Fixed Point Approach To The Hyers-Ulam-Rassias Approximation Of Homomorphisms And Derivations On Non-Archimedean Random Lie $C^*$-Algebras

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Abstract. In this paper, using a fixed point method, we prove the generalized Hyers-Ulam stability of random homomorphisms in random $C^*$-algebras and random Lie $C^*$-algebras and of derivations on Non-Archimedean random $C^*$-algebras and Non-Archimedean random Lie $C^*$-algebras for the following $m$-variable additive functional equation:

$$
\sum_{i=1}^{m} f(x_i) = \frac{1}{2m} \left[ \sum_{i=1}^{m} f(mx_i + \sum_{j=1, j \neq i}^{m} x_j) + f\left(\sum_{i=1}^{m} x_i\right) \right].
$$

Keywords: Additive functional equation, fixed point, Non-Archimedean random space, homomorphism in $C^*$-algebras and Lie $C^*$-algebras, generalized Hyers-Ulam stability, derivation on $C^*$-algebras and Lie $C^*$-algebras

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations was originated from a question of Ulam [34] concerning about the stability of group homomorphisms: Let $(G_1, \ast)$ be a group and let $(G_2, \circ, d)$ be a metric group (a metric which is defined on a set with group property) with the metric

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1 Corresponding author: h.azadikenary@gmail.com
Received: 24 June 2012
Revised: 1 July 2012
Accepted: 16 July 2013
\(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta(\epsilon) > 0\) such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality \(d(h(x + y), h(x) \circ h(y)) < \delta\) for all \(x, y \in G_1\), then there is a homomorphism \(H : G_1 \to G_2\) with \(d(h(x), H(x)) < \epsilon\) for all \(x \in G_1\)? If the answer is affirmative, we would say that the equation of homomorphism \(H(x \circ y) = H(x) \circ H(y)\) is stable (see also [17, 22]).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([1]–[28]). By a *non-Archimedean field* we mean a field \(K\) equipped with a function (valuation) \(|\cdot|\) from \(K\) into \([0, \infty)\) such that \(|r| = 0\) if and only if \(r = 0\), \(|rs| = |r||s|\), and \(|r + s| \leq \max\{|r|, |s|\}\) for all \(r, s \in K\). Clearly \(|1| = |-1| = 1\) and \(|n| \leq 1\) for all \(n \in \mathbb{N}\). By the trivial valuation we mean the mapping \(|\cdot|\) taking everything but 0 into 1 and \(|0| = 0\). Let \(X\) be a vector space over a field \(K\) with a non-Archimedean non-trivial valuation \(|\cdot|\). A function \(\|\cdot\| : X \to [0, \infty)\) is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) \(\|x\| = 0\) if and only if \(x = 0\); (ii) for any \(r \in K, x \in X\), \(\|rx\| = |r|\|x\|\); (iii) the strong triangle inequality (ultrametric) holds; namely,

\[
\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).
\]

Then \((X, \|\cdot\|)\) is called a non-Archimedean normed space. From the fact that

\[
\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m),
\]

holds, a sequence \(\{x_n\}\) is Cauchy if and only if \(\{x_{n+1} - x_n\}\) converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number \(x\), there exists a unique integer \(n_x \in \mathbb{Z}\) such that \(x = \frac{a}{b}p^{n_x}\), where \(a\) and \(b\) are integers not divisible by \(p\). Then \(|x|_p := p^{-n_x}\) defines a non-Archimedean norm on \(\mathbb{Q}\). The completion of \(\mathbb{Q}\) with respect to the metric \(d(x, y) = |x - y|_p\) is denoted by \(\mathbb{Q}_p\), which is called the \(p\)-adic number field.

A non-Archimedean Banach algebra is a complete non-Archimedean algebra \(\mathcal{A}\) which satisfies \(\|ab\| \leq \|a\| \cdot \|b\|\) for all \(a, b \in \mathcal{A}\). For more detailed definitions of non-Archimedean Banach algebras, we provide the following studied for further reading [11, 33].

