leads us to the consequence of the equality below.

$$\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} - gv\kappa.$$

□

Theorem 3.3. *Let $\frac{\partial \alpha}{\partial t} = fT + gN + hB$ be a smooth flow of the curve α in three dimensional Lie group G . The flow is inextensible if and only if*

$$\frac{\partial f}{\partial s} = g\kappa. \tag{3.4}$$

Proof. Let us assume that the curve flow is inextensible. Combining (3.2) and (3.3), we have

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\frac{\partial f}{\partial u} - gv\kappa \right) du = 0,$$

which leads us to the fact that

$$\frac{\partial f}{\partial u} - gv\kappa = 0.$$

By using the last equation with (3.1), we get

$$\frac{\partial f}{\partial s} = g\kappa.$$

On the contrary, following similar way as above, the proof can be completed.

Now, suppose that the curve α is an arc-length parametrized curve in three dimensional Lie group G . That is, $v = 1$ and the local coordinate u corresponds to the curve arclength s . □

Lemma 3.4. *Let α be a arc-length parameterized curve with the Frenet vectors $\{T, N, B\}$ in three dimensional Lie group G . If the flow of the curve α is inextensible, then the derivatives of $\{T, N, B\}$ with respect to t are*

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) N + \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) B, \\ \frac{\partial N}{\partial t} &= - \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) T + \psi B, \\ \frac{\partial B}{\partial t} &= - \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) T - \psi N,\end{aligned}$$

where $\psi = \langle \frac{\partial N}{\partial t}, B \rangle$.

Proof. For $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are commutative, we have

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial \alpha}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial \alpha}{\partial t} \right) = \frac{\partial}{\partial s} (fT + gN + hB) \\ &= \frac{\partial f}{\partial s} T + f\kappa N + \frac{\partial g}{\partial s} N + g(-\kappa T + (\tau - \tau_G)B) + \frac{\partial h}{\partial s} B + h(-(\tau - \tau_G)N) \\ &= \left(\frac{\partial f}{\partial s} - g\kappa \right) T + \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) N + \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) B.\end{aligned}$$

Substituting (3.4) into the last equation, we get

$$\frac{\partial T}{\partial t} = \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) N + \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) B.$$

Also, since $\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$, we get

$$\begin{aligned}0 &= \frac{\partial}{\partial t} \langle T, N \rangle = \left\langle \frac{\partial T}{\partial t}, N \right\rangle + \left\langle T, \frac{\partial N}{\partial t} \right\rangle = \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) + \left\langle T, \frac{\partial N}{\partial t} \right\rangle, \\ 0 &= \frac{\partial}{\partial t} \langle T, B \rangle = \left\langle \frac{\partial T}{\partial t}, B \right\rangle + \left\langle T, \frac{\partial B}{\partial t} \right\rangle = \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) + \left\langle T, \frac{\partial B}{\partial t} \right\rangle, \\ 0 &= \frac{\partial}{\partial t} \langle N, B \rangle = \left\langle \frac{\partial N}{\partial t}, B \right\rangle + \left\langle N, \frac{\partial B}{\partial t} \right\rangle = \psi + \left\langle N, \frac{\partial B}{\partial t} \right\rangle.\end{aligned}$$

From the above equations, we obtain that

$$\frac{\partial N}{\partial t} = - \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) T + \psi B,$$

and

$$\frac{\partial B}{\partial t} = - \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) T - \psi N.$$

□

Lemma 3.4 includes the following special cases for derivatives of $\{T, N, B\}$ with respect to t :

(i) If G is Abelian group then $\tau_G = 0$. Thus we have that

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(f\kappa + \frac{\partial g}{\partial s} - h\tau \right) N + \left(g\tau + \frac{\partial h}{\partial s} \right) B, \\ \frac{\partial N}{\partial t} &= - \left(f\kappa + \frac{\partial g}{\partial s} - h\tau \right) T + \psi B, \\ \frac{\partial B}{\partial t} &= - \left(g\tau + \frac{\partial h}{\partial s} \right) T - \psi N.\end{aligned}$$

