Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2026.31111.1811
Caspian J Math Sci. **15**(2)(2026), 434-466 (Research Article)

Operational matrix approach based on linear cardinal B-splines for stochastic Itô- Volterra integral equations

Payam Moradpour[1], Khosro Sayevand [1] [1] and Iman Masti [1]
[1] Faculty of Mathematical Sciences and Statistics, Malayer University, Malayer, Iran

Abstract. In many modeling that occurs in various disciplines, including physics and biological mathematics, the stochastic factor is used to increase the accuracy of the model and make it more consistent with the reality of the phenomena, which has led to the creation of various classes of stochastic equations. One of the equations in this family is the linear or nonlinear stochastic Itô-Volterra integral equations (SIVIEs). This study investigated a method based on the operational matrix from linear cardinal B-spline functions by means of the Simpson's rule for a family of the linear or nonlinear SIVIEs due to the Brownian motion. To this end, we have created matrix forms for the Volterra integral as well as the stochastic Itô-integral. Then we obtained the operational matrix of product, which we found to be not particularly computationally complex. By stating several theorems, we show that the method is convergent, and as a consequence of these theorems, we prove that the order of convergence of this method is $\mathbb{O}(\frac{1}{2^{2I}})$ and some convergence verification notes for different values of I to confirm the theoretical rates are demonstrated.

[1]Corresponding author: payammoradpour19@gmail.com, ksayehvand@malayeru.ac.ir, iman.masty@gmail.com,
Received: 24 January 2026
Revised: 19 May 2026
Accepted: 25 May 2026
How to Cite: Moradpour, Payam; Sayevand, Khosro; Masti, Iman. Operational matrix approach based on linear cardinal B-splines for stochastic Itô- Volterra integral equations. Casp.J. Math. Sci.,**15**(2)(2026), 434-466.
This work is licensed under a Creative Commons Attribution 4.0 International License.
Copyright © 2026 by University of Mazandaran. Subbmited for possible open access publication under the terms and conditions of the Creative Commons Attribution(CC BY) license(https://craetivecommons.org/licenses/by/4.0/)

Keywords: Linear Cardinal B-spline polynomials, Stochastic Itô-Volterra integral equations, Error analysis, Operational matrix.

2000 Mathematics subject classification: 65D07, 60H05, 45D05.

1. Introduction

During the last three decades, deterministic chaos theory has rapidly attracted mathematicians, engineers, economists, physicists, biologists, etc. But, this type of chaos can only be understood as quasi-chaos, where all states of a system can be predicted and recreated by experimenting. In addition, many experiments in the natural sciences have provided hard evidence of stochastic behavior. Deterministic models can represent idealized situations and can often be corrected by adding stochastic influences. Because of the irregularity of Brownian motion, stochastic differential equations can only be interpreted in terms of stochastic integral equations. Kiyosi Itô (1915-2008) discovered the difficulty of investigating differential equations with a noise term that does not converge to a unique limit in the mean square sense. To avoid this difficulty, Itô developed a new approach and created a formula called the Itô formula, which is the basis of the Itô calculus and an introduction to stochastic differential equations (SDE) [1, 2]. Stochastic differential equations are basically equations of motion for dynamic systems that evolve according to a probabilistic description. In recent years, stochastic ordinary differential equations have been widely used in modeling complex phenomena in various scientific fields. Among these applications are population dynamics, the spread of bacteriophage infections, and other biological systems in which the role of uncertainty and random fluctuations is important [3]. Stochastic differential equations (SDEs) governed by Brownian processes provide powerful analytical tools for studying dynamical systems under uncertainty. These studies have diverse applications in economics, biology, mechanics, physics, chemistry, and finance, and play a fundamental role in analyzing the behavior of complex systems with random noise [4, 5, 6].

For example, the geometric Brownian motion:

$$dZ(x) = \alpha Z(x)dx + \beta Z(x)dB(x),\ Z(0) = Z_0, \tag{1.1}$$

which is a stochastic process commonly used to model the behavior of asset prices in finance, including in the Black-Scholes-Merton option pricing model. Another important example is Feller's branching diffusion, which is a stochastic process commonly used in mathematical biology to model population dynamics, particularly in the context of

branching processes[7]:

$$dZ(x) = \alpha Z(x)dx + \sigma\sqrt{Z(x)}dB(x),\ Z(0) = Z_0 > 0. \qquad (1.2)$$

For example, the Fokker-Planck stochastic equation is used in statistical mechanics to study the behavior of particles under the influence of second-order forces and drag forces. These equations allow for a more accurate analysis of the dynamics of physical systems and provide a better understanding of the effects of noise and chaos in these systems [8]:

$$d^{\beta}Z(x) = k_1(t,x)Z(x)dx + k_2(t,x)Z(x)dB(x),\ Z(0) = Z_0,\ \beta \in \mathbb{R}. \quad (1.3)$$

To solve this class of equations, various methods have been proposed. The most important of which are the second-order Chebyshev wavelet Galerkin method [9] and the shifted Legendre spectral collocation method [10]. In the framework of numerical methods, the matrix form has been used as an efficient tool for solving various problems. For this purpose, researchers in [11] have used the operational matrix based on the dual Bernstein polynomial to approximate fractional derivatives and also the stochastic integral. Other advanced methods such as collocation-Galerkin and the shifted Jacobi polynomials have also been investigated in order to improve the accuracy and computational efficiency, and techniques based on the Galerkin method and other developed numerical methods have played an important role in solving these equations [12, 13]. For more details and about discussing the additional relevant works to better contextualize these contributions see [14, 15, 16, 17, 18].

Since solving the SDEs can only be done in a particular cases, developing stable numerical techniques for SDEs is a major and rapidly increasing area of research. Due to their high accuracy, spectral methods have been widely used in the last decades. Three main types of spectral methods can be applied: collocation, tau, and Galerkin. The choice of the type of method depends fundamentally on the application. Usually, the matrices generated by the discretization matrices have an increasing condition number, and thus computational rounding errors. This reduces the expected theoretical exponential accuracy. Furthermore, the discretization matrices are usually fully populated, making it difficult to use efficient algebraic solvers.

Several attempts have been made to eliminate these problems. The operational matrix method is a powerful numerical technique used to solve various types of differential equations, especially those involving fractional derivatives. The key advantages of the method include the following:

Operational matrices significantly reduce the computational cost compared to traditional numerical methods, especially for higher-order derivatives or fractional derivatives (Efficiency).

Depending on the choice of basis functions and the number of terms used, the method can achieve high accuracy (Accuracy).

This method can be used for systems of different sizes and complexities (Flexibility).

Computational software greatly facilitates the work with this method (Use of computational tools).

It can be applied to a wide range of problems, including: fractional differential equations, integro-differential equations, delay differential equations, nonlinear problems (Versatility).

Among the considerations that should be taken into account for this method is the choice of basis functions. The choice of basis functions significantly impacts the accuracy and efficiency of the method, such as polynomials Legendre, Chebyshev, Bernstein, Bernoulli, wavelets, or other suitable functions [11, 19, 20, 21]. In summary, the operational matrix method provides a powerful and versatile framework for solving a wide range of differential equations. By leveraging the properties of operational matrices, this method offers significant advantages in terms of computational efficiency and accuracy, making it a valuable tool in various scientific and engineering disciplines.

With these ideas in mind, this study investigates a method based on the operational matrix of B-spline cardinal linear functions for numerically solving linear and nonlinear SIVIEs. We consider the following SIVIE:

$$z(x) = g(x) + \int_0^x \lambda_1(x,t)\Upsilon_1(t,z(t))dt + \int_0^x \lambda_2(x,t)\Upsilon_2(t,z(t))dB(t), \tag{1.4}$$

where $x, z(x)$, $g(x) \in D = [0,T]$ and $\Upsilon_i(t,z(t)), \lambda_i(x,t) \in D \times D$, for $i = 1,2$, are the stochastic processes defined on the probability space $(\Omega, \mathbb{F}, P)$. Also, $B = \{B(t), t \geq 0\}$ is a Brownian process.

Also, many functions or polynomials, such as the approach based on Legendre wavelets Galerkin method [22] and piecewise spectral-collocation method [23], the approach based on operational matrices of Chebyshev cardinal functions [24] and the explicit finite difference scheme and Bernstein polynomials [25], were used to derive solutions of SIVIEs.

In this work, the choice of linear cardinal B-spline basis functions is driven by a deliberate balance between the sparsity of operational matrices, numerical stability, simplicity of implementation, and robustness against the limited smoothness induced by stochastic noise. Unlike global polynomial bases such as Legendre and Chebyshev polynomials,

which produce dense matrices with rapidly growing condition numbers and whose spectral accuracy deteriorates when confronted with the non-smooth solutions typical of stochastic equations, linear B-splines, with their compact support, yield sparse and nearly banded operational matrices whose condition number remains bounded independently of the scale, enabling stable system solutions even for a large number of basis functions. Furthermore, in contrast to Bernstein polynomials, which despite good conditioning suffer from dense matrices and slow algebraic convergence, and wavelets, which although sparse require complex multiresolution constructions and boundary adaptations, linear cardinal B-splines admit simple closed-form expressions, require no numerical quadrature, offer a remarkably straightforward implementation, and provide a robust and reliable second-order convergence for stochastic problems. This combination of features makes linear cardinal B-splines a highly suitable choice for the numerical solution of stochastic Volterra integral equations, particularly in applications requiring repeated execution over numerous Brownian motion realizations.

2. Basic definitions

2.1. **Cardinal B-spline functions.**

Definition 2.1. The nth-order cardinal B-spline $B_n(x)$ for $n \geq 2$ can be defined as follows:

$$B_n(x) = (B_{n-1} * B_1)(x) = \int_{-\infty}^{\infty} B_{n-1}(x-t)B_1(t)dt = \int_0^1 B_{n-1}(x-t)dt, \tag{2.1}$$

here $B_1(x) = \chi_{[0,1]}(x)$ is the characteristic function on $[0, 1]$.

Lemma 2.2. *The following formula can be obtained recursively through calculations:*

$$B_n(x) = \frac{x}{n-1}B_{n-1}(x) + \frac{n-x}{n-1}B_{n-1}(x-1), \tag{2.2}$$

where $Supp[B_n(x)] = [0, n]$.

