Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2026.31072.1810
Caspian J Math Sci. **15**(2)(2026), 489-502 (Research Article)

# Rings such that, for each unit $u$, $u^n - 1$ belongs to the $\sqrt{J(R)}$

G.Karamali [1] and O.Hasanzadeh [2]
[1,2] Department of Basic Sciences, Shahid Sattari Aeronautical University of Science and Technology, South Mehrabad, Tehran, Iran.

Abstract. We introduce and systematically study the new class of $n$-$\sqrt{JU}$ rings, where a ring $R$ is called $n$-$\sqrt{JU}$ if for any unit $u \in U(R)$, $u^n - 1$ belongs to $\sqrt{J(R)}$ for some positive integer $n \geq 2$. This class provides a natural unified generalization of $n$-$UU$, $n$-$UJ$, and $\sqrt{JU}$ rings. Key structural properties are investigated, including stability under homomorphic images, direct products, and corner rings. Furthermore, we obtain complete characterizations for regular odd-$n$-$\sqrt{JU}$ rings, for division rings and fields, and for semi-local rings. Finally, we examine the persistence of this property in various ring extensions, including skew (formal) power series rings, polynomial rings over 2-primal rings, and several matrix constructions such as triangular and generalized matrix rings.

Keywords: $n$-$\sqrt{JU}$ ring, $\sqrt{JU}$ ring, (semi-)regular ring, clean ring.

*2000 Mathematics subject classification:* 16S34, 16U60.

[1]Corresponding author: rezakaramali918@gmail.com

[2]hasanzadeomiid@gmail.com
Received: 21 January 2026
Revised: 11 April 2026
Accepted: 13 April 2026
**How to Cite:** Karamali, Gholamreza; Hasanzadeh, Omid. Rings such that, for each unit $u$, $u^n - 1$ belongs to the $\sqrt{J(R)}$. Casp.J. Math. Sci.,**15**(2)(2026), 489-502.
This work is licensed under a Creative Commons Attribution 4.0 International License.
Copyright © 2026 by University of Mazandaran. Subbmited for possible open access publication under the terms and conditions of the Creative Commons Attribution(CC BY) license(https://craetivecommons.org/licenses/by/4.0/)

## 1. Introduction and Motivation

In this paper, we consider an associative ring $R$ with identity, not assumed to be commutative. For such a ring, we use the notation $U(R)$, $Nil(R)$, $C(R)$, and $Id(R)$ to denote the sets of units, nilpotent elements, central elements, and idempotent elements of $R$, respectively. The Jacobson radical of $R$ is denoted by $J(R)$. Moreover, we write $T_n(R)$ for the rings of all upper triangular $n \times n$ matrices over $R$. A ring is said to be abelian if every idempotent element of the ring is central.

A ring $R$ is called $n$-UJ if for every unit $u \in U(R)$ one has $u - u^n \in J(R)$, where $n \geq 2$ is a fixed integer; equivalently, $u^n - 1 \in J(R)$ for all $u \in U(R)$. This notion was first introduced by Danchev in 2019 [5], and independently by Koşan et al. in 2020 [11]. Note that for $n = 1$ these rings generalize the so-called JU rings, originally defined by Danchev in 2016 [3] and later reformulated under the name UJ rings in [10]. It was further shown that, for $(2n)$-UJ rings, the classes of semi-regular, exchange, and clean rings all coincide.

In a related direction, as observed in [8], a ring $R$ is called a UU-ring if every unit differs from $1_R$ by a nilpotent element; equivalently, $U(R) = 1 + Nil(R)$.

Analogously, for a fixed integer $n \geq 2$, a ring $R$ is called $n$-UU if, for every unit $u \in U(R)$, one has $u^n - 1 \in Nil(R)$. This notion was introduced by Danchev (see [4]) and was further examined in more detail in [7]. It was shown in [7] that a ring $R$ is $(n-1)$-UU and strongly $\pi$-regular if and only if $R$ is strongly $n$-nil-clean. Hence, $n$-UU rings form a natural generalization of UU-rings.

In [21], the set

$$\sqrt{J(R)} = \{\, z \in R : z^n \in J(R), \text{ for some } n \geq 1 \,\}$$

was introduced as a subset of $R$, which is not necessarily a subring and properly contains $J(R)$. It is also evident that $Nil(R)$ is a proper subset of $\sqrt{J(R)}$.

Recently, in [19] the authors defined a ring to be a $\sqrt{J}U$-ring provided that $U(R) = 1 + \sqrt{J(R)}$. The class of $\sqrt{J}U$-rings properly contains both $UU$-rings and $JU$-rings, and thus forms a natural generalization of these two classes.

The purpose of this paper is to investigate a new class of rings, called $n$-$\sqrt{J}U$ rings, defined by the condition that for each $u \in U(R)$ one has $u^n - 1 \in \sqrt{J(R)}$, where $n \geq 2$ is a fixed integer. It is clear that when $n = 1$ this definition recovers the class of $\sqrt{J}U$-rings. Hence, $n$-$\sqrt{J}U$ rings naturally generalize the classes of $\sqrt{J}U$, $UU$, $JU$, $n$-$UU$, and $n$-$UJ$ rings.