(ii) If G is SO^3 then $\tau_G = \frac{1}{2}$. Thus we have that

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(f\kappa + \frac{\partial g}{\partial s} - h\left(\tau - \frac{1}{2}\right) \right) N + \left(g\left(\tau - \frac{1}{2}\right) + \frac{\partial h}{\partial s} \right) B, \\ \frac{\partial N}{\partial t} &= - \left(f\kappa + \frac{\partial g}{\partial s} - \left(\tau - \frac{1}{2}\right) \right) T + \psi B, \\ \frac{\partial B}{\partial t} &= - \left(g\left(\tau - \frac{1}{2}\right) + \frac{\partial h}{\partial s} \right) T - \psi N.\end{aligned}$$

(iii) If G is SU^2 then $\tau_G = 1$. Thus we have that

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - 1) \right) N + \left(g(\tau - 1) + \frac{\partial h}{\partial s} \right) B, \\ \frac{\partial N}{\partial t} &= - \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - 1) \right) T + \psi B, \\ \frac{\partial B}{\partial t} &= - \left(g(\tau - 1) + \frac{\partial h}{\partial s} \right) T - \psi N.\end{aligned}$$

Theorem 3.5. *Let the curve flow $\frac{\partial \alpha}{\partial t} = fT + gN + hB$ be inextensible in three dimensional Lie group G . Then, there exists the following system of partial differential equations:*

$$\frac{\partial \kappa}{\partial t} = g\kappa^2 + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} - 2\frac{\partial h}{\partial s}(\tau - \tau_G) - h\frac{\partial(\tau - \tau_G)}{\partial s} - g(\tau - \tau_G)^2,$$

$$\kappa\psi = (\tau - \tau_G) \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) + \frac{\partial}{\partial s} (g(\tau - \tau_G)) + \frac{\partial^2 h}{\partial s^2},$$

$$\frac{\partial(\tau - \tau_G)}{\partial t} = g\kappa(\tau - \tau_G) + \kappa\frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s}.$$

Proof. Since $\frac{\partial}{\partial s} \frac{\partial T}{\partial t} = \frac{\partial}{\partial t} \frac{\partial T}{\partial s}$, we have

$$\begin{aligned}
\frac{\partial}{\partial s} \frac{\partial T}{\partial t} &= \frac{\partial}{\partial s} \left[\left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) N + \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) B \right] \\
&= \left(\frac{\partial f}{\partial s} \kappa + f \frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} - \frac{\partial h}{\partial s} (\tau - \tau_G) - h \frac{\partial(\tau - \tau_G)}{\partial s} \right) N \quad (3.5) \\
&\quad + \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) (-\kappa T + (\tau - \tau_G) B) \\
&\quad + \left(\frac{\partial g}{\partial s} (\tau - \tau_G) + g \frac{\partial(\tau - \tau_G)}{\partial s} + \frac{\partial^2 h}{\partial s^2} \right) B - \\
&\quad \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) (\tau - \tau_G) N
\end{aligned} \quad (3.6)$$

and

$$\frac{\partial}{\partial t} \frac{\partial T}{\partial s} = \frac{\partial}{\partial t} (\kappa N) = \frac{\partial \kappa}{\partial t} N + \kappa \frac{\partial N}{\partial t}. \quad (3.7)$$

Therefore, we followed (3.5) and (3.7) that

$$\frac{\partial \kappa}{\partial t} = g\kappa^2 + f \frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} - 2 \frac{\partial h}{\partial s} (\tau - \tau_G) - h \frac{\partial(\tau - \tau_G)}{\partial s} - g(\tau - \tau_G)^2$$

and

$$\kappa\psi = (\tau - \tau_G) \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - \tau_G) \right) + \frac{\partial}{\partial s} (g(\tau - \tau_G)) + \frac{\partial^2 h}{\partial s^2}.$$