Definition 2.3. In explicit form, linear B-spline function $B_2(x)$ can be defined as follows:

$$B_2(x) = \begin{cases} x & x \in [0,1] \\ 2-x & x \in [1,2] \\ 0 & o.w. \end{cases}$$

Definition 2.4. The two-scale relationship for linear B-splines functions $B_2(x)$ for $x \geq 0$ is defned as follows:

$$B_{i,j}(x) = B_2(2^i x - j), \tag{2.3}$$

where $i, j \in \mathbb{Z}$.

It can easily be shown that

$$supp[B_{i,j}(x)] = [2^{-i}j, 2^{-i}(2+j)],\ i, j \in \mathbb{Z}. \tag{2.4}$$

Define $V_i = \{j : [2^{-i}j, 2^{-i}(2+j)] \cap (0,1) \neq \oslash\}$ for $i \in \mathbb{Z}$. It can be derived that $min\{V_i\} = -1$ and $max\{V_i\} = 2^i - 1$ where $i \in \mathbb{Z}$. Since the support of $B_{i,j}(x)$ may not be in range of $[0,1]$, to take the scaling functions into the interval $[0,1]$, we let

$$\varphi_{i,j}(x) = B_{i,j}(x).\chi_{[0,1]}(x),\ i \in \mathbb{Z}, j \in V_i. \tag{2.5}$$

As a result, one can write

$$\varphi_{i,j}(x) = \begin{cases} \begin{cases} 1 - 2^i x & x \in [0, 2^{-i}) \\ 0 & o.w \end{cases} & j = -1 \\ \begin{cases} 2^i x - j & x \in [2^{-i}j, 2^{-i}(j+1)) \\ 2 - 2^i x + j & x \in [2^{-i}(j+1), 2^{-i}(j+2)) \\ 0 & o.w \end{cases} & j = 0, \cdots, 2^i - 2 \\ \begin{cases} 2^i x - 2^i + 1 & x \in [1 - 2^{-i}, 1) \\ 0 & o.w \end{cases} & j = 2^i - 1 \end{cases} \tag{2.6}$$

2.2. **Itô integral.** The probability space (Ω, F, P) is the set that contains all possible outcomes of a random experiment or process. This mathematical space allows us to predict and analyze the random behavior of systems. A probability space consists of three main components:

The sample space Ω is the set of all possible outcomes of a random experiment, The subset set F is the set of the sample space that contains measurable sets, and te probability function P, is a function of the subset F to real numbers that assigns to each event a probability of its occurrence.

Brownian motion is a stochastic process named after the Scottish scientist. It refers to the random motion of particles suspended in a liquid or gas under the influence of molecular collisions. Brownian motion is generally characterized by the following characteristics [26]:

Brownian motion is completely random, meaning that the path of the particles is unpredictable and its behavior can only be modeled by the laws of probability (Randomness).

The motion of the particles changes direction and speed at any given moment, so that their motion appears irregular and without a specific pattern (Complexity and irregularity).

Definition 2.5. [7]-[10] For $v \in [0,T]$, Brownian motion is accurately modeled using a stochastic process or $B(v)$, which has certain properties, some of which are:

- Starting from zero, meaning that: $B(0) = 0$,
- Random return, meaning that: $B(v) - B(u) \sim N(0, v-u)$,
- Independence, meaning that: For $v > u$, $B(v) - B(u)$ is independent of the past,
- Continuous but inseparable paths.

A random integral is a type of integral used in probability theory and stochastic processes. These integrals are used to model and analyze systems that exhibit random behavior, such as Brownian motion and other random processes. Unlike ordinary integrals, which involve continuous and predictable functions, in a random integral, the stochastic processes are introduced as a component of the integral, which introduces complications in the analysis and calculations. One of the most common and important types of stochastic integrals is the Itô integral. The Itô integral is used particularly in stochastic calculus and the theory of stochastic processes to model stochastic systems with properties such as discontinuity and randomness [7].

Definition 2.6. Let $f \in \Delta(A,B)$ and φ_m is a sequence such that:

$$E\left[\int_A^B \left(f(a,b) - \varphi_m(a,b)\right)^2 da\right] = 0, \ as \ m \ \to \infty. \tag{2.7}$$

Then:

$$\int_A^B f(a,b)dB(b) = \lim_{m\to\infty} \int_A^B \varphi_m(a,b)dB(b). \tag{2.8}$$

Proposition 2.7. *One of the main equations that models Brownian motion is the stochastic integral equation, which is as follows:*

$$\int_0^x f(w)dB(w) = f(x)B(x) - \int_0^x B(w)df(w). \tag{2.9}$$

Proof. See ([7]). □

Proposition 2.8.

$$E\left[\left(\int_U^V g(v,l)dBr_v(l)\right)^2\right] = E\left[\int_U^V g^2(v,l)dv\right], \ g \in \Delta(U,V).$$

Proof. See ([7]). □

2.3. **Approximation of functions.** Let us consider the set of cardinal B-spline functions of I-th degree:

$$\phi_I(x) = \left[\varphi_{I,-1}(x), \varphi_{I,0}(x), ..., \varphi_{I,2^I-1}(x)\right]^T \subset L^2[0,1], \tag{2.10}$$

and assume that

$$W = span\{\varphi_{I,-1}(x), \varphi_{I,0}(x), ..., \varphi_{I,2^I-1}(x)\}.$$

We also know that, since W is a finite-dimensional vector space, for any arbitrary function g in the space $L^2[0,1]$, $g_I \in W$ is the best approximation to g in the space W. In other words,

$$\forall h \in W, \ \| g - g_I \|_2 \ \leq \ \| g - h \|_2 \ . \tag{2.11}$$

Since $g_I \in W$, $\mathcal{L} = [l_{-1}, l_0, \cdots, l_{2^I-1}]^T$ will be a singular vector that:

$$g(x) \simeq g_I(x) = \sum_{j=-1}^{2^I-1} l_j \varphi_{I,j}(x) = \mathcal{L}^T \phi_I(x). \tag{2.12}$$

The function $\varphi_{I,j}(x)$ can be expressed as follows:

$$\varphi_{I,j}\left(\frac{m+1}{2^I}\right) = \delta_{j,m} = \begin{cases} 1, & j = m \\ 0, & j \neq m \end{cases} \qquad m = -1, 0, ..., 2^I - 1. \tag{2.13}$$

Therefore the coefficients l_j in Eq. (2.12) are determined as:

$$l_j = g\left(\frac{j+1}{2^I}\right), \ j = -1, 0, \cdots, 2^I - 1. \tag{2.14}$$

Let e_m be the $(m+2)$-th column of unit matrix of order $2^I + 1$, so it is easy to verify that

$$\phi_I(\mu_m) = e_m, \ \mu_m = \frac{m+1}{2^I}, \ m = -1, 0, ..., 2^I - 1. \tag{2.15}$$

Similarly, for function $\lambda \in L^2([0,1] \times [0,1])$ as:

$$\lambda(x,t) \simeq \lambda_m(x,t) = \sum_{p=-1}^{2^I-1} \sum_{q=-1}^{2^I-1} \varphi_{p,m}(x) k_{pq} \varphi_{q,m}(t) = \phi^T(x) \mathcal{K} \phi(t), \tag{2.16}$$

where $\mathcal{K} = [k_{pq}]_{(2^I+1)\times(2^I+1)}$ and

$$k_{pq} = \lambda(\frac{p+1}{2^I}, \frac{q+1}{2^I}), \ p, q = -1, 0, \cdots, 2^I - 1. \tag{2.17}$$

3. Description of the method

In this section, we first construct the integer and the stochastic integral of linear cardinal B-spline functions on $[0,1]$. Then, a numerical approach based on matrix operators will be presented.

3.1. **Operational matrices.**

Theorem 3.1. *Let $\phi_I(x)$ be linear cardinal B-spline vector defined in Eq.* (2.10)*. Then, the operational matrix of integration based on the linear cardinal B-spline vector is defined as follows:*

$$\int_0^x \phi_I(t)dt = \mathbf{I}\phi_I(x), \tag{3.1}$$

and $\mathbf{I}$ *is defined as:*

$$\mathbf{I} = \begin{pmatrix} 0 & \zeta & \zeta & \zeta & \dots & \zeta \\ 0 & \zeta & 2\zeta & 2\zeta & \dots & 2\zeta \\ 0 & 0 & \zeta & 2\zeta & \dots & 2\zeta \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \zeta & 2\zeta \\ 0 & 0 & 0 & 0 & 0 & \zeta \end{pmatrix}, \tag{3.2}$$

where $\zeta = 2^{-I-1}$.

Proof. Let $x \in [0,1]$, then define

$$\mathcal{I}\varphi_j(x) = \int_0^x \varphi_j(t)dt. \tag{3.3}$$

In order to calculate the aforementioned integral, three cases are analyzed and evaluated in detail:

Case 1: if $j = -1$:

In this case, if $x \in [0, 2^{-I}]$ then Eq. (2.6) is calculated as follows:

$$\mathcal{I}\varphi_{I,-1}(x) = \int_0^x \varphi_{I,-1}(t)dt = \int_0^x (1 - 2^I t)dt = x - 2^{I-1}x^2. \tag{3.4}$$

If $x > 2^{-I}$ then Eq. (2.6) can be obtained as:

$$\mathcal{I}\varphi_{I,-1}(x) = \int_0^{2^{-I}} \varphi_{I,-1}(t)dt = \int_0^{2^{-I}} (1 - 2^I t)dt = 2^{-I-1}. \tag{3.5}$$

Case 2: if $j = 0, 1, \cdots, 2^I - 2$:

In this case, if $x \in [0, 2^{-I}j]$ then:

$$\mathcal{I}\varphi_{I,j}(x) = 0. \tag{3.6}$$

If $x \in [2^{-I}j, 2^{-I}(j+1)]$ then:

$$\mathcal{I}\varphi_{I,j}(x) = \int_{2^{-I}j}^x (2^I t - j)dt = 2^{I-1}x^2 - jx + 2^{-I-1}j^2. \tag{3.7}$$

If $x \in [2^{-I}(j+1), 2^{-I}(j+2)]$ then:

$$\begin{aligned}\mathcal{I}\varphi_{I,j}(x) &= \int_{2^{-I}j}^{2^{-I}(j+1)} (2^I t - j)dt + \int_{2^{-I}(j+1)}^{x} (2 - (2^I t - j))dt \\ &= (2+j)x - 2^{I-1}x^2 - 2^{-I-1}(j^2 + 4j + 2).\end{aligned} \tag{3.8}$$