If \(\mathcal{U}\) is a non-Archimedean Banach algebra, then an involution on \(\mathcal{U}\) is a mapping \(t \to t^*\) from \(\mathcal{U}\) into \(\mathcal{U}\) which satisfies

(i) \(t^{**} = t\) for \(t \in \mathcal{U}\); (ii) \((\alpha s + \beta t)^* = \overline{\alpha} s^* + \overline{\beta} t^*\); (iii) \((st)^* = t^* s^*\) for \(s, t \in \mathcal{U}\).
If, in addition $\|t^*t\| = \|t\|^2$ for $t \in \mathcal{U}$, then $\mathcal{U}$ is a non-Archimedean $C^*$-algebra.

We recall a fundamental result in a fixed point theory. Let $\Omega$ be a set. A function $d : \Omega \times \Omega \rightarrow [0, \infty]$ is called a generalized metric on $\Omega$ if $d$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in \Omega$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \Omega$.

Theorem 1.1. Let $(\Omega, d)$ be a complete generalized metric space and let $J : \Omega \rightarrow \Omega$ be a contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $\Gamma = \{y \in \Omega | d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.

In this paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms and derivations in non-Archimedean random $C^*$-algebras and non-Archimedean random Lie $C^*$-algebras for the following additive functional equation (see [?])

$$\sum_{i=1}^{m} f(x_i) = \frac{1}{2m} \left[ \sum_{i=1}^{m} f \left( m x_i + \sum_{j=1, j \neq i}^{m} x_j \right) + f \left( \sum_{i=1}^{m} x_i \right) \right]$$

(1.1)

2. RANDOM SPACES

In the section, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [9, 31, 32]. Throughout this paper, $\Delta^+$ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \rightarrow [0, 1]$ such that $F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. $D^+$ is a subset of $\Delta^+$ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function $f$ at the point $x$, that is, $l^- f(x) = \lim_{t \rightarrow x^-} f(t)$. The space $\Delta^+$ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $\Delta^+$ in this order is the distribution function $\varepsilon_0$ given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (briefly, a continuous $t$-norm) if $T$ satisfies the
following conditions:
(a) $T$ is commutative and associative; (b) $T$ is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$; (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous $t$-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz $t$-norm).

**Definition 2.2.** ([32]) A non-Archimedean random normed space (briefly, NA-RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm and $\mu$ is a mapping from $X$ into $D^+$ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$; (RN2) $\mu_{ax}(t) = \mu_x\left(\frac{t}{|a|}\right)$ for all $x \in X$, $\alpha \neq 0$; (RN3) $\mu_{x+y}(t) \geq T(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t \geq 0$.

Every normed space $(X, \|\|)$ defines a non-Archimedean random normed space $(X, \mu, T_M)$, where $\mu_x(t) = \frac{t}{t + \|x\|}$ for all $t > 0$, and $T_M$ is the minimum $t$-norm. This space is called the induced random normed space.

**Definition 2.3.** ([32]) A non-Archimedean random normed algebra $(X, \mu, T, T')$ is a non-Archimedean random normed space $(X, \mu, T)$ with algebraic structure such that

(RN-4) $\mu_{xy}(t) \geq T'(\mu_x(t), \mu_y(t))$ for all $x, y \in X$ and all $t > 0$, in which $T'$ is a continuous $t$-norm.

Every non-Archimedean normed algebra $(X, \|\|)$ defines a non-Archimedean random normed algebra $(X, \mu, T_M)$, where $\mu_x(t) = \frac{t}{t + \|x\|}$ for all $t > 0$ if

$$\|xy\| \leq \|x\|\|y\| + t\|y\| + t\|x\|$$

$(x, y \in X; \ t > 0)$. This space is called the induced non-Archimedean random normed algebra.

**Definition 2.4.** (1) Let $(X, \mu, T_M)$ and $(Y, \mu, T_M)$ be non-Archimedean random normed algebras. An $\mathbb{R}$-linear mapping $f : X \rightarrow Y$ is called a homomorphism if

$$f(xy) = f(x)f(y)$$

for all $x, y \in X$.