Since $\frac{\partial}{\partial s} \frac{\partial B}{\partial t} = \frac{\partial}{\partial t} \frac{\partial B}{\partial s}$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial s} \frac{\partial B}{\partial t} &= \frac{\partial}{\partial s} \left[- \left(g(\tau - \tau_G) + \frac{\partial h}{\partial s} \right) T - \psi N \right] \\
&= \left(- \frac{\partial g}{\partial s} (\tau - \tau_G) - g \frac{\partial(\tau - \tau_G)}{\partial s} - \frac{\partial^2 h}{\partial s^2} \right) T \\
&\quad + \left(-g(\tau - \tau_G) - \frac{\partial h}{\partial s} \right) \kappa N - \frac{\partial \psi}{\partial s} N - \psi (-\kappa T + (\tau - \tau_G) B)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t} \frac{\partial B}{\partial s} &= \frac{\partial}{\partial t} (-(\tau - \tau_G) N) \\
&= - \frac{\partial(\tau - \tau_G)}{\partial t} N - (\tau - \tau_G) \frac{\partial N}{\partial t}.
\end{aligned}$$

Hence

$$\frac{\partial(\tau - \tau_G)}{\partial t} = g\kappa(\tau - \tau_G) + \kappa \frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s}.$$

No other new formulas are obtained from the relation $\frac{\partial}{\partial s} \frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \frac{\partial N}{\partial s}$. \square

Theorem 3.5 includes the following special cases for system of partial differential equations:

(i) If G is Abelian group then $\tau_G = 0$. Thus we have that

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &= g\kappa^2 + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} - 2\frac{\partial h}{\partial s}\tau - h\frac{\partial \tau}{\partial s} - g\tau^2, \\ \kappa\psi &= \tau \left(f\kappa + \frac{\partial g}{\partial s} - h\tau \right) + \frac{\partial}{\partial s}(g\tau) + \frac{\partial^2 h}{\partial s^2}, \\ \frac{\partial \tau}{\partial t} &= g\kappa\tau + \kappa\frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s}.\end{aligned}$$

(ii) If G is SO^3 then $\tau_G = \frac{1}{2}$. Thus we have that

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &= g\kappa^2 + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} - 2\frac{\partial h}{\partial s}\left(\tau - \frac{1}{2}\right) - h\frac{\partial(\tau - \frac{1}{2})}{\partial s} - g\left(\tau - \frac{1}{2}\right)^2, \\ \kappa\psi &= \left(\tau - \frac{1}{2}\right) \left(f\kappa + \frac{\partial g}{\partial s} - h\left(\tau - \frac{1}{2}\right) \right) + \frac{\partial}{\partial s}\left(g\left(\tau - \frac{1}{2}\right)\right) + \frac{\partial^2 h}{\partial s^2}, \\ \frac{\partial(\tau - \frac{1}{2})}{\partial t} &= g\kappa\left(\tau - \frac{1}{2}\right) + \kappa\frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s}.\end{aligned}$$

(iii) If G is SU^2 then $\tau_G = 1$. Thus we have that

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &= g\kappa^2 + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} - 2\frac{\partial h}{\partial s}(\tau - 1) - h\frac{\partial(\tau - 1)}{\partial s} - g(\tau - 1)^2, \\ \kappa\psi &= (\tau - 1) \left(f\kappa + \frac{\partial g}{\partial s} - h(\tau - 1) \right) + \frac{\partial}{\partial s}(g(\tau - 1)) + \frac{\partial^2 h}{\partial s^2}, \\ \frac{\partial(\tau - 1)}{\partial t} &= g\kappa(\tau - 1) + \kappa\frac{\partial h}{\partial s} + \frac{\partial \psi}{\partial s}.\end{aligned}$$

4. CONCLUSION

The above noted special cases are showed us, the study is a generalization of inextensible flows of curves defined by Kwon [8] in Eculidean 3-space. Also, the similar conditions for inextensible flows of curves can investigate using the Lie group structure in different spaces or on other manifolds in addition to this study.

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