A ring $R$ is said to be regular (resp., unit-regular) in the sense of von Neumann if, for every $a \in R$, there exists $x \in R$ (resp., $x \in U(R)$) such that $axa = a$. The ring $R$ is said to be strongly regular if, for every $a \in R$, one has $a \in a^2R$.

Recall that a ring $R$ is called an exchange ring if, for each $a \in R$, there exists an idempotent $e \in aR$ with $e^2 = e$ such that $1 - e \in (1 - a)R$. A ring is said to be clean if every element of $R$ is a sum of an idempotent and a unit (cf. [17]). It is well known that every clean ring is exchange, whereas the converse is generally not true; however, it holds in the abelian case (see [17, Proposition 1.8]).

In this aspect, a ring $R$ is called semi-regular provided that $R/J(R)$ is regular and idempotents lift modulo $J(R)$. Semi-regular rings are always exchange rings, but the converse is typically false (see, for instance, [17]).

Let $\alpha : R \to R$ be a ring endomorphism. The ring $R[x; \alpha]$ denotes the skew polynomial ring over $R$, where multiplication is defined by the rule $xr = \alpha(r)x$ for all $r \in R$. In particular, when $\alpha = 1_R$, we recover the ordinary polynomial ring $R[x] = R[x; 1_R]$. Likewise, $R[[x; \alpha]]$ denotes the skew formal power series ring over $R$, consisting of all formal power series in the indeterminate $x$ with coefficients from $R$, with multiplication determined by $xr = \alpha(r)x$ for all $r \in R$. In the special case $\alpha = 1_R$, we obtain the usual formal power series ring $R[[x]] = R[[x; 1_R]]$.

Let $Nil_*(R)$ denote the prime radical (also called the lower nil-radical) of a ring $R$, i.e., the intersection of all prime ideals of $R$. It is well known that $Nil_*(R)$ is a nil-ideal of $R$. A ring $R$ is called 2-primal if its lower nil-radical coincides with the set of all nilpotent elements of $R$. In particular, both reduced and commutative rings are 2-primal.

Recall also that a ring $R$ is said to be Dedekind-finite if, for any $a, b \in R$, the equality $ab = 1$ implies $ba = 1$. Equivalently, all one-sided inverses in the ring are automatically two-sided.

Our principal work in what follows is planned thus: In Section 2, we lay the foundation by examining basic properties of $n$-$\sqrt{J}U$ rings stability under direct products, homomorphic images, and corner rings (see Proposition 2.4) and show, notably, that any odd-$n$-$\sqrt{J}U$ ring must be Dedekind-finite (Proposition 2.12). Section 3 is devoted to our central findings. We first establish a key bridge: $R$ is $n$-$\sqrt{J}U$ precisely when $R/J(R)$ is an $n$-$UU$ ring. This equivalence unlocks several structural characterizations: we describe completely which division rings, regular rings, and semi-local rings belong to this class (Theorem 3.1). In particular, for odd $n$, the exchange, clean, and semi-regular properties coincide within $n$-$\sqrt{J}U$ rings (Theorem 3.1). Finally, we explore how the property behaves across familiar ring constructions, including skew

power series rings, skew polynomial rings over 2-primal rings, and various matrix rings (triangular, Morita context, etc.) see Proposition 3.8, Theorem 3.10, Proposition 3.12 and 3.13.

## 2. Fundamental Structural Properties $n$-$\sqrt{J}U$ rings

We begin this section by restating the main definition introduced earlier, in order to emphasize its central role in the development of our results.

**Definition 2.1.** A ring $R$ is called $n$-$\sqrt{J}U$ if, for each $u \in U(R)$, $u^n - 1 \in \sqrt{J(R)}$, where $n \geq 2$ is a fixed integer.

It is easy to see that if $n = 1$, the ring reduces to a $\sqrt{J}U$ ring.

Below we present an example of a ring that is $n$-$\sqrt{J}U$ but not $n$-$UJ$. This demonstrates that the newly introduced class properly contains the class of all $n$-$UJ$ rings.

**Example 2.2.** (1) Let $R$ be the $\mathbb{F}_2$-algebra generated by $x, y$ with the only relation $x^2 = 0$. Then, $U(R) = 1 + \mathbb{F}_2 x + xRx$, so $R$ is an $UU$ ring by [8, Example 2.5] and hence is $n$-$UU$ ring. Since $Nil(R) \subseteq \sqrt{J(R)}$, we get that $R$ is a $n$-$\sqrt{J}U$ ring. Moreover, $J(R) = 0$, so $R$ is not a $n$-$UJ$ ring.

(2) Every $n$-$UU$ ring is also $n$-$\sqrt{J}U$, but again, the converse is not valid in general. For example, the ring $R = \mathbb{F}_2[[x]]$ is a $\sqrt{J}U$ ring and hence it is 3-$\sqrt{J}U$, but it is not 3-$UU$ since its Jacobson radical $J(R)$ is not nil (see [7, Proposition 2.12]).