Else if $x > 2^{-I}(j+2)$ then:

$$\begin{aligned}\mathcal{I}\varphi_{I,j}(x) &= \int_{2^{-I}j}^{2^{-I}(j+1)} (2^I t - j)dt + \int_{2^{-I}(j+1)}^{2^{-I}(j+2)} (2 - (2^I t - j))dt \\ &= 2^{-I}.\end{aligned} \tag{3.9}$$

Case 3: $j = 2^I - 1$:

In this case, if $x \in [0, 2^{-I}j]$ then:

$$\mathcal{I}\varphi_{I,j}(x) = 0. \tag{3.10}$$

If $x \in [1 - 2^{-I}, 1]$ then:

$$\mathcal{I}\varphi_{I,2^I-1}(x) = \int_{1-2^{-I}}^{x} (2^I t - 2^I + 1)dt = 2^{I-1}x^2 - jx + 2^{-I-1}j^2. \tag{3.11}$$

Else if $x > 2^{-I}(j+2)$ then:

$$\mathcal{I}\varphi_{I,2^I-1}(x) = \int_{1-2^{-I}}^{1} (2^I t - 2^I + 1)dt = 2^{-I-1}. \tag{3.12}$$

Now, by using Eqs. (3.4)-(3.12), $\mathcal{I}\varphi_{I,j}(x)$ is equalto

$$\begin{cases} \begin{cases} 1 - 2^{I-1}x^2 & x \in [0, 2^{-I}) \\ 2^{-I-1} & x \geqslant 2^{-I} \\ 0 & o.w \end{cases} & j = -1 \\ \\ \begin{cases} 2^{I-1}x^2 - jx + 2^{-I-1}j^2 & x \in [2^{-I}j, 2^{-I}(j+1)) \\ 2^{-I-1} & x \in [2^{-I}(j+1), 2^{-I}(j+2)) \\ 2^{-I} & x \geqslant 2^{-I}(j+2) \\ 0 & o.w \end{cases} & j = 0, \cdots, 2^I - 2 \\ \\ \begin{cases} 2^{I-1}x^2 - (2^I - 1)x + 2^{-I-1}(2^I - 1)^2 & x \in [1 - 2^{-I}, 1) \\ 2^{-I-1} & x \geqslant 1 \\ 0 & o.w \end{cases} & j = 2^I - 1 \end{cases} \tag{3.13}$$

By using Eq. (2.10) and Eq. (3.13), it can be obtained:

$$\mathcal{I}\phi_I(x) = \mathbf{I}\phi_I(x), \tag{3.14}$$

where

$$I_{i,j} = \mathcal{I}(\varphi_j(\mu_i)),\ \mu_i = \frac{i+1}{2^I},\ i,j = -1,0,...,2^I-1. \tag{3.15}$$

Therefore, it can be proven that the operational matrix of integration based on the linear cardinal B-spline vector, $\mathcal{I}$ of order $(2^I+1)\times(2^I+1)$ is defined as follows:

$$\mathbf{I} = \begin{pmatrix} 0 & \zeta & \zeta & \zeta & \dots & \zeta \\ 0 & \zeta & 2\zeta & 2\zeta & \dots & 2\zeta \\ 0 & 0 & \zeta & 2\zeta & \dots & 2\zeta \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \zeta & 2\zeta \\ 0 & 0 & 0 & 0 & 0 & \zeta \end{pmatrix}, \tag{3.16}$$

where $\zeta = 2^{-I-1}$.

■

□

Theorem 3.2. *Let $\phi_I(x)$ be linear cardinal B-spline vector defined in Eq.* (2.10)*. Then, the stochastic operational matrix of integration based on the linear cardinal B-spline vector is defined as follows:*

$$\int_0^x \phi_I(t)dB(t) = \mathbf{I_s}\phi_I(x), \tag{3.17}$$

and $\mathbf{I_s}$ is defined as:

$$\mathbf{I_s} = \begin{pmatrix} 0 & \gamma & \gamma & \gamma & \dots & \gamma \\ 0 & \delta_0 & \gamma_0 & \gamma_0 & \dots & \gamma_0 \\ 0 & 0 & \delta_1 & \gamma_1 & \dots & \gamma_1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \delta_{2^I-2} & \gamma_{2^I-2} \\ 0 & 0 & 0 & 0 & 0 & \vartheta \end{pmatrix}. \tag{3.18}$$

Proof. Let $x \in [0,1]$, then define

$$\mathcal{I}^{\int}\varphi_{I,j}(x) = \int_0^x \varphi_{I,j}(t)dB(t), \tag{3.19}$$

in order to calculate the aforementioned integral, three cases are analyzed and evaluated in detail:

Case 1: if $j = -1$:

In this case, if $x \in [0, 2^{-I}]$ then Eq. (2.6) is calculated as follows:

$$\mathcal{I}^{\int}\varphi_{I,-1}(x) = \int_0^x \varphi_{I,-1}(t)dB(t) = \int_0^x (1-2^I t)dB(t) = B(x) - 2^I\int_0^t tdB(t). \tag{3.20}$$

Therefore, by using property 2.7, we have:

$$\mathcal{I}^{\int}\varphi_{I,-1}(x) = B(x) - 2^I\left(xB(x) - \int_0^t B(t)dt\right). \tag{3.21}$$

Now, we can use Simpson's rule of integration to calculate the integral at 3.21. So we have:

$$\mathcal{I}^{\int}\varphi_{I,-1}(x) = B(x)\left(1 - \frac{2^{I+1}x}{3}\right) + \frac{2^{I+2}x}{3}B(\frac{x}{2}). \tag{3.22}$$

If $x > 2^{-I}$ then Eq. (2.6) can be obtained as:

$$\mathcal{I}^{\int}\varphi_{I,-1}(x) = \int_0^{2^{-I}}(1 - 2^I t)dB(t) = B(2^{-I}) - 2^I\int_0^{2^{-I}} tdB(t). \tag{3.23}$$

Therefore, by using property 2.7, we have:

$$\mathcal{I}^{\int}\varphi_{I,-1}(x) = B(2^{-I}) - 2^I\left(xB(x)|_0^{2^{-I}} - \int_0^{2^{-I}} B(t)dt\right). \tag{3.24}$$

Now, we can use Simpson's rule of integration. So we have:

$$\mathcal{I}^{\int}\varphi_{I,-1}(x) = \frac{4}{3}B(2^{-I-1}) + \frac{1}{3}B(2^{-I}). \tag{3.25}$$

Case 2: if $j = 0, 1, \cdots, 2^I - 2$:

In this case, if $x \in [0, 2^{-I}j]$ then:

$$\mathcal{I}^{\int}\varphi_{I,j}(x) = 0. \tag{3.26}$$

If $x \in [2^{-I}j, 2^{-I}(j+1)]$ then by using property 2.7 and use Simpson's rule, we have:

$$\begin{aligned}\mathcal{I}^{\int}\varphi_{I,j}(x) &= \int_{2^{-I}j}^{x}(2^I t - j)dB(t)\\ &= \frac{2^I t - j}{3}\left(2B(x) - B(2^{-I}j) - 4B(\frac{2^I t + j}{2^{I+1}})\right).\end{aligned} \tag{3.27}$$

If $x \in [2^{-I}(j+1), 2^{-I}(j+2)]$ then:

$$\begin{aligned}\mathcal{I}^{\int}\varphi_{I,j}(x) &= \int_{2^{-I}j}^{2^{-I}(j+1)}(2^I t - j)dB(t) + \int_{2^{-I}(j+1)}^{x}(2 - (2^I t - j))dB(t)\\ &= \frac{5 + 2j - 2^{I+1}x}{3}B(x) - \frac{1}{3}B(2^{-I}j) + \frac{2^I j - 2 - j}{3}B((j+1)2^{-I})\\ &- \frac{4}{3}B((2j+1)2^{-I-1}) + \frac{2^{I+2}}{3}\left(x - 2^{-I}(j+1)\right)B(\frac{x + 2^{-I}(j+1)}{2}).\end{aligned} \tag{3.28}$$

Else if $x > 2^{-I}(j+2)$ then:

$$\begin{aligned}\mathcal{I}^{\int}\varphi_{I,j}(x) &= \int_{2^{-I}j}^{2^{-I}(j+1)}(2^I t - j)dB(t) + \int_{2^{-I}(j+1)}^{2^{-I}(j+2)}(2-(2^I t - j))dB(t)\\ &= \frac{-1}{3}B(2^{-I}j) + \frac{1}{3}B(2^{-I}(j+2)) - \frac{4}{3}B(2^{-I-1}(2j+1))\\ &+ \frac{4}{3}B(2^{-I-1}(2j+3)).\end{aligned} \tag{3.29}$$

Case 3: $j = 2^I - 1$:

In this case, if $x \in [0, 2^{-I}j]$ then:

$$\mathcal{I}^{\int}\varphi_{I,j}(x) = 0. \tag{3.30}$$

If $x \in [1-2^{-I}, 1]$ then:

$$\begin{aligned}\mathcal{I}^{\int}\varphi_{I,2^I-1}(x) &= \int_{1-2^{-I}}^{x}(2^I t - 2^I + 1)dB(t)\\ &= \frac{2^I x - j}{3}\left(2B(x) - B(2^{-I}j) - 4B(\frac{x+2^{-I}j}{2})\right).\end{aligned} \tag{3.31}$$

Else if $x > 1$ then:

$$\begin{aligned}\mathcal{I}^{\int}\varphi_{I,2^I-1}(x) &= \int_{1-2^{-I}}^{1}(2^I t - 2^I + 1)dB(t)\\ &= \frac{1}{3}\left(2B(2^{-I}(j+1)) - B(2^{-I}j) - 4B(2^{-I-1}(2j+1))\right).\end{aligned} \tag{3.32}$$