(2) An $\mathbb{R}$-linear mapping $f : X \rightarrow X$ is called a derivation if

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in X$. 
Definition 2.5. Let \((U, \mu, T, T')\) be a non-Archimedean random Banach algebra, then an involution on \(U\) is a mapping \(u \rightarrow u^*\) from \(U\) into \(U\) which satisfies

(i) \(u^{**} = u\) for \(u \in U\); (ii) \((\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*\); (iii) \((uv)^* = v^* u^*\)
for \(u, v \in U\).

If, in addition \(\mu_{u^* u}(t) = T'(\mu_u(t), \mu_u(t))\) for \(u \in U\) and \(t > 0\), then \(U\) is a non-Archimedean random \(C^*\)-algebra.

Definition 2.6. Let \((X, \mu, T)\) be an NA-RN-space.

(1) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to \(x\) in \(X\) if, for every \(\epsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N\) such that \(\mu_{x_n - x}(\epsilon) > 1 - \lambda\) whenever \(n \geq N\).

(2) A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if, for every \(\epsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N\) such that \(\mu_{x_n - x_m}(\epsilon) > 1 - \lambda\) whenever \(n \geq m \geq N\).

(3) An RN-space \((X, \mu, T)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent to a point in \(X\).

3. Stability of homomorphisms and derivations in non-Archimedean random \(C^*\)-algebras

Throughout this section, assume that \(\mathcal{A}\) is a non-Archimedean random \(C^*\)-algebra with norm \(\mu^A\) and that \(\mathcal{B}\) is a non-Archimedean random \(C^*\)-algebra with norm \(\mu^B\).

Theorem 3.1. Let \(V\) and \(W\) be real vector spaces. A mapping \(f : V \rightarrow W\) with \(f(0) = 0\) satisfies in the following functional equation

\[
\sum_{i=1}^{m} f(x_i) = \frac{1}{2m} \left[ \sum_{i=1}^{m} f \left( mx_i + \sum_{j=1, j \neq i}^{m} x_j \right) + f \left( \sum_{i=1}^{m} x_i \right) \right]
\]

if and only if \(f\) is additive.

For a given mapping \(f : \mathcal{A} \rightarrow \mathcal{B}\), we define

\[
D_{\mu} f(x_1, ..., x_m) := \sum_{i=1}^{m} \mu f(x_i) - \frac{1}{2m} \sum_{i=1}^{m} f \left( \mu mx_i + \sum_{j=1, j \neq i}^{m} x_j \right) + \mu f \left( \sum_{i=1}^{m} \mu x_i \right)
\]

for a fixed positive integer \(m\) with \(m \geq 2\) and for all \(\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}\) and all \(x_1, ..., x_m \in \mathcal{A}\).
Note that a $C$-linear mapping $H : A \to B$ is called a $^\ast$-homomorphism in non-Archimedean random $C^\ast$-algebras if $H$ satisfies

$$H(xy) = H(x)H(y) \quad \text{and} \quad H(x^\ast) = H(x)^\ast$$

for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random $C^\ast$-algebras for the functional equation $D_\lambda f(x_1, \ldots, x_m) = 0$.

**Theorem 3.2.** Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : A^m \to D^+, \psi : A^2 \to D^+$ and $\eta : A \to D^+$ such that $|m| < 1$ is far from zero and

$$\mu^B_{D_\lambda f(x_1, \ldots, x_m)}(t) \geq \varphi_{x_1, \ldots, x_m}(t), \quad (3.1)$$

$$\mu^B_{f(xy) - f(x)f(y)}(t) \geq \psi_{x,y}(t), \quad (3.2)$$

$$\mu^B_{f(x^\ast) - f(x)^\ast}(t) \geq \eta_x(t), \quad (3.3)$$

for all $\lambda \in T^1$, all $x_1, \ldots, x_m, x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that $|m| < 1$ is far from zero and

$$\varphi_{mx_1, \ldots, mx_m}(|m|Lt) \geq \varphi_{x_1, \ldots, x_m}(t), \quad (3.4)$$

$$\psi_{mx,my}(|m|^2Lt) \geq \psi_{x,y}(t), \quad (3.5)$$

$$\eta_{mx}(|m|Lt) \geq \eta_x(t), \quad (3.6)$$

for all $x, y, x_1, \ldots, x_m \in A$ and $t > 0$, then there exists a unique random $^\ast$-homomorphism $H : A \to B$ such that

$$\mu^B_{f(x) - H(x)}(t) \geq \varphi_{x,0,\ldots,0}((|m| - |m|L)t) \quad (3.7)$$

for all $x \in A$ and $t > 0$.