**Lemma 2.3.** *Let $R$ be a ring.*

1. *If $a \in \sqrt{J(R)}$, and $b \in J(R)$, then $a + b \in \sqrt{J(R)}$.*
2. *If $a \in \sqrt{J(R)}$ and $b \in R$ with $ab = ba$, then $ab \in \sqrt{J(R)}$.*
3. *For every $a \in R$ and $n \in \mathbb{N}$, we have $a^n \in \sqrt{J(R)}$ if and only if $a \in \sqrt{J(R)}$.*

**Proposition 2.4.**

1. *Let $R$ be a $n$-$\sqrt{J}U$ ring, where $n$ is an odd number. Then, $2 \in \sqrt{J(R)}$.*
2. *A direct product $\prod_{i \in I} R_i$ of rings $R_i$ is $n$-$\sqrt{J}U$ if, and only if, each direct component $R_i$ is $n$-$\sqrt{J}U$.*
3. *Let $R$ be an $n$-$\sqrt{J}U$ ring. If $T$ is an epimorphic image of $R$ such that all units of $T$ lift to units of $R$, then $T$ is $n$-$\sqrt{J}U$.*
4. *Let $R$ be a $n$-$\sqrt{J}U$ ring and let $e$ be an idempotent of $R$. Then, $eRe$ is a $n$-$\sqrt{J}U$ ring.*

*Proof.* (1) It is clear.

(2) By the equalities $\sqrt{J(\prod_{i\in I} R_i)} = \prod_{i\in I}\sqrt{J(R_i)}$ by [19, Proposition 2.2] and $U(\prod_{i\in I} R_i) = \prod_{i\in I} U(R_i)$ are fulfilled, the result follows at once.

(3) Suppose that $f : R \to T$ is a ring epimorphism. Let $v \in U(T)$. Then, there exists $u \in U(R)$ such that $v = f(u)$ and $u^n = 1+j \in 1+\sqrt{J(R)}$. Thus, we have

$$v^n = (f(u))^n = f(u^n) = f(1+j) = f(1) + f(j) \in 1 + \sqrt{J(T)},$$

as asked for.

(4) Let $u \in U(eRe)$. Thus, $u + (1-e) \in U(R)$. By hypothesis,

$$(u + (1-e))^n = u^n + (1-e) = 1 + j \in 1 + \sqrt{J(R)}.$$

So, we have $u^n - e \in \sqrt{J(R)} \cap eRe = \sqrt{J(eRe)}$ by [19, Proposition 2.11].

□

**Proposition 2.5.** *A division ring $D$ is $n$-$\sqrt{J}U$ if, and only if, $u^n = 1$ for every $u \in U(D)$.*

*Proof.* This follows immediately from the fact that, for any division ring $D$, $\sqrt{J(D)} = 0$. □

**Lemma 2.6.** *Suppose $\mathbb{F}$ is a field. Then, $\mathbb{F}$ is $n$-$\sqrt{J}U$ if, and only if, $\mathbb{F}$ is finite and $(|\mathbb{F}| - 1)|n$.*

*Proof.* Since $\mathbb{F}$ is a field, $\sqrt{J(\mathbb{F})} = 0$. Thus, $\mathbb{F}$ is $n$-$\sqrt{J}U$ if and only if $u^n = 1$ for all $u \in \mathbb{F}^* = F - \{0\}$. Assume that $\mathbb{F}$ is $n$-$\sqrt{J}U$. Then every element of $\mathbb{F}^*$ is a root of $x^n - 1$, so $|\mathbb{F}^*| \leq n$ and $\mathbb{F}$ is finite. Since $\mathbb{F}^*$ is cyclic, for a generator $a$ we have $a^n = 1$, and hence

$$|\mathbb{F}^*| = |\mathbb{F}| - 1 \mid n.$$

Conversely, if $\mathbb{F}$ is finite and $(|\mathbb{F}| - 1) \mid n$, then $u^{|\mathbb{F}|-1} = 1$ for all $u \in \mathbb{F}^*$, which implies $u^n = 1$. Therefore, $\mathbb{F}$ is $n$-$\sqrt{J}U$. □

**Lemma 2.7.** *Let $D$ be a division ring and $n \geq 2$. If $D$ is $n$-$\sqrt{J}U$, then $D$ is a finite field and $(|D| - 1)|n$.*

*Proof.* Certainly, $\sqrt{J(D)} = 0$. Hence, for every $a \in D^* = D - \{0\}$, we have $a^n = 1$, and consequently $a = a^{n+1}$. By the celebrated Jacobson's Theorem [14, Theorem 12.10], it follows that $D$ is commutative. Therefore, $D$ is a field, as expected. The second assertion follows immediately from Lemma 2.6. □

**Example 2.8.** Consider the ring $\mathbb{Z}$. Since $U(\mathbb{Z}) = \{1, -1\}$, we have $\sqrt{J(\mathbb{Z})} = 0$. Thus, $\mathbb{Z}$ is an $n$-$\sqrt{J}U$ ring. However, for any prime $p$, the ring $\mathbb{Z}_p$ is not $n$-$\sqrt{J}U$ for every $n$, unless $p-1$ divides $n$ by Lemma 2.6.

