Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2026.31192.1813
Caspian J Math Sci. **15**(2)(2026), 518-531 (Research Article)

Join-simple Filter Graph of a Lattice

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Abstract. Let $\mathcal{L}$ be a distributive lattice with 1. The join-simple filter graph $\mathbb{SG}(\mathcal{L})$ of $\mathcal{L}$ is a simple undirected graph whose vertices are all nontrivial filters of $\mathcal{L}$ and any two distinct vertices F and G are adjacent if and only if $F \cap G$ is a simple filter of $\mathcal{L}$. The clique number, connectedness, split character, planarity, and domination number of $\mathbb{SG}(\mathcal{L})$, as well as their relations to algebraic properties of $\mathcal{L}$, are explored.

Keywords: lattices, simple filter, join-simple graph.

2020 Mathematics subject classification: Primary 97H50, 06A11; Scondary 16G30, 06D25

1. Introduction

The study of graphs associated with algebraic structures has become an important research topic and has received considerable attention from researchers over the past decades. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. One of the central questions in this area is to characterize the graphs that can be associated

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Received: 03 February 2026
Revised: 19 June 2026
Accepted: 23 June 2026
How to Cite: Khoramdel, Mehdi; Ebrahimi Atani, Shahabaddin. Join-simple Filter Graph of a Lattice. Casp.J. Math. Sci.,**15**(2)(2026), 518-531.
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with algebraic structures. Efforts toward answering this question typically involve the study of graph invariants, with the aim of establishing criteria and uncovering structural properties that govern the admissibility of graph representations of algebraic objects. One of the most important graphs which have been studied is the intersection graph. Bosak [3] defined the intersection graph of semigroups. *Csàkàny* and *Pollàk* [6] studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [5]. The intersection minimal ideal graph of a ring, i.e. a simple graph whose vertices are nontrivial ideals of a ring R and two vertices I, J are adjacent if the intersection of corresponding ideals is a minimal ideal, was investigated by Barman and Rajkhowa in [4]. The authors in [1] defined a novel intersection graph on a left R-module M, whose vertices are precisely the nonzero submodules of M. Two distinct vertices N and K are connected by an edge if and only if their intersection $N \cap K$ forms a simple submodule of M. The intersection graph of ideals of rings, submodules of modules and lattices has been investigated by several authors ([2, 10, 11, 12, 13, 14]).

Let $\mathcal{L}$ be a distributive lattice with 1. Motivated by works done in [1] and [4], we investigate a graph associated to a lattice $\mathcal{L}$ called the join-simple filter graph of $\mathcal{L}$. This will result in characterization of lattices in terms of some specific properties of those graphs. The join-simple filter graph of $\mathcal{L}$ is a simple graph $\mathbb{SG}(\mathcal{L})$ whose vertices are all nontrivial filters and two distinct vertices are adjacent if and only if the join of the corresponding filters is a simple filter of $\mathcal{L}$.

Here is a brief outline of the article. Among many results in this paper, the first, Preliminaries section contains elementary observations needed later on. In Section 2 and Section 3, We characterize the lattices for which the join-simple filter graphs are connected, complete bipartite and star. The concepts of planarity, clique number, domination number and split character are also investigated. *Most of the results in this article are observed in Artinian lattices.*

2. PRELIMINARIES

Let G be a simple graph with vertex set $\mathcal{V}$ (G) and edge set $\mathcal{E}$ (G). The degree of a vertex v of the graph G, denoted by $\deg_G(v)$, is the number of edges incident to v. The (open) neighborhood $N(v)$ of a vertex v of $\mathcal{V}(G)$ is the set of vertices which are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices. If the vertices of G can be partitioned into two disjoint sets V_1 and V_2 with

every vertex of V_1 is adjacent to any vertex of V_2 and no two vertices belonging to same set are adjacent, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. If one of the partite sets contains exactly one element, then the graph becomes a star graph. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. If a graph G have K_5 or $K_{3,3}$ as its subgraph, then G is not planar [15]. Let u and v be elements of $\mathcal{V}(G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is $\text{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\text{gr}(G) = \infty$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. A subset S of $\mathcal{V}(G)$ is said to be an independent set if no two vertices of S are adjacent. If $\mathcal{V}(G)$ can be partitioned in an independent set and a clique, then G is said to be split. A set $D \subseteq \mathcal{V}(G)$ is said to be a dominating set, if every vertex not in D is adjacent to at least one of the members of D. The cardinality of smallest dominating set is the domination number of the graph G and is denoted by $\gamma(G)$. Note that a graph whose vertex set is empty is