Now, by using Eqs. (3.20)-(3.32), $\mathcal{I}^{\int}\varphi_{I,j}(x)$ is equal to

$$\begin{cases} for\ \ j=-1: \\ \\ \begin{cases} B(x)\left(1-\dfrac{2^{I+1}x}{3}\right)+\dfrac{2^{I+2}x}{3}B(\dfrac{x}{2}) & x\in[0,2^{-I}) \\ \dfrac{4}{3}B(2^{-I-1})+\dfrac{1}{3}B(2^{-I}) & x\geqslant 2^{-I} \\ 0 & o.w \end{cases} \\ \\ for\ \ j=0,1,\cdots,2^I-2: \\ \\ \begin{cases} \dfrac{2^It-j}{3}\left(2B(x)-B(2^{-I}j)-4B(\dfrac{2^It+j}{2^{I+1}})\right) \\ where\ x\in[2^{-I}j,2^{-I}(j+1)) \\ \\ \dfrac{5+2j-2^{I+1}x}{3}B(x)-\dfrac{1}{3}B(2^{-I}j)+\dfrac{2^Ij-2-j}{3}B((j+1)2^{-I}) \\ -\dfrac{4}{3}B((2j+1)2^{-I-1})+\dfrac{2^{I+2}}{3}\left(x-2^{-I}(j+1)\right)B(\dfrac{x+2^{-I}(j+1)}{2}) \\ where\ x\in[2^{-I}(j+1),2^{-I}(j+2)) \\ \\ \dfrac{-1}{3}B(2^{-I}j)+\dfrac{1}{3}B(2^{-I}(j+2))-\dfrac{4}{3}B(2^{-I-1}(2j+1))+\dfrac{4}{3}B(2^{-I-1}(2j+3)) \\ where\ x\geqslant 2^{-I}(j+2) \\ \\ 0 \qquad\qquad o.w \end{cases} \\ \\ for\ \ j=2^I-1: \\ \begin{cases} \dfrac{2^Ix-2^I+1}{3}\left(2B(x)-B(1-2^{-I})-4B(\dfrac{x+1-2^{-I}}{2})\right) & x\in[1-2^{-I},1) \\ \dfrac{1}{3}\left(2B(1)-B(1-2^{-I})-4B(1-2^{-I-1})\right) & x\geqslant 1 \\ 0 & o.w \end{cases} \end{cases} \tag{3.33}$$

By using Eq. (2.10) and Eq. (3.33), it can be obtained:

$$\mathcal{I}^{\int}\phi_I(x)=\mathbf{I_s}\phi_I(x), \tag{3.34}$$

where

$$I_{i,j}=\mathcal{I}^{\int}(\varphi_j(\mu_i)),\ \mu_i=\frac{i+1}{2^I},\ i,j=-1,0,...,2^I-1. \tag{3.35}$$

Therefore, it can be proven that the stochastic operational matrix of integration based on the linear cardinal B-spline vector, $\mathbf{I_s}$ of order $(2^I+$

$1) \times (2^I + 1)$ is defined as follows:

$$\mathbf{I_s} = \begin{pmatrix} 0 & \gamma & \gamma & \gamma & \cdots & \gamma \\ 0 & \delta_0 & \gamma_0 & \gamma_0 & \cdots & \gamma_0 \\ 0 & 0 & \delta_1 & \gamma_1 & \cdots & \gamma_1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \delta_{2^I-2} & \gamma_{2^I-2} \\ 0 & 0 & 0 & 0 & 0 & \vartheta \end{pmatrix}, \tag{3.36}$$

where

$$\begin{aligned} &\gamma = \frac{1}{3}\left(4B(2^{-I-1}) + B(2^{-I})\right), \\ &\gamma_j = \frac{1}{3}\left(4B(2^{-I}(j+2)) - B(2^{-I}j) - 4B(2^{-I-1}(2j+1)) + 4B(2^{-I-1}(2j+3))\right), \\ &\delta_j = \frac{1}{3}\left(2B(2^{-I}(j+1)) - B(2^{-I}j) - 4B(2^{-I-1}(2j+1))\right), \\ &\vartheta = \frac{1}{3}\left(2B(1) - B(2^{-I}(2^I-1)) - 4B(2^{-I-1}(2^{I+1}-1))\right), \\ &\text{for} \quad j = 0, 1, ..., 2^I - 2. \end{aligned} \tag{3.37}$$

■

□

Note 3.3. [27] *For obtaining the values of $B(x)$ in Eq. (3.37) we apply the Definition (2.5), where:*

$$B(\mu) - B(\xi) \simeq \sqrt{(\mu - \xi)}N(0,1), \;\; \xi < \mu. \tag{3.38}$$

Let us set $\delta t = \dfrac{T}{N}$ for some positive integer N and write B_j for denoting $B(t_j)$ with $t_j = j\delta t$. From definition (2.5) we have $B(0) = 0$ and this definition also states that:

$$B_j = B_{j-1} + dB_j, \;\; j = 0, 1, \cdots, N. \tag{3.39}$$

Theorem 3.4. *Let $\phi_I(x)$ be linear cardinal B-spline vector defined in Eq. (2.10) and $C = [c_{-1}, c_0, \cdots, c_{2^I-1}]^T$ is an arbitrary vector. The operational matrix of product based on the linear cardinal B-spline vector is defined as follows:*

$$\phi_I(x)\phi_I^T(x)C = \widehat{C}\phi_I(x), \tag{3.40}$$

where

$$\widehat{C} = diag[c_{-1}, c_0, \cdots, c_{2^I-1}]. \tag{3.41}$$

Proof. We have $\phi_I(x)\phi_I^T(x)C$ is equal to

$$\begin{pmatrix} \varphi_{-1}(x)\varphi_{-1}(x) & \varphi_{-1}(x)\varphi_0(x) & \varphi_{-1}(x)\varphi_1(x) & \dots & \varphi_{-1}(x)\varphi_{2^I-1}(x) \\ \varphi_0(x)\varphi_{-1}(x) & \varphi_0(x)\varphi_0(x) & \varphi_0(x)\varphi_1(x) & \dots & \varphi_0(x)\varphi_{2^I-1}(x) \\ \varphi_1(x)\varphi_{-1}(x) & \varphi_1(x)\varphi_0(x) & \varphi_1(x)\varphi_1(x) & \dots & \varphi_1(x)\varphi_{2^I-1}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi_{2^I-1}(x)\varphi_{-1}(x) & \varphi_{2^I-1}(x)\varphi_0(x) & \varphi_{2^I-1}(x)\varphi_1(x) & \dots & \varphi_{2^I-1}(x)\varphi_{2^I-1}(x) \end{pmatrix} C, \tag{3.42}$$

where C is

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_{2^I-1} \end{pmatrix}. \tag{3.43}$$

Now, we approximate all functions $\varphi_i(x)\varphi_j(x)$ for $i, j = -1, 0, \cdots, 2^I - 1$ by the linear cardinal B-splines. Therefore,

$$\varphi_i(x)\varphi_j(x) = \sum_{k=-1}^{2^I-1} l_k \varphi_k(x), \tag{3.44}$$

where

$$l_k = \varphi_i(\mu_k)\varphi_j(\mu_k),\ \mu_k = \frac{k+1}{2^I}, \tag{3.45}$$

and

$$\begin{aligned} \varphi_i(x)\varphi_j(x) &= \sum_{k=-1}^{2^I-1} \varphi_i(\mu_k)\varphi_j(\mu_k)\varphi_k(x) = \sum_{k=-1}^{2^I-1} \delta_{ik}\delta_{jk}\varphi_k(x) \\ &= \begin{pmatrix} \varphi_{-1}(x) & 0 & 0 & \cdots & 0 \\ 0 & \varphi_0(x) & 0 & \cdots & 0 \\ 0 & 0 & \varphi_1(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_{2^I-1}(x) \end{pmatrix}. \end{aligned} \tag{3.46}$$

Therefore,

$$\begin{aligned} \phi_I(x)\phi_I^T(x)C &= \begin{pmatrix} \varphi_{-1}(x) & 0 & 0 & \cdots & 0 \\ 0 & \varphi_0(x) & 0 & \cdots & 0 \\ 0 & 0 & \varphi_1(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_{2^I-1}(x) \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_{2^I-1} \end{pmatrix} \\ &= \widehat{C}\phi_I(x). \end{aligned} \tag{3.47}$$

■

□

3.2. **Scheme for the numerical solution.** A new technique for solving SIVIE (1.4) using the linear cardinal B-spline vector is presented. Assume know that:

$$\begin{aligned} z(x) &= g(x) + \int_0^x \lambda_1(x,t)\Upsilon_1(t,z(t))dt + \int_0^x \lambda_2(x,t)\Upsilon_2(t,z(t))dB(t), \\ x &\in D = [0,T]. \end{aligned} \tag{3.48}$$

To obtain the solution of this equation, we expand $z(x), \lambda_1(x,t)$ and $\lambda_2(x,t)$ with linear cardinal B-spline so that:

$$\begin{aligned} &z(x) = Z^T\phi_I(x), \\ &g(x) = G^T\phi_I(x), \\ &\lambda_p(x,t) = \phi_I^T(x)\lambda_p\phi_I(t),\ p = 1,2, \end{aligned} \tag{3.49}$$

with Z, G, $\phi_I(x)$, λ_1 and λ_2 as defined in the previous section. Given different definitions for $\Upsilon_p(t,z(t))$ for $p = 1,2$, we will have different forms of the equation. So we consider cases for $\Upsilon_p(t,z(t))$ for $p = 1,2$.

Case 1: For the first kind, we consider $\Upsilon_p(t,z(t)) = z(t)$ for $p = 1,2$. Therefore, SIVIE (1.4) will be linear and

$$z(x) = g(x) + \int_0^x \lambda_1(x,t)z(t)dt + \int_0^x \lambda_2(x,t)z(t)dB(t),\ x \in D = [0,T]. \tag{3.50}$$

By substituting Eqs. (3.49) into Eq. (3.48) we have $Z^T\phi_I(x)$ is equal to

$$G^T\phi_I(x) + \phi_I^T(x)\lambda_1\int_0^x \phi_I(t)\phi_I^T(t)Zdt + \phi_I^T(x)\lambda_2\int_0^x \phi_I(t)\phi_I^T(t)ZdB(t). \tag{3.51}$$

Therefore, by substituting Eqs. (3.40) into Eq. (3.51) one will set

$$Z^T\phi_I(x) = G^T\phi_I(x) + \phi_I^T(x)\lambda_1\widehat{Z}\int_0^x \phi_I(t)(t)dt + \phi_I^T(x)\lambda_2\widehat{Z}\int_0^x \phi_I(t)dB(t). \tag{3.52}$$

Using Theorems 3.1 and 3.2

$$Z^T\phi_I(x) = G^T\phi_I(x) + \phi_I^T(x)\lambda_1\widehat{Z}\mathbf{I}\phi_I(x) + \phi_I^T(x)\lambda_2\widehat{Z}\mathbf{I_s}\phi_I(x). \tag{3.53}$$

Also, from the initial conditions (1.4):

$$Z^T\phi_I(0) = 0. \tag{3.54}$$

By substituting points $\mu_j = 2^{-I}(j+1)$ for $j = 0,1,\cdots,2^I-1$ in Eq. (3.53) and using Eq. (2.15), we have:

$$Z^Te_j = G^Te_j + e_j^T\lambda_1\widehat{Z}\mathbf{I}e_j + e_j^T\lambda_2\widehat{Z}\mathbf{I_s}e_j, \tag{3.55}$$

and also:

$$Z^T e_{-1} = 0. \tag{3.56}$$

This problem is related to the system of $(2^I + 1)$ algebraic equations in linear B-spline space, which by solving the system of equations and calculating the vector Z, an approximate solution to Eq. (1.4) is obtained.