**Theorem 2.9.** *Let $I \subseteq J(R)$ be an ideal of a ring $R$. Then, $R$ is $n$-$\sqrt{J}U$ if, and only if, so is $R/I$.*

*Proof.* Let $R$ be $n$-$\sqrt{J}U$ and $u + I \in U(R/I)$. Then $u \in U(R)$ and $u^n = 1 + j$ for some $j \in \sqrt{J(R)}$. So $(u+I)^n = u^n + I = (1+I) + (j+I)$ where $j + I \in \sqrt{J(R)}/I = \sqrt{J(R/I)}$ by [19, Proposition 2.3].

Conversely, let $R/I$ be $n$-$\sqrt{J}U$ and $u \in U(R)$. Then $u + I \in U(R/I)$ so $(u + I)^n = (1 + I) + (j + I)$ for some $j + I \in \sqrt{J(R/I)}$. Hence $u^n + I = (1 + j) + I$ and $u^n - (1 + j) \in I \subseteq J(R)$. Thus $u^n = 1 + j'$ for some $j' \in \sqrt{J(R)}$. □

**Proposition 2.10.** *For any ring $R \neq \{0\}$ and any integer $n \geq 2$, the ring $M_n(R)$ is not a $(2k-1)$-$\sqrt{J}U$ ring whenever $k \geq 1$.*

*Proof.* Since $M_2(R)$ is isomorphic to a corner ring of $M_n(R)$ for $n \geq 2$, it suffices to show that $M_2(R)$ is not a $(2k-1)$-$\sqrt{J}U$. To this goal, consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in U(M_2(R)).$$

Thus, $A^{2k-1} = A$ or $A^{2k-1} = -A$. Now, let $M_2(R)$ is $(2k-1)$-$\sqrt{J}U$. If firstly $A^{2k-1} = A$, then we conclude that

$$B = A - I = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \in \sqrt{J(M_2(R))}.$$

But, we know that $B$ is a unit and this is a contradiction by [19, Proposition 2.1].

If now $A^{2k-1} = -A$, it can be concluded that $I \in \sqrt{J(M_2(R))}$ and again this is a contraposition. So, $M_2(R)$ is really not a $(2k-1)$-$\sqrt{J}U$ ring, as desired. □

**Example 2.11.** Consider $R = M_2(\mathbb{Z}_2)$. We have

$$U(R) = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\}.$$

A simple calculation shows that for any $u \in U(R)$, $u^6 - 1 \in \sqrt{J(R)}$. Hence $R$ is a 6-$\sqrt{J}U$ ring. In general, $M_n(R)$ $(n \geq 2)$ is not $n$-$\sqrt{J}U$ when $n$ is odd. However, this is not true in general for even $n$.

A set of nonzero elements $\{e_{ij} \mid 1 \leq i, j \leq n\}$ in a ring $R$ is called a system of matrix units if

$$e_{ij}e_{st} = \delta_{js}e_{it} \quad \text{for all } i, j, s, t,$$

where $\delta_{js}$ is 1 if $j = s$ and 0 otherwise. If we set $e := \sum_{i=1}^{n} e_{ii}$, then $e$ is an idempotent. Also, the ring $eRe$ is basically a matrix ring $M_n(S)$,

where

$$S = \{r \in eRe \mid re_{ij} = e_{ij}r \text{ for all } i, j\}.$$

**Proposition 2.12.** *Let $k \geq 1$. Every $(2k-1)$-$\sqrt{JU}$ ring is Dedekind-finite.*

*Proof.* Suppose $R$ is not Dedekind-finite. Then there exist $a, b \in R$ with $ab = 1$ but $ba \neq 1$. Set $e_{ij} = a^i(1 - ba)b^j$ and $e = \sum_{i=1}^{n} e_{ii}$. Then $eRe \cong M_n(S)$ for some nonzero ring $S$. By Proposition 2.4(4), $eRe$ is $(2k-1)$-$\sqrt{JU}$, so $M_n(S)$ is too, which contradicts Proposition 2.10. Hence, $R$ is Dedekind-finite. □

Recall that a ring $R$ is called semi-local if $R/J(R)$ is left Artinian, or equivalently, if $R/J(R)$ is semisimple.

**Proposition 2.13.** *Let $R$ be a ring and $n \geq 1$. Then, the following two conditions are equivalent for a semi-local ring:*

(1) *$R$ is a $(2n-1)$-$\sqrt{JU}$ ring.*
(2) *$R/J(R) \cong \prod_{i=1}^{m} \mathbb{F}_{p^{k_i}}$, where $(p^{k_i} - 1)|n$ and $\mathbb{F}_{p^{k_i}}$ is a field with $p^{k_i}$ elements.*

*Proof.* (1) $\Longrightarrow$ (2) Since $R$ is semi-local, $R/J(R)$ is semisimple, so we can write

$$R/J(R) \cong \prod_{i=1}^{m} M_{n_i}(D_i),$$

where each $D_i$ is a division ring. Using Theorem 2.9 and Proposition 2.10, we get

$$R/J(R) \cong \prod_{i=1}^{m} D_i.$$

Then, by Lemma 2.7, each $D_i \cong \mathbb{F}_{p^{k_i}}$ with $p^{k_i} - 1$ dividing $n$, as claimed.