**Proof.** It follows from (3.4), (3.5), (3.6) and $L < 1$ that

$$\lim_{n \to \infty} \varphi_{mx_1, \ldots, mx_m}(|m|^n t) = 1, \quad (3.8)$$

$$\lim_{n \to \infty} \psi_{mx,my}(|m|^{2n} t) = 1, \quad (3.9)$$

$$\lim_{n \to \infty} \eta_{mx}(|m|^n t) = 1, \quad (3.10)$$

for all $x, y, x_1, \ldots, x_m \in A$ and $t > 0$. Let us define $\Omega$ to be the set of all mappings $g : A \to B$ and introduce a generalized metric on $\Omega$ as follows:

$$d(g,h) = \inf\{k \in (0, \infty) : \mu^B_{g(x) - h(x)}(kt) > \varphi_{x,0,\ldots,0}(t), \forall x \in A, t > 0\}.$$
It is easy to show that \((\Omega, d)\) is a generalized complete metric space (see [S]). Now we consider the function \(J : \Omega \rightarrow \Omega\) defined by \(Jg(x) = \frac{1}{m}g(mx)\) for all \(x \in A\) and \(g \in \Omega\). Note that for all \(g, h \in \Omega\) we have
\[
d(g, h) < k \implies \mu^B(g(x) - h(x))(kt) > \phi_{x,0,...,0}(t) \implies \mu^B(\frac{1}{m}g(mx) - \frac{1}{m}h(mx))(kt)
\]
> \phi_{x,0,...,0}(\langle m | t) \implies \mu^B(\frac{1}{m}g(mx) - \frac{1}{m}h(mx))(kL)
\]
> \phi_{x,0,...,0}(t) \implies d(Jg, Jh) < kL.

From this it is easy to see that \(d(Jg, Jk) \leq Ld(g, h)\) for all \(g, h \in \Omega\), that is, \(J\) is a self-function of \(\Omega\) with the Lipschitz constant \(L\). Putting \(\mu = 1\), \(x = x_1\) and \(x_2 = x_3 = ... = x_m = 0\) in (3.1), we have \(\mu^B(\frac{1}{m}x,0,...,0(t) \geq \phi_{x,0,...,0}(t)\) for all \(x \in A\) and \(t > 0\). Then
\[
\mu^B(f(x) - \frac{1}{m}f(mx))(t) \geq \phi_{x,0,...,0}(\langle m | t)
\]
for all \(x \in A\) and \(t > 0\), that is, \(d(Jf, f) \leq \frac{1}{m}\) \(< \infty\). Now, from the fixed point alternative, it follows that there exists a fixed point \(H\) of \(J\) in \(\Omega\) such that
\[
H(x) = \lim_{n \to \infty} |m|^{-n}f(m^n x)
\]
for all \(x \in A\), since \(\lim_{n \to \infty} d(J^n f, H) = 0\). On the other hand, it follows from (3.1), (3.8), and (3.11) that
\[
\mu^B(\frac{1}{m}\phi_{m^n x_1,...,m^n x_m}(\langle m | n t) = 1
\]
By a similar method noted above, we get \(\lambda H(mx) = H(m\lambda x)\) for all \(\lambda \in \mathbb{T}^1\) and all \(x \in A\). Thus one can show that the mapping \(H : A \to B\) is \(C\)-linear. It follows from (3.2), (3.9) and (3.11) that
\[
\mu^B_H(\frac{1}{m}\phi_{m^n x_1,...,m^n x_m}(\langle m | n t) = 1
\]
for all \(x, y \in A\). So \(H(xy) = H(x)H(y)\) for all \(x, y \in A\). Thus \(H : A \to B\) is a homomorphism, satisfying (3.7) intended. Also by (3.3), (3.10), (3.11) and by a similar method, we have \(H(x^*) = H(x)^*\).