Case 2: For the second kind, we consider $\Upsilon_p(t, z(t)) = z^k(t)$ for $p = 1, 2$ and $k \in \mathbb{N}$. Therefore, SIVIE (1.4) will be nonlinear and

$$z(x) = g(x) + \int_0^x \lambda_1(x,t) z^k(t) dt + \int_0^x \lambda_2(x,t) z^k(t) dB(t),\ x \in D = [0, T]. \tag{3.57}$$

From Theorem 3.4 it is easily verified that:

$$z^k(t) = Z_k \phi_I(t),\ k \in \mathbb{N}. \tag{3.58}$$

By substituting Eqs. (3.49) and (3.58) into Eq. (3.57) we have $Z^T \phi_I(x)$ is equal to

$$G^T \phi_I(x) + \phi_I^T(x)\lambda_1 \int_0^x \phi_I(t)\phi_I^T(t) Z_k^T dt + \phi_I^T(x)\lambda_2 \int_0^x \phi_I(t)\phi_I^T(t) Z_k^T dB(t). \tag{3.59}$$

Therefore, by substituting Eqs. (3.40) into Eq. (3.59) we have $Z^T \phi_I(x)$ is equal to

$$G^T \phi_I(x) + \phi_I^T(x)\lambda_1 \widehat{Z_k^T} \int_0^x \phi_I(t)(t) dt + \phi_I^T(x)\lambda_2 \widehat{Z_k^T} \int_0^x \phi_I(t) dB(t). \tag{3.60}$$

Using Theorems 3.1 and 3.2,

$$Z^T \phi_I(x) = G^T \phi_I(x) + \phi_I^T(x)\lambda_1 \widehat{Z_k^T} \mathbf{I} \phi_I(x) + \phi_I^T(x)\lambda_2 \widehat{Z_k^T} \mathbf{I_s} \phi_I(x). \tag{3.61}$$

Also, from the initial conditions (1.4):

$$Z^T \phi_I(0) = 0. \tag{3.62}$$

By substituting points $\mu_j = 2^{-I}(j + 1)$ for $j = 0, 1, \cdots, 2^I - 1$ in Eq. (3.53) and using Eq. (2.15), we have:

$$Z^T e_j = G^T e_j + e_j^T \lambda_1 \widehat{Z_k^T} \mathbf{I} e_j + e_j^T \lambda_2 \widehat{Z_k^T} \mathbf{I_s} e_j, \tag{3.63}$$

and also:

$$Z^T e_{-1} = 0. \tag{3.64}$$

This problem corresponds to a system of $(2^I + 1)$ nonlinear equations in $(2^I + 1)$ unknown linear cardinal B-spline coefficients. After solving the system of equations and computing the vector Z , we obtain the approximate solution of Eq. (1.4).

4. Convergence analysis

Let us now examine the convergence and error analysis of the proposed scheme for solving SIVIE. We get an upper error bound on the following theorem.

Theorem 4.1. *Let $g(x) \in C^2[0,1]$ and $g_I(x)$ is approximate solutions of g be the linear cardinal B-spline. Then the upper bound error for the method is*

$$E_I(x) = \|g(x) - g_I(x)\|_2 \leq 2^{-2I-1}\|g^{''}(x)\|_2. \tag{4.1}$$

Proof. Let $D_j = \{x|2^{-I}j \leq x \leq 2^{-I}(j+1)\}$ for $j = 0, 1, \cdots, 2^I - 1$ and define

$$E_j(x) = \begin{cases} g(x) - g_I(x) & x \in D_j \\ 0 & o.w. \end{cases} \tag{4.2}$$

Therefore,

$$\begin{aligned}
&E_j(x) = g(x) - g_I(x) \\
&= g(x) - \sum_{p=-1}^{2^I-1} g(2^{-I}(p+1))\varphi_p(x) \\
&= g(x) - \left[g(2^{-I}j)\varphi_{j-1}(x) + g(2^{-I}(j+1))\varphi_j(x)\right] \\
&= g(x) - \left[g(2^{-I}j)(-2^I x + j + 1) + g(2^{-I}(j+1))(2^I x - j)\right] \\
&= g(x) - \left[g(2^{-I}j) - g(2^{-I}j)(2^I x - j) + g(2^{-I}(j+1))(2^I x - j)\right] \\
&= g(x) - \left[g(2^{-I}j) + 2^I(x - 2^{-I}j)\left(g(2^{-I}(j+1)) - g(2^{-I}j)\right)\right] \\
&= g(x) - \left[g(2^{-I}j) + (x - 2^{-I}j)\left(\frac{g(2^{-I}(j+1)) - g(2^{-I}j)}{2^{-I}}\right)\right] \\
&\simeq g(x) - \left[g(2^{-I}j) + (x - 2^{-I}j)g^{'}(2^{-I}j)\right].
\end{aligned} \tag{4.3}$$

By writing the Taylor expansion for function $g(x)$ in Eq. (4.3) in terms of $(x - 2^{-I}j)$, we have:

$$E_j(x) \simeq \frac{(x - 2^{-I}j)^2}{2} g^{''}(\zeta_j),\ \zeta_j \in (2^{-I}j, 2^{-I}(j+1)). \tag{4.4}$$

Since $|x - 2^{-I}j| \leq 2^{-I}$:

$$|E_j(x)| \simeq 2^{-2I-1}|g^{''}(\zeta_j)|, \tag{4.5}$$

Thus

$$\|E(x)\| = \sup_{x \in D_j} |E_j| \simeq 2^{-2I-1}\|g^{''}(\zeta_j)\|. \tag{4.6}$$

■

□

Lemma 4.2. *Suppose $g_I(x) = G\phi_I(x)$ is the best approximation for the function $g(x) \in C^2[0,1]$ by the linear cardinal B-spline functions. In this case, we have:*

$$\|E(x)\| = \|g(x) - g_I(x)\| = \mathbb{O}\left(\frac{1}{2^{2I}}\right). \tag{4.7}$$

Proof. By applying Theorem 4.1, the result is easily achieved.

■

□

Theorem 4.3. *Suppose $\lambda_I(x,t)$ is the best approximation for the function $\lambda(x,t) \in C^2([0,1]\times[0,1])$ by the linear cardinal B-spline functions. In this case, $\|E(x,t)\|$ is equal to*

$$\|\lambda(x,t)-\lambda_I(x,t)\| \leq 2^{-2I-1}\left(\|\frac{\partial^2\lambda(x,t)}{\partial x^2}\| + 2\|\frac{\partial^2\lambda(x,t)}{\partial x\partial t}\| + \|\frac{\partial^2\lambda(x,t)}{\partial t^2}\|\right). \tag{4.8}$$

Proof. Let $D_{i,j} = \left\{(x,t)|2^{-I}i \leq x \leq 2^{-I}(i+1), 2^{-I}j \leq t \leq 2^{-I}(j+1)\right\}$ for $i,j = 0,1,\cdots,2^I-1$ and define

$$E_{i,j}(x,t) = \begin{cases} \lambda(x,t) - \lambda_I(x,t) & x \in D_{i,j} \\ 0 & o.w. \end{cases} \tag{4.9}$$

Therefore, $E_{i,j}(x)$ is equal to

$$\begin{aligned}
&= \lambda(x,t) - \lambda_I(x,t) \\
&= \lambda(x,t) - \sum_{p=-1}^{2^I-1}\sum_{q=-1}^{2^I-1}\lambda(2^{-I}(p+1), 2^{-I}(q+1))\varphi_p(x)\varphi_q(t) \\
&= \lambda(x,t) - [\lambda(2^{-I}i, 2^{-I}j)\varphi_{i-1}(x)\varphi_{j-1}(t) + \lambda(2^{-I}(i+1), 2^{-I}j)\varphi_i(x)\varphi_{j-1}(t) \\
&+ \lambda(2^{-I}i, 2^{-I}(j+1))\varphi_{i-1}(x)\varphi_j(t) + \lambda(2^{-I}(i+1), 2^{-I}(j+1))\varphi_i(x)\varphi_j(t)].
\end{aligned} \tag{4.10}$$

By writing the Taylor expansion for function $\lambda(x,t)$ in Eq. (4.10) in terms of $(x - 2^{-I}j)$, we have:

$$E_{i,j}(x,t) \simeq \frac{1}{2}\left(\frac{\partial}{\partial x}(x - 2^{-I}i) + \frac{\partial}{\partial t}(t - 2^{-I}j)\right)^2 \lambda(\zeta_i, \eta_j), \tag{4.11}$$

where $\zeta_i \in (2^{-I}i, 2^{-I}(i+1)), \eta_j \in (2^{-I}j, 2^{-I}(j+1))$.