(2) $\Longrightarrow$ (1) By Lemma 2.6, each $\mathbb{F}_{p^{k_i}}$ is $(2n-1)$-$\sqrt{JU}$. Then, Proposition 2.4(2) gives that $\prod_{i=1}^{m} \mathbb{F}_{p^{k_i}}$ is $(2n-1)$-$\sqrt{JU}$, so $R/J(R)$ is $(2n-1)$-$\sqrt{JU}$. Hence, by Theorem 2.9, $R$ is $(2n-1)$-$\sqrt{JU}$, as required. □

A ring $R$ is called *reduced* if it contains no non-zero nilpotent elements, that is, $Nil(R) = 0$.

**Lemma 2.14.** *Let $R$ be a $(2n-1)$-$\sqrt{JU}$ ring with $n \geq 1$. If $J(R) = 0$ and every nonzero right ideal of $R$ has a nonzero idempotent, then $R$ has no nonzero nilpotent elements; in other words, $R$ is reduced.*

*Proof.* Suppose, on the contrary, that $R$ is not reduced. Then there is a nonzero element $a \in R$ with $a^2 = 0$. By [15, Theorem 2.1], there exists an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some nontrivial

ring $T$. But, by Proposition 2.4(4), $eRe$ is $(2n-1)$-$\sqrt{J}U$, so $M_2(T)$ would also be $(2n-1)$-$\sqrt{J}U$, which is impossible. Hence, $R$ must be reduced. □

## 3. Main results

A ring $R$ is called *$\pi$-regular* if, for each $a$ in $R$, $a^n \in a^n R a^n$ for some integer $n$. So, regular rings are always $\pi$-regular. Also, a ring $R$ is said to be *strongly $\pi$-regular* provided that, for any $a \in R$, there exists $n$ depending on $a$ such that $a^n \in a^{n+1}R$.

**Theorem 3.1.** *Let $R$ be a ring and $n \geq 1$. The following are equivalent:*

(1) *$R$ is a regular $(2n-1)$-$\sqrt{J}U$ ring.*
(2) *$R$ is a $\pi$-regular, reduced $(2n-1)$-$\sqrt{J}U$ ring.*
(3) *Every $x \in R$ satisfies $x^{2n} = x$.*

*Proof.* (1) ⇒ (2) If $R$ is regular, then $J(R) = 0$ and every nonzero right ideal has a nonzero idempotent. By Lemma 2.14, $R$ is reduced. Also, every regular ring is $\pi$-regular. So (ii) holds.

(2) ⇒ (3) Reduced rings are abelian, and $Nil(R) = J(R)$ by [1, Lemma 5], so $Nil(R) = J(R) = \sqrt{J(R)} = 0$. For any $x \in R$, [9, Proposition 2.5] gives an idempotent $e$ and a unit $u$ with $x = e + u$ and $ex = xe = 0$. Then

$$x = x - xe = x(1-e) = u(1-e) = (1-e)u.$$

Since $R$ is $(2n-1)$-$\sqrt{J}U$, $u^{2n-1} = 1$, so

$$x^{2n-1} = ((1-e)u)^{2n-1} = u^{2n-1}(1-e) = 1 \cdot (1-e) = 1 - e.$$

Thus,

$$x = x(1-e) = xx^{2n-1} = x^{2n}.$$

(3) ⇒ (1) If $x^{2n} = x$ for all $x$, $R$ is clearly regular. For any unit $u$, $u^{2n} = u$ implies $u^{2n-1} = 1$, so $R$ is $(2n-1)$-$\sqrt{J}U$. □

**Corollary 3.2.** *Let $n \geq 1$. The following are equivalent for a ring $R$:*

(1) *$R$ is a regular $(2n-1)$-$\sqrt{J}U$ ring.*
(2) *$R$ is strongly regular and $(2n-1)$-$\sqrt{J}U$.*
(3) *$R$ is unit-regular and $(2n-1)$-$\sqrt{J}U$.*
(4) *Every $x \in R$ satisfies $x^{2n} = x$.*

**Proposition 3.3.** *Let $k \geq 1$. A ring $R$ is $\sqrt{J}U$ if and only if:*

(1) $2 \in \sqrt{J(R)}$,
(2) *$R$ is a $2^k$-$\sqrt{J}U$ ring.*

*Proof.* ($\Rightarrow$) If $R$ is a $\sqrt{J}U$ ring, then $-1 = 1 + j$ for some $j \in \sqrt{J(R)}$, which gives $-2 \in \sqrt{J(R)}$ and so $2 \in \sqrt{J(R)}$. Also, any $\sqrt{J}U$ ring is automatically $2^k$-$\sqrt{J}U$.

($\Leftarrow$) Let $u \in U(R)$. By (2), $u^{2^k} \in 1 + \sqrt{J(R)}$, so

$$(u-1)^{2^k} = 1 + u^{2^k} + j$$

for some $j \in \sqrt{J(R)}$, which shows $(u-1)^{2^k} \in \sqrt{J(R)}$. Hence $u - 1 \in \sqrt{J(R)}$, and $R$ is $\sqrt{J}U$. □

**Theorem 3.4.** *Let $R$ be a $(2n-1)$-$\sqrt{J}U$ ring. The following are equivalent:*

(1) *$R$ is an exchange ring.*
(2) *$R$ is a clean ring.*
(3) *$R$ is a semi-regular ring.*

*Proof.* (2) $\Rightarrow$ (1) This is clear.