**Corollary 3.3.** Let \(r > 1\) and \(\theta\) be nonnegative real numbers, and let \(f : A \to B\) be a mapping with \(f(0) = 0\) such that
\[
\mu^B_{D_\lambda f(x_1,...,x_m)}(t) \geq \frac{t}{t + \theta \cdot (\|x_1\|^r_A + \|x_2\|^r_A + \cdots + \|x_m\|^r_A)},
\]
\[
\mu^B_{f(xy)-f(x)f(y)}(t) \geq \frac{t}{t + \theta \cdot (\|x\|^r_A \cdot \|y\|^r_A)}, \quad \mu^B_{f(x^*)-f(x)^*}(t) \geq \frac{t}{t + \theta \cdot \|x\|^r_A}
\]

\[\Box\]
for all $\lambda \in \mathbb{T}_1$, all $x_1, \ldots, x_m, x, y \in A$ and $t > 0$. Then there exists a unique $^*-$homomorphism $H : A \to B$ such that

$$\mu^B_{f(x)-H(x)}(t) \geq \frac{(|m| - |m|^r)t}{(|m| - |m|^r) + \theta|m| - |m|^r\|x\|_A^r}$$

for all $x \in A$ and $t > 0$.

**Proof.** The proof follows from Theorem 3.2 by taking

$$\varphi_{x_1, \ldots, x_m}(t) = \frac{t}{t + \theta \cdot (\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)},$$

$$\psi_{x,y}(t) := \frac{t}{t + \theta \cdot (\|x\|_A^r \cdot \|y\|_A^r)},$$

$$\eta_x(t) = \frac{t}{t + \theta \cdot \|x\|_A^r}$$

for all $x_1, \ldots, x_m, x, y \in A$, $L = |m|^{r-1}$ and $t > 0$, we get the desired result. $\square$

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random $C^*$-algebras for the functional equation $D_{\lambda}f(x_1, \ldots, x_m) = 0$.

**Theorem 3.4.** Let $f : A \to A$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : A^m \to D^+$, $\psi : A^2 \to D^+$ and $\eta : A \to D^+$ such that $|m| < 1$ is far from zero and

$$\mu^A_{D_{\lambda}f(x_1, \ldots, x_m)}(t) \geq \varphi_{x_1, \ldots, x_m}(t), \mu^A_{f(x+y)-f(x)-f(y)}(t) \geq \psi_{x,y}(t),$$

$$\mu^A_{f(x^*)-f(x)}(t) \geq \eta_x(t),$$

for all $\lambda \in \mathbb{T}_1$ and all $x_1, \ldots, x_m, x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that (3.4), (3.5) and (3.6) hold, then there exists a unique derivation $\delta : A \to A$ such that

$$\mu^A_{f(x)-\delta(x)}(t) \geq \varphi_{x,0,\ldots,0}(|m| - |m|L)t$$

for all $x \in A$ and $t > 0$.

4. STABILITY OF HOMOMORPHISMS AND DERIVATIONS IN NON-ARCHIMEDEAN LIE $C^*$-ALGEBRAS

A non-Archimedean random $C^*$-algebra $C$, endowed with the Lie product $[x, y] := \frac{xy - yx}{2}$ on $C$, is called a Lie non-Archimedean random $C^*$-algebra.

**Definition 4.1.** Let $A$ and $B$ be random Lie $C^*$-algebras. A $C$-linear mapping $H : A \to B$ is called a non-Archimedean Lie $C^*$-algebra homomorphism if $H([x,y]) = [H(x), H(y)]$ for all $x, y \in A$. 
Throughout this section, assume that $\mathcal{A}$ is a non-Archimedean random Lie $C^*$-algebra with norm $\mu^\mathcal{A}$ and that $\mathcal{B}$ is a non-Archimedean random Lie $C^*$-algebra with norm $\mu^\mathcal{B}$.

We prove the generalized Hyers-Ulam stability of homomorphisms in non-Archimedean random Lie $C^*$-algebras for the functional equation $D_\lambda f(x_1, \ldots, x_m) = 0$.