Since $|x-2^{-I}i| \leq 2^{-I}$ and $|t-2^{-I}j| \leq 2^{-I}$:

$$|E_{i,j}(x,t)| \simeq 2^{-2I-1}\left(|\frac{\partial^2\lambda(\zeta_i,\eta_j)}{\partial x^2}| + 2|\frac{\partial^2\lambda(\zeta_i,\eta_j)}{\partial x\partial t}| + |\frac{\partial^2\lambda(\zeta_i,\eta_j)}{\partial t^2}|\right). \tag{4.12}$$

Thus

$$\begin{aligned}\|E(x,t)\| &= \sup_{(x,t)\in D_{i,j}} |E_{i,j}(x,t)| \\ &\leq 2^{-2I-1} \sup_{(x,t)\in[0,1]\times[0,1]} \left(|\frac{\partial^2\lambda(x,t)}{\partial x^2}| + 2|\frac{\partial^2\lambda(x,t)}{\partial x\partial t}| + |\frac{\partial^2\lambda(x,t)}{\partial t^2}|\right).\end{aligned} \tag{4.13}$$

Finally

$$\|E(x,t)\| \leq 2^{-2I-1}\left(\|\frac{\partial^2\lambda(x,t)}{\partial x^2}\| + 2\|\frac{\partial^2\lambda(x,t)}{\partial x\partial t}\| + \|\frac{\partial^2\lambda(x,t)}{\partial t^2}\|\right). \tag{4.14}$$

■

□

Lemma 4.4. *Suppose $\lambda_I(x,t)$ is the best approximation for the function $\lambda(x,t) \in C^2([0,1]\times[0,1])$ by the linear cardinal B-spline functions. In this case, we have:*

$$\|E(x,t)\| = \|\lambda(x,t) - \lambda_I(x,t)\| = \mathbb{O}\left(\frac{1}{2^{2I}}\right). \tag{4.15}$$

Proof. By applying Theorem 4.3, the result is easily achieved.

■

□

Theorem 4.5. *Let $z(x)$ and $z_I(x)$ be the exact and approximate solutions of Eq. (1.4), and*

$$\begin{aligned}(i)\ \|z\|_2 &\leq M, \\ (ii)\ \|\lambda_p\|_2 &\leq M_p \text{ for } p=1,2, \\ (iii)\ \|\Upsilon_i - \Upsilon_{iI}\|_2 &\leq L_i\|z-z_I\|_2 \text{ for } i=0,1 \text{ Lipschitz condition},\end{aligned} \tag{4.16}$$

then:

$$\|e_I\|_2 \leq \frac{\xi + M(\gamma_1+\gamma_2)}{1 - L_1(M_1+\gamma_1) - L_2(M_2+\gamma_2)}. \tag{4.17}$$

Proof. We have:

$$z(x) = g(x) + \int_0^x \lambda_1(x,t)\Upsilon_1(t,z(t))dt + \int_0^x \lambda_2(x,t)\Upsilon_2(t,z(t))dB(t). \tag{4.18}$$

The approximate equation of Eq. (4.18) is

$$z_I(x) = g_I(x) + \int_0^x \lambda_{1I}(x,t)\Upsilon_1(t, z_I(t))dt + \int_0^x \lambda_{2I}(x,t)\Upsilon_2(t, z_I(t))dB(t). \tag{4.19}$$

Let e_I represent the error function $e_I(v) = z(v) - z_I(v)$. Therefore, from Eqs. (4.18) and (4.19) we have:

$$\begin{aligned} e_I(x) = \{g(x) - g_I(x)\} &+ \int_0^x \{\lambda_1(x,t)\Upsilon_1(t, z(t)) - \lambda_{1I}(x,t)\Upsilon_1(t, z_I(t))\}dt \\ &+ \int_0^x \{\lambda_2(x,t)\Upsilon_2(t, z(t)) - \lambda_{2I}(x,t)\Upsilon_2(t, z_I(t))\}dB(t). \end{aligned} \tag{4.20}$$

From Eq. (4.20)

$$e_I(x) = J_1(x) + J_2(x) + J_3(x), \tag{4.21}$$

where

$$J_1(x) = g(x) - g_I(x), \tag{4.22}$$

$$J_2(x) = \int_0^x \{\lambda_1(x,t)\Upsilon_1(t, z(t)) - \lambda_{1I}(x,t)\Upsilon_1(t, z_I(t))\}dt, \tag{4.23}$$

and

$$J_3(x) = \int_0^x \{\lambda_2(x,t)\Upsilon_2(t, z(t)) - \lambda_{2I}(x,t)\Upsilon_2(t, z_I(t))\}dB(t). \tag{4.24}$$

By using Theorem 4.1 we have:

$$\|J_1\|_2 = \|g - g_I\|_2 \leqslant \xi,\ \xi = \frac{1}{2^{2I+1}}\|g''(x)\|_2. \tag{4.25}$$

Therefore, we now estimate $J_2(x)$ and $J_3(x)$.
For J_2 we have:

$$\begin{aligned} \|J_1\|_2^2 &= \int_0^1 \left| \int_0^x \{\lambda_1(x,t)\Upsilon_1(t, z(t)) - \lambda_{1I}(x,t)\Upsilon_1(t, z_I(t))\}dt \right|^2 dx \\ &\leq \int_0^1 \left(\int_0^x |\lambda_1(x,t)\Upsilon_1(t, z(t)) - \lambda_{1I}(x,t)\Upsilon_1(t, z_I(t))|dt \right)^2 dx \\ &\leq \int_0^1 \int_0^1 |\lambda_1(x,t)\Upsilon_1(t, z(t)) - \lambda_{1I}(x,t)\Upsilon_1(t, z_I(t))|^2 \, dtdx \\ &= \|\lambda_1\Upsilon_1 - \lambda_{1I}\Upsilon_{1I}\|_2^2. \end{aligned}$$

Therefore:

$$\|J_2\|_2 \leq \|\lambda_1\Upsilon_1 - \lambda_{1I}\Upsilon_{1I}\|_2. \tag{4.26}$$

For J_3 we have:

$$\|J_2\|_2^2 = \int_0^1 \left| \int_0^x \{\lambda_2(x,t)\Upsilon_2(t,z(t)) - \lambda_{2I}(x,t)\Upsilon_2(t,z_I(t))\} dB(t) \right|^2 dx$$
$$\leq \int_0^1 \left(\int_0^x |\lambda_2(x,t)\Upsilon_2(t,z(t)) - \lambda_{2I}(x,t)\Upsilon_2(t,z_I(t))| dB(t) \right)^2 dx.$$

By using Eq. (2.8) it comes:

$$E\|J_2\|_2^2 \leq \int_0^1 E\left(\int_0^1 |\lambda_2(x,t)\Upsilon_2(t,z(t)) - \lambda_{2I}(x,t)\Upsilon_2(t,z_I(t))|^2 dt \right) dx.$$

Therefore:

$$\|J_2\|_2 \leq \|\lambda_2\Upsilon_2 - \lambda_{2I}\Upsilon_{2I}\|_2. \tag{4.27}$$

From Eq. (4.21) yields:

$$\|e_I\|_2 = \|z - z_I\|_2 \leq \|g - g_I\|_2 + \|\lambda_1\Upsilon_1 - \lambda_{1I}\Upsilon_{1I}\|_2 + \|\lambda_2\Upsilon_2 - \lambda_{2I}\Upsilon_{2I}\|_2. \tag{4.28}$$

On the other hand, for $i = 0, 1$

$$\begin{aligned} \|\lambda_i\Upsilon_i - \lambda_{iI}\Upsilon_{iI}\|_2 &= \|\lambda_i\Upsilon_i - \lambda_i\Upsilon_{iI} + \lambda_i\Upsilon_{iI} + \lambda_{iI}\Upsilon_{iI}\|_2 \\ &= \|\lambda_i(\Upsilon_i - \Upsilon_{iI}) + \lambda_i(\Upsilon_{iI} + \Upsilon_i) - \lambda_{iI}(\Upsilon_{iI} - \Upsilon_i) + \lambda_i\Upsilon_i - \lambda_{iI}\Upsilon_i\|_2 \\ &\leq \|\lambda_i\|_2\|\Upsilon_i - \Upsilon_{iI}\|_2 + \|\lambda_i - \lambda_{iI}\|_2(\|\Upsilon_i - \Upsilon_{iI}\|_2 + \|\Upsilon_i\|_2). \end{aligned} \tag{4.29}$$

Following the assumptions (*iii*):

$$\|\lambda_i\Upsilon_i - \lambda_{iI}\Upsilon_{iI}\|_2 \leq L_i\|\lambda_i\|_2\|z - z_I\|_2 + \|\lambda_i - \lambda_{iI}\|_2(L_i\|z - z_I\|_2 + \|z_i\|_2), \tag{4.30}$$

from Theorem 4.3, we have $\|\lambda_i - \lambda_{iI}\|_2 \leq \gamma_i$ for $i = 1, 2$, therefore:

$$\|e_I\|_2 \leq \frac{\xi + M(\gamma_1 + \gamma_2)}{1 - L_1(M_1 + \gamma_1) - L_2(M_2 + \gamma_2)}. \tag{4.31}$$

■

□

5. Numerical feedbacks

In this section, the performance, accuracy, and efficiency of the proposed method have been evaluated through the analysis of three numerical examples. It should be noted that due to the presence of integrals and random coefficients in the examples, the final results are constantly changing. Therefore, in order to achieve valid numerical results, the process of executing the examples has been repeated at least 20 times using MATLAB software and finally the most desirable results have been presented.

In the follow-up, the numerical convergence rate of the proposed method is examined. For each scale I, let $h = 2^{-I}$ be the approximate mesh size, and define the discrete maximum error over a fixed set of test points $\mathcal{T}$ as

$$E_I = \max_{x\in\mathcal{T}} \big|z(x) - z_I(x)\big|, \tag{5.1}$$

where $z(x)$ is the exact solution and $z_I(x)$ is the numerical solution obtained by the proposed method. Due to the stochastic nature of the problem, each E_I is reported after averaging over 20 independent Brownian motion realizations. According to Theorem 4.1, the approximation error behaves as $\mathcal{O}(2^{-2I}) = \mathcal{O}(h^2)$, i.e., the method is expected to exhibit second-order convergence. To verify this numerically, we assume the error follows the model $E_I \approx C\, h^p = C\, 2^{-pI}$, where p is the numerical convergence rate. For two successive scales I_1 and I_2 with $I_2 > I_1$, the order is estimated by

$$p = \frac{\log_2(E_{I_1}) - \log_2(E_{I_2})}{I_2 - I_1}. \tag{5.2}$$

In particular, when $I_2 = I_1 + 2$, Eq. (5.2) simplifies to

$$p = \frac{1}{2}\,\log_2\!\left(\frac{E_{I_1}}{E_{I_2}}\right). \tag{5.3}$$

Example 5.1. Consider the following nonlinear SIVIE [22, 23]:

$$z(x) = \frac{1}{10} - \frac{1}{900}\left(\int_0^x z(t)dt - \int_0^x z^3(t)dt\right) + \frac{1}{30}\int_0^x (1 - z(t))^2\, dB(t), \tag{5.4}$$

where $x \in [0, 1)\,, z(0) = 1$.