(1) $\Rightarrow$ (2) If $R$ is an exchange $(2n-1)$-$\sqrt{J}U$ ring, then by Lemma 2.14, $R$ is reduced and hence abelian. Abelian exchange rings are clean, so (ii) holds.

(1) $\Rightarrow$ (3) If $R$ is exchange, then $R/J(R)$ is exchange and idempotents lift modulo $J(R)$. By Theorem 2.9, $R/J(R)$ is $(2n-1)$-$\sqrt{J}U$, so we can assume $J(R) = 0$. Since $R$ is exchange, every nonzero one-sided ideal has a nonzero idempotent. By Lemma 2.14, $R$ is reduced and abelian, so $Nil(R) = J(R) = \sqrt{J(R)} = 0$. Also, $v^{2n-1} = 1$ for every $v \in U(R)$, and the result follows from [11, Theorem 2.19].

(3) $\Rightarrow$ (1) Every semi-regular ring is exchange, so this is clear. □

**Theorem 3.5.** *Let $R$ be a ring. Then, $R$ is a $n$-$\sqrt{J}U$ ring if, and only if, $R/J(R)$ is a $n$-$UU$ ring.*

*Proof.* Suppose that $R$ is $n$-$\sqrt{J}U$ and $u + J(R) \in U(R/J(R))$. So, $u \in U(R)$ and hence $u^n = 1 + j$ where $j \in \sqrt{J(R)}$. Thus, $(u + J(R))^n = u^n + J(R) = (1 + j) + J(R) = (1 + J(R)) + (j + J(R))$ where $j + J(R) \in Nil(R/J(R))$ (since $j \in \sqrt{J(R)}$, we have $j^k \in J(R)$ for some $k$). Conversely, let $R/J(R)$ is $n$-$UU$ and $u \in U(R)$. Hence, $u + J(R) \in U(R/J(R))$. Then, $(u + J(R))^n = (1 + J(R)) + (q + J(R))$ where $q + J(R) \in Nil(R/J(R))$ and hence $(q + J(R))^m = J(R)$ for some $m$. Therefore, $q^m \in J(R)$ and hence $q \in \sqrt{J(R)}$. On the other hand $u^n - (1 + q) \in J(R)$ and so $u^n = 1 + q + j$ where $j \in J(R)$. But $q + j \in \sqrt{J(R)}$ by Lemma 2.3(1). Therefore, $R$ is $n$-$\sqrt{J}U$, as required. □

**Corollary 3.6.** *Let $R$ be a ring with $J(R) = 0$. Then, $R$ is a $n$-$\sqrt{J}U$ ring if, and only if, $R$ is a $n$-$UU$ ring.*

**Corollary 3.7.** *Let $R$ be a ring with $J(R)$ nil. Then, $R$ is a $n$-$\sqrt{J}U$ ring if, and only if, $R$ is a $n$-$UU$ ring.*

**Proposition 3.8.** *The ring $R[[x;\alpha]]$ is $n$-$\sqrt{J}U$ if and only if $R$ is $n$-$\sqrt{J}U$.*

*Proof.* Let $I = R[[x;\alpha]]x$. It is easy to see that $I$ is an ideal of $R[[x;\alpha]]$. Also, $J(R[[x;\alpha]]) = J(R)+I$, so $I \subseteq J(R[[x;\alpha]])$. Since $R[[x;\alpha]]/I \cong R$, the result follows directly from Theorem 2.9. □

**Corollary 3.9.** *The ring $R[[x]]$ is $n$-$\sqrt{J}U$ if, and only if, so is $R$.*

**Theorem 3.10.** *Let $R$ be a 2-primal ring and $\alpha$ an endomorphism of $R$ such that $R$ is $\alpha$-compatible. The following are equivalent:*

(1) *$R[x;\alpha]$ is an $n$-$\sqrt{J}U$ ring.*
(2) *$R$ is an $n$-$\sqrt{J}U$ ring.*

*Proof.* (2) ⇒ (1) Take $f = \sum_{i=0}^{m} a_i x^i \in U(R[x;\alpha])$. By [6, Corollary 2.14], we have $a_0 \in U(R)$ and $a_i \in Nil(R)$ for $i \geq 1$. Since $R$ is $n$-$\sqrt{J}U$, $1 - a_0^n \in \Delta(R)$. Then there exists $g \in Nil_*(R)[x;\alpha] \subseteq \sqrt{J(R[x;\alpha])}$ such that

$$f^n = a_0^n + gx \in 1 + \sqrt{J(R[x;\alpha])},$$

so $f^n \in 1 + \sqrt{J(R[x;\alpha])}$.

(1) ⇒ (2) Take any $u \in U(R) \subseteq U(R[x;\alpha])$. Then

$$u^n \in 1 + \sqrt{J(R[x;\alpha])} = 1 + \sqrt{J(R)} + \sqrt{J(R[x;\alpha])}x,$$

which shows $u^n \in 1 + \sqrt{J(R)}$. Hence $R$ is $n$-$\sqrt{J}U$. □

**Corollary 3.11.** *Let $R$ be a 2-primal ring. Then, the following are equivalent:*

(1) *$R[x]$ is a $n$-$\sqrt{J}U$ ring.*
(2) *$R$ is a $n$-$\sqrt{J}U$ ring.*

Let $R$ be a ring and $M$ an $R$-bimodule. The trivial extension of $R$ by $M$ is

$$T(R,M) = \{(r,m) \mid r \in R, m \in M\},$$

with addition done component-wise and multiplication defined by

$$(r,m)(s,n) = (rs, rn + ms).$$

It is easy to see that $T(R,M)$ is isomorphic to the subring

$$\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M \right\}$$

of the $2 \times 2$ matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$.