**Theorem 4.2.** Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : \mathcal{A}^m \to D^+$ and $\psi : \mathcal{A}^2 \to D^+$ such that (3.1) and (3.3) hold and

$$\mu^\mathcal{B}_f([x,y]) - [f(x), f(y)](t) \geq \psi_{x,y}(t)$$

for all $\lambda \in \mathbb{T}$, all $x, y \in \mathcal{A}$ and $t > 0$. If there exists an $L < 1$ and (3.4), (3.5) and (3.6) hold, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that (3.7) hold.

**Proof.** By the same reasoning as in the proof of Theorem 3.2, we can find the mapping $H : \mathcal{A} \to \mathcal{B}$ given by

$$H(x) = \lim_{n \to \infty} |m|^{-n} f(m^n x)$$

(4.2)

for all $x \in \mathcal{A}$. It follows from (3.5) and (4.2) that

$$\mu^\mathcal{B}_H([x,y]) - [H(x), H(y)](t) = \lim_{n \to \infty} \mu^\mathcal{B}_f(m^{2n}[x,y] - [f(m^n x), f(m^n y)])(|m|^{2n} t) \geq \lim_{n \to \infty} \psi_{m^n x, m^n y}(|m|^{2n} t) = 1$$

for all $x, y \in \mathcal{A}$ and $t > 0$, then $H([x, y]) = [H(x), H(y)]$ for all $x, y \in \mathcal{A}$. Thus $H : \mathcal{A} \to \mathcal{B}$ is a Lie $C^*$-algebra homomorphism satisfying (3.7), as intended. $\square$

**Corollary 4.3.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ such that

$$\mu^\mathcal{B}_f(x_1, \ldots, x_m)(t) \geq \frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}$$

$$\mu^\mathcal{B}_f([x,y]) - [f(x), f(y)](t) \geq \frac{t}{t + \theta \cdot \|x\|_A^r}$$

for all $\lambda \in \mathbb{T}$, all $x_1, \ldots, x_m, x, y \in \mathcal{A}$ and $t > 0$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\mu^\mathcal{B}_f(x) - H(x)(t) \geq \frac{(|m| - |m|^r) t}{(|m| - |m|^r) t + \theta \cdot \|x\|_A^r}$$

for all $x \in \mathcal{A}$ and $t > 0$.

**Proof.** The proof follows from Theorem 4.2 and a method similar to Corollary 3.3. $\square$
Definition 4.4. Let $\mathcal{A}$ be a non-Archimedean random $C^*$-algebra. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a Lie derivation if $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$.

We prove the generalized Hyers-Ulam stability of derivations on non-Archimedean random $C^*$-algebras for the functional equation $D_\lambda f(x_1, \ldots, x_m) = 0$.

Theorem 4.5. Let $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ for which there are functions $\varphi : \mathcal{A}^m \to \mathbb{D}^+$ and $\psi : \mathcal{A}^2 \to \mathbb{D}^+$ such that (3.1) and (3.3) hold and
\[
\mu^A_{|f([x,y])-[f(x),y]-[x,f(y)]|}(t) \geq \psi_{x,y}(t),
\]
for all $x, y \in \mathcal{A}$. If there exists an $L < 1$ and (3.4), (3.5) and (3.6) hold, then there exists a unique Lie derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that such that (3.7) hold.

Proof. By the same reasoning as the proof of Theorem 4.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3.7); the mapping $\delta : \mathcal{A} \to \mathcal{A}$ is given by
\[
\delta(x) = \lim_{n \to \infty} |m|^{-n} f(m^n x)
\]
for all $x \in \mathcal{A}$. It follows from (3.5) and (4.4) that
\[
\mu^A_{|\delta([x,y])-[\delta(x),y]-[x,\delta(y)]|}(t) = \lim_{n \to \infty} \mu^A_{|f(m^n[x,y])-[f(m^n x),m^n y]-[m^n x,f(m^n y)]|}(|m|^{2n} t)
\]
\[
\geq \lim_{n \to \infty} \psi_{m^n x,m^n y}(|m|^{2n} t) = 1
\]
for all $x, y \in \mathcal{A}$ and $t > 0$, then $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$. Thus $\delta : \mathcal{A} \to \mathcal{A}$ is a Lie derivation satisfying (3.7). \□

References