The exact answer will be

$$z(x) = tanh\left(\frac{1}{30}B(x) + arctanh(\frac{1}{10})\right). \tag{5.5}$$

For different values of I, a comprehensive comparison between the approximate solution and the exact solution of the problem is performed, which is shown in Figures 1 and 2. These figures help to better understand the efficiency of the method. In addition, Table 1 includes the absolute error for different values of x with for $I = 4$, $I = 6$, $I = 8$ and $I = 10$. The comparison presented in Table 2, shows the fundamental difference between the proposed method and some existing methods such as the Legendre wavelets Galerkin method [22] and piecewise spectral-collocation method [23]. In such a way that the specific characteristics and influential parameters of each method were examined independently. By analyzing the data in this table, one can see a significant difference in the accuracy and efficiency of the method compared

to other methods. Using the maximum errors extracted from Table 1, the computed convergence orders are presented in Table 3. As can be seen from Table 3, the computed orders fluctuate around $p = 2$, which is in good agreement with the theoretical second-order convergence established in Theorem 4.1. The slight variations are attributed to the inherent randomness of the stochastic differential equation and the finite number of Monte Carlo realizations (20 runs). Averaging the three computed orders gives $\bar{p} \approx 1.98$, confirming the expected convergence rate.

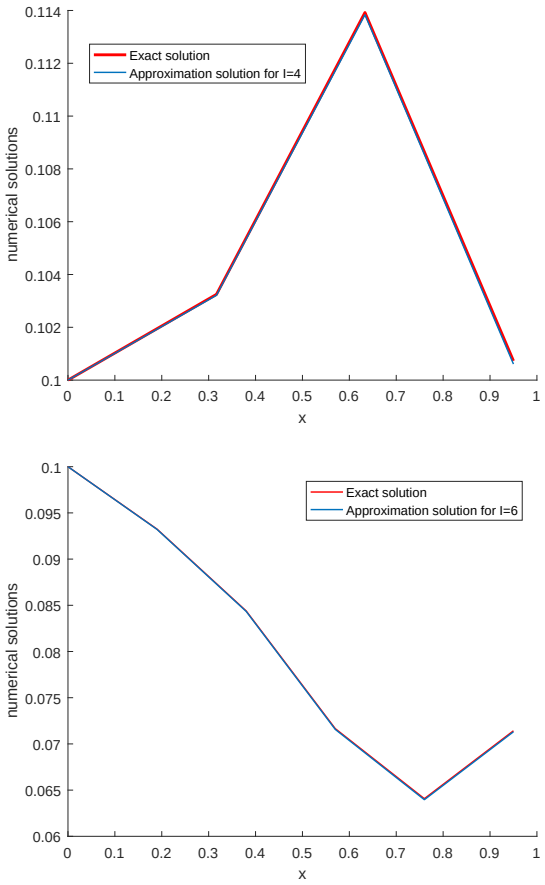


FIGURE 1. Approximate solutions and exact solutions of Example 5.1 for $I = 4$ and $I = 6$.

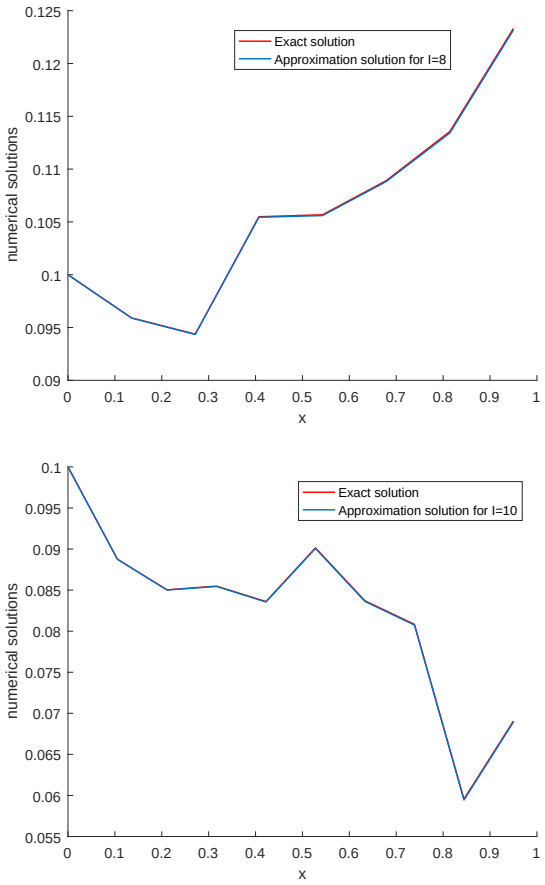


FIGURE 2. Approximate solutions and exact solutions of Example 5.1 for $I = 8$ and $I = 10$.

TABLE 1. The absolute error of Example 5.1 for $I = 4$, $I = 6$, $I = 8$ and $I = 10$ with different values of x.

x	$I = 4$	$I = 6$	$I = 8$	$I = 10$
0.1000	3.5892×10^{-5}	3.1554×10^{-6}	1.7045×10^{-7}	1.3562×10^{-8}
0.3000	8.5586×10^{-5}	4.9955×10^{-6}	3.0742×10^{-7}	2.5902×10^{-8}
0.5000	1.2199×10^{-5}	8.5285×10^{-6}	5.1794×10^{-7}	3.7774×10^{-8}
0.7000	4.9552×10^{-4}	1.1081×10^{-5}	7.8466×10^{-7}	1.3155×10^{-7}
0.9000	8.6064×10^{-5}	1.1057×10^{-5}	1.4685×10^{-6}	1.1864×10^{-8}

Example 5.2. Consider the following nonlinear SIVIE [24]:

$$z(x) = 1 + \int_0^x \frac{1}{32} z(t)dt - \int_0^x z^3(t)dt + \int_0^x 0.25 z(t)dB(t), \ x \in [0, 1], \tag{5.6}$$

TABLE 2. The comparison of the obtained results for Example 5.1.

Methods	Conditions	Maximum Errors
Legendre wavelets [22]	$M = 48$	1.954×10^{-3}
Piecewise spectral-collocation [23]	$M = 12, N = 12$	5.700×10^{-4}
Proposed method	$I = 4$	1.2199×10^{-5}
Proposed method	$I = 6$	1.1057×10^{-6}
Proposed method	$I = 10$	1.1864×10^{-8}

TABLE 3. Convergence analysis for Example 5.1.

I	$h = 2^{-I}$	L_∞ error	Order p
4	0.0625	4.9552×10^{-4}	–
6	0.015625	1.1081×10^{-5}	2.74
8	0.00390625	1.4685×10^{-6}	1.46
10	0.0009765625	1.3155×10^{-7}	1.74

where $z(0) = 1$.

The exact answer will be

$$z(x) = \frac{e^{0.25B(x)}}{\sqrt{1 + 2\int_0^x e^{0.5B(t)}dt}}. \tag{5.7}$$

At first, considering different values of $I = 4$ and $I = 6$ we solve the equation, which the absolute errors are reported in Table 4, for different values of x. The comparison presented in Table 4, shows the fundamental difference between the proposed method and some existing methods such as the operational matrices of Chebyshev cardinal functions [24]. Comparisons have been made within the specific conditions of each method and by considering their specific parameters. By analyzing the data in this table, one can see a significant difference in the accuracy and efficiency of the method compared to other methods. Using the maximum errors extracted, the computed convergence orders are presented in Table 5. The computed convergence orders remain remarkably stable at $p \approx 2.32$ across all mesh refinements. This value slightly exceeds the theoretical second-order rate established in Theorem 4.1, indicating that the proposed linear cardinal B-spline method performs particularly well for this class of nonlinear stochastic Volterra integral equations with smooth exact solutions. The consistency of the order over multiple scales also confirms the robustness of the method and the reliability of the numerical results.

TABLE 4. Comparing errors of Example 5.2.

	Method in [24]	Proposed method	
v	$N = 10$	$I = 4$	$I = 6$
0.1000	8.2714×10^{-2}	6.5350×10^{-5}	8.8895×10^{-6}
0.3000	1.7747×10^{-2}	8.6064×10^{-5}	7.4322×10^{-6}
0.5000	5.9106×10^{-2}	1.2409×10^{-4}	1.0453×10^{-5}
0.7000	4.8138×10^{-2}	9.2100×10^{-5}	1.7366×10^{-6}
0.9000	8.8528×10^{-2}	2.6482×10^{-4}	2.1378×10^{-6}

TABLE 5. Convergence analysis for Example 5.2.

I	$h = 2^{-I}$	L_∞ error	Order p
4	0.0625	2.6482×10^{-4}	–
6	0.015625	1.0453×10^{-5}	2.33
8	0.00390625	4.1373×10^{-7}	2.31
10	0.0009765625	1.6451×10^{-8}	2.32

Example 5.3. Consider the following linear SIVIE [25]:

$$z(x) = -\frac{1}{8} - \int_0^x \frac{1}{4} t z(t) dt - \int_0^x \frac{1}{40} z(t) dB(t), \ x \in [0, 1], \tag{5.8}$$

where $z(0) = 0$.

The exact answer will be

$$z(x) = -\frac{1}{8} e^{-\left(\frac{1}{40}B(x) + \frac{1}{8}x^3 + \frac{1}{3200}x^2\right)}. \tag{5.9}$$

For $I = 6$, a comprehensive comparison between the approximate solution and the exact solution of the problem is performed, which is shown in Figures 3. These figures help to better understand the efficiency of the method. In addition, Table 6 includes the absolute error for different values of x with for $I = 4$ and $I = 6$. The comparison presented in Table 6, shows the fundamental difference between the proposed method and some existing methods such as the explicit finite difference scheme and Bernstein polynomials [25]. Comparisons have been made within the specific conditions of each method and by considering their specific parameters. By analyzing the data in this table, one can see a significant difference in the accuracy and efficiency of the method compared to other methods.