Also, the units of $T(R,M)$ are exactly

$$U(T(R,M)) = T(U(R),M).$$

**Proposition 3.12.** *Let $R$ be a ring and $M$ an $R$-bimodule. Then:*

(1) *The trivial extension $T(R,M)$ is $n$-$\sqrt{J}U$ if and only if $R$ is $n$-$\sqrt{J}U$.*
(2) *The upper triangular matrix ring $T_n(R)$ is $n$-$\sqrt{J}U$ if and only if $R$ is $n$-$\sqrt{J}U$.*

*Proof.* (1) Let $A = T(R,M)$ and $I := T(0,M)$. It is easy to check that $I \subseteq J(A)$ and $A/I \cong R$. Then the result follows from Theorem 2.9.

(2) Let

$$I := \{(a_{ij}) \in T_n(R) \mid a_{ii} = 0\}.$$

Then $I \subseteq J(T_n(R))$ and $T_n(R)/I \cong R^n$. Applying Theorem 2.9 and Proposition 2.4(2) gives the result.

□

Let $A$ and $B$ be rings, and let $M$ be an $(A,B)$-bimodule and $N$ a $(B,A)$-bimodule. Suppose we have bilinear maps

$$\phi : M \otimes_B N \to A \quad \text{and} \quad \psi : N \otimes_A M \to B$$

satisfying

$$\mathrm{Id}_M \otimes_B \psi = \phi \otimes_A \mathrm{Id}_M, \quad \mathrm{Id}_N \otimes_A \phi = \psi \otimes_B \mathrm{Id}_N.$$

For $m \in M$ and $n \in N$, set

$$mn := \phi(m \otimes n) \in A, \quad nm := \psi(n \otimes m) \in B.$$

Then the 4-tuple

$$R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

becomes an associative ring under usual matrix addition and multiplication. This ring is called a Morita context ring.

The two-sided ideals $\mathrm{Im}\,\phi$ and $\mathrm{Im}\,\psi$, denoted $MN$ and $NM$, are called the trace ideals of the Morita context.

**Proposition 3.13.** *Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring with $MN \subseteq J(A)$ and $NM \subseteq J(B)$. Then $R$ is a $(2n-1)$-$\sqrt{J}U$ ring if and only if both $A$ and $B$ are $(2n-1)$-$\sqrt{J}U$.*

*Proof.* By [18, Lemma 3.1], we have

$$J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix},$$

so

$$R/J(R) \cong A/J(A) \times B/J(B).$$

The result now follows from Theorem 2.9 and Proposition 2.4(2). □

Let $R$ and $S$ be rings, and let $M$ be an $(R,S)$-bimodule such that

$$(rm)s = r(ms) \quad \text{for all } r \in R, m \in M, s \in S.$$

Then we can define

$$T(R,S,M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

with the usual matrix addition and multiplication. This is called a formal triangular matrix ring.

Notice that in Proposition 3.13, if we take $N = 0$, we get this case of a triangular matrix ring.

**Corollary 3.14.** *Let $R$ and $S$ be rings, and let $M$ be an $(R,S)$-bimodule. Then the formal triangular matrix ring $T(R,S,M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is n-$\sqrt{J}U$ if and only if both $R$ and $S$ are n-$\sqrt{J}U$.*

Let $R$ be a ring and let $s$ be a central element of $R$. Consider the set

$$\begin{pmatrix} R & R \\ R & R \end{pmatrix}$$

with component-wise addition and multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

This forms a ring, denoted by $K_s(R)$.

A Morita context

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

with $A = B = M = N = R$ is called a generalized matrix ring over $R$. As observed by Krylov [13], a ring $S$ is a generalized matrix ring over $R$ if and only if $S = K_s(R)$ for some $s \in C(R)$.

Here, $MN = NM = sR$, so

$$MN \subseteq J(A) \iff s \in J(R), \quad NM \subseteq J(B) \iff s \in J(R).$$

**Corollary 3.15.** *Let $R$ be a ring and $s \in C(R) \cap J(R)$. Then $K_s(R)$ is n-$\sqrt{J}U$ if and only if $R$ is n-$\sqrt{J}U$.*

Following Tang and Zhou [20], for $n \geq 2$ and $s \in C(R)$, the $n \times n$ formal matrix ring over $R$, denoted by $M_n(R;s)$, is the set of all $n \times n$ matrices over $R$ with the usual addition, and multiplication defined as follows:

For $(a_{ij})$ and $(b_{ij})$ in $M_n(R;s)$,

$$(a_{ij})(b_{ij}) = (c_{ij}) \quad \text{where} \quad c_{ij} = \sum_k s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here, $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$, and $\delta_{ij}$, $\delta_{ik}$, $\delta_{jk}$ are the usual Kronecker delta symbols.

**Corollary 3.16.** *Let $R$ be a ring and $s \in C(R) \cap J(R)$. Then, for any $k \geq 1$, $M_n(R;s)$ is $k$-$\sqrt{J}U$ if and only if $R$ is $k$-$\sqrt{J}U$.*

A Morita context

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

is called trivial if the context products are zero, that is, $MN = (0)$ and $NM = (0)$.

According to [12], in this case we have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $T(A \times B, M \oplus N)$ is the trivial extension of $A \times B$ by $M \oplus N$.

We can now start the proof of the next result.

**Corollary 3.17.** *Let $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a trivial Morita context. Then it is $n$-$\sqrt{J}U$ if and only if both $A$ and $B$ are $n$-$\sqrt{J}U$.*

*Proof.* It is clear that we have the isomorphisms

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}.$$

The result then follows directly from Proposition 3.12 and Proposition 2.4(2). □

## 4. OPEN PROBLEMS

A ring $R$ is called $UQ$ if $U(R) = 1 + QN(R)$ (see [6]).

**Problem 4.1.** Study the rings $R$ in which, for every $u \in U(R)$, we have

$$u^n = 1 + q \quad \text{for some } q \in QN(R),$$

that is, the n-$UQ$ rings.

**Problem 4.2.** Characterize semi-potent $n$-$\sqrt{J}U$ rings for an arbitrary $n \geq 1$.

**Problem 4.3.** Let $R$ be a ring and $G$ a multiplicative group. Determine necessary and sufficient conditions under which the group ring $RG$ is $n$-$\sqrt{J}U$.

**Acknowledgement.** The authors express their sincere gratitude to the expert referee for the numerous competent suggestions made which lead to a substantial improvement of the exposition.

## References

[1] A. Badawi, *On abelian $\pi$-regular rings*, Commun. Algebra, **25**(4)(1997), 1009–-1021.
[2] W. Chen, *On constant products of elements in skew polynomial rings*, Bull. Iran. Math. Soc, **41**(2)(2015), 453–462.
[3] P. V. Danchev, *Rings with Jacobson units*, Toyama Math. J, **38**(1)(2016), 61–74.
[4] P. V. Danchev, *On exchange $\pi$-UU unital rings*, Toyama Math. J, **39**(1)(2017), 1–7.
[5] P. V. Danchev, *On exchange $\pi$-JU unital rings*, Rend. Sem. Mat. Univ. Pol. Torino, **77**(1)(2019), 13–23.
[6] P. V. Danchev, A. Javan, O. Hasanzadeh, A. Moussavi, *Rings with $u-1$ quasinilpotent for each unit $u$*, J. Algebra Appl. **24**(10) (2025).
[7] P. V. Danchev, A. Javan and A. Moussavi, *Rings with $u^n-1$ nilpotent for each unit $u$*, J. Algebra Appl, **25**(2026).
[8] P.V. Danchev and T.Y. Lam, *Rings with unipotent units*, Publ. Math. Debrecen, **88**(3-4)(2016), 449–-466.
[9] A. J. Diesl, *Nil clean rings*, J. Algebra, **383**(2013), 197–211.
[10] M. T. Koşan, A. Leroy and J. Matczuk, *On UJ-rings*, Commun. Algebra, **46**(5)(2018), 2297–2303.
[11] M. T. Koşan, T. C. Quynh, T. Yildirim and J. Žemlička, *Rings such that, for each unit $u$, $u-u^n$ belongs to the Jacobson radical*, Hacettepe J. Math. Stat, **49**(4)(2020), 1397–1404.
[12] M. T. Koşan, Y. C. Quynh and J. Žemlička, *UNJ-Rings*, J. Algebra Appl, **19**(2020).
[13] P. A. Krylov, *Isomorphism of generalized matrix rings*, Algebra Logic, **47**(4)(2008), 258-262.
[14] T. Y. Lam, *A First Course in Noncommutative Rings*, Second Edition, Springer Verlag, New York, (2001).
[15] J. Levitzki, *On the structure of algebraic algebras and related rings*, Trans. Am. Math. Soc, **74**(1953), 384–409.
[16] A. R. Nasr-Isfahani, *On skew triangular matrix rings*, Commun. Algebra, **39**(11)(2011), 4461–4469.
[17] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Am. Math. Soc, **229**(1977), 269-–278.
[18] G. Tang, C. Li, and Y. Zhou, *Study of Morita contexts*, Commun. Algebra, **42**(4)(2014), 1668-1681.
[19] S. Saini and D. Udar, *$\sqrt{J}U$ rings*, Czechoslovak Math. J. **75**(4) (2025). doi:10.21136/CMJ.2025.0117-25.
[20] G. Tang and Y. Zhou, *A class of formal matrix rings*, Linear Algebra Appl, **438**(2013), 4672–4688.
[21] Z. Wang and J. Chen, *Pseudo Drazin inverses in associative rings and Banach algebras*, Linear Algebra & Appl. **437**(6) (2012), 1332–1345.
[22] Y. Zhou, *On clean group rings*, Advances in Ring Theory, Treads in Mathematics, Birkhauser, Verlag Basel/Switzerland, (2010), 335–345.