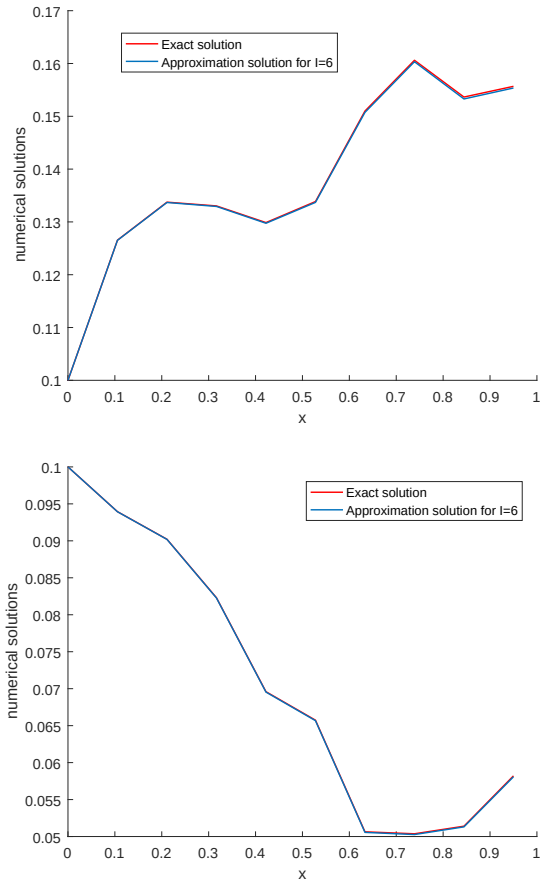


FIGURE 3. Approximate solutions and exact solutions of Example 5.3 for Various performances as $I = 6$.

TABLE 6. Comparing errors of Example 5.3

	Method in [25]	Proposed method	
x	$N = 16$	$I = 4$	$I = 6$
0.1000	6.2500×10^{-8}	3.4224×10^{-8}	6.6547×10^{-9}
0.3000	1.2500×10^{-7}	3.3212×10^{-8}	5.2103×10^{-9}
0.5000	1.8750×10^{-7}	9.6805×10^{-8}	6.3143×10^{-10}
0.7000	2.5000×10^{-7}	1.2497×10^{-8}	2.8374×10^{-9}
0.9000	3.1250×10^{-7}	9.8515×10^{-8}	1.4747×10^{-10}

6. Conclusion

This study developed a computationally efficient framework for solving linear and nonlinear stochastic Itô-Volterra integral equations driven by Brownian motion, using an operational matrix approach based on linear cardinal B-spline functions combined with Simpson's rule. We established matrix representations for both the deterministic Volterra integral and the stochastic Itô integral, and derived the product operational matrix with modest computational complexity. A rigorous convergence analysis proved that the method achieves an accuracy order of $\mathcal{O}\left(\frac{1}{2^{2I}}\right)$, and numerical examples confirmed its efficiency and high accuracy compared to existing techniques. The numerical results confirm that the proposed method consistently achieves at least second-order convergence, in accordance with the theoretical error bound established in Theorem 4.1. In certain cases (Examples 5.2), the observed convergence rate even exceeds the theoretical prediction, demonstrating the robustness and efficiency of the linear cardinal B-spline operational matrix approach.

The main contributions and advantages of this work are:

- The method is mesh-free, eliminating the need for domain discretization.
- High accuracy is attainable with only a small number of basis functions.
- The scheme reduces the original linear or nonlinear SIVIEs to algebraic systems that can be solved by standard solvers.
- It provides an efficient framework for solving SIVIEs and can be extended to similar problems with an Itô-Volterra structure.

Nevertheless, the current study has certain limitations. The analysis assumes a standard Brownian motion driver; extensions to more general stochastic processes (e.g., jump-diffusions or fractional Brownian motion) require additional theoretical justification. Moreover, while the operational matrices are compact, applying the method to very high-dimensional systems may increase computational cost, and the present formulation focuses on integral equations rather than differential equations. Future work will therefore focus specifically on adapting this method to stochastic fractional differential equations, as well as extending it to more complex problems such as multidimensional stochastic fractional differential-integral equations.

7. Declarations

Ethical Statement

Not Applicable.

Consent for Publication

Not Applicable.

Funding

This research received no funding.

Availability of Data and Materials

Ph.D. thesis dissertations are public documents, but at the moment are not internet-accessible. If one is interested in revising the investigation, information can be requested from the authors.

Conflict of Interest

The authors declare no competing interests.

Authors' Contributions

Conceptualization, K. S. and I. M.; Formal analysis, K. S., I. M. and P. M; Investigation, K. S., I. M. and P. M.; Methodology, K. S., I. M. and P. M.; Validation, K. S., I. M.; Visualization, K. S.; Writing-original draft, K. S., I. M. and P. M.; Writing-review-editing, K. S.; I. M.; All authors contributed equally to this paper.

Institutional Review Board Statement

Not Applicable.

Acknowledgments

The authors wish to express their cordial thanks to the anonymous referees and editor of the journal for useful suggestions and comments.

References

[1] F. C. Klebaner, Introduction to stochastic calculus with applications, World Scientific Publishing Company, 2012.

[2] K. Itô, On a stochastic integral equation, *Proceedings of the Japan Academy,* **22***(1-4) (1946), 32-35.*

[3] M. Carletti, K. Burrage, and P. M. Burrage, Numerical simulation of stochastic ordinary differential equations in bio mathematical modelling, *Mathematics and Computers in Simulation,* **64***(2) (2004), 271-277.*

[4] N. Ikeda, and S. Watana, Stochastic differential equations and diffusion processes, Elsevier, 2004.

[5] P. Kloeden, and A. Neuenkirch, Convergence of numerical methods for stochastic differential equations in mathematical finance: In recent developments in computational finance: Foundations, *Algorithms and Applications (2013), 49-80.*

[6] M. Khodabin, K. Maleknejad, M. Rostami, and M. Nouri, Interpolation solution in generalized stochastic exponential population growth model, *Applied Mathematical Modelling,* **36** *(3) (2012), 1023-1033.*

[7] B. Oksendal, Stochastic differential equations: an introduction with applications, Springer Science & Business Media, 2013.

[8] L. P. Kadanoff, Statistical physics: statics, dynamics and renormalization, World Scientific, 2000.

[9] F. Mohammadi, Efficient Galerkin solution of stochastic fractional differential equations using second kind Chebyshev wavelets, *Boletim da Sociedade Paranaense de Mathematica,* **35** *(1) (2015), 195-215.*

[10] Z. Taheri, S. Javadi, and E. Babolian, Numerical solution of stochastic fractional integro-differential equation by the spectral collocation method, Journal of Computational and Applied Mathematics, **321** (2017) 336-347.

[11] K. Sayevand, J. Tenreiro Machado, and I. Masti, On dual Bernstein polynomials and stochastic fractional integro-differential equations, Mathematical Methods in the Applied Sciences, **43** (17) (2020), 9928-9947.

[12] I. Masti, and K. Sayevand, On collocation-Galerkin method and fractional B-spline functions for a class of stochastic fractional integro-differential equations, *Mathematics and Computers in Simulation,* **216** *(2024), 263-287.*

[13] M. Kamrani, Convergence of Galerkin method for the solution of stochastic fractional integro differential equations, *Optik,* **127***(20) (2016), 10049-10057.*

[14] E. Aryani, A. Babaei, and A. Valinejad, A numerical technique for solving nonlinear fractional stochastic integro-differential equations with n-dimensional Wiener process, *Computational Methods for Differential Equations,* **10** *(1)(2022), 61-67.*

[15] S. Banihashemi, H. Jafari, and A. Babaei, A stable collocation approach to solve a neutral delay stochastic differential equation of fractional order, *Journal of Computational and Applied Mathematics,* **403** *2022), 113845.*

[16] Z. Moniri, A, Babaei, and B. Parsa Moghaddam, Robust numerical framework for simulating 2D fractional time-space stochastic diffusion equation driven by spatio-temporal noise: L1-FFT hybrid approach, *Communications in Nonlinear Science and Numerical Simulation,* **146** *(2025), 108791.*

[17] L. He, S. Banihashemi, H. Jafari , and A. Babaei, Numerical treatment of a fractional order system of nonlinear stochastic delay differential equations using a computational scheme, *Chaos, Solitons & Fractals,* **149** *(2021) 111018.*

[18] B. P. Moghaddam, A. Babaei, A. Dabiri, and A. Galhano, Fractional stochastic partial differential equations: Numerical advances and practical applications-A state of the art review, *Symmetry* **16** *(5) (2025), 563.*

[19] A. Pourdarvish, K. Sayevand, I. Masti, and S. Kumar, Orthonormal Bernoulli polynomials for solving a class of two dimensional stochastic Volterra–Fredholm integral equations, *International Journal of Applied and Computational Mathematics,* **8** *(1) (2022), 31.*

[20] I. Masti, K. Sayevand, and H. Jafari, On analyzing two dimensional fractional order brain tumor model based on orthonormal Bernoulli polynomials and Newton's method, *An International Journal of Optimization and Control: Theories & Applications,* **14** *(1) (2024), 12-19.*

[21] I. Masti, K. Sayevand , and H. Jafari, On epidemiological transition model of the Ebola virus in fractional sense, *Journal of Applied Analysis and Computation,* **14***(3) (2024), 1625-1647.*
[22] M. H. Heydari, M. R. Hooshmandasl, A. Shakiba, and C. Cattani, Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations, *Nonlinear Dynamics,* **85** *(2016), 1185-1202.*
[23] F. Mohammadi, Numerical treatment of nonlinear stochastic Itô-Volterra integral equations by piecewise spectral-collocation method, *Journal of Computational and Nonlinear Dynamics,* **14** *(3) (2019), 031007.*
[24] Z. Rebiha, New numerical method for solving nonlinear stochastic integral equations, *Vladikavkaz Mathematical Journal,* **22** *(4)(2020), 68-86.*
[25] E. O. Oduselu-Hassan, I. N. Njoseh, and J. Tsetimi, A Numerical approximation of the stochastic Itô-Volterra integral equation, *Asian Research Journal of Mathematics,* **19** *(11) (2023), 61-68.*
[26] P. Mörters , and Y. Peres, Brownian Motion (Vol. 30), Cambridge University Press, 2010.
[27] D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rview,* **43** *(3)(2001), 525-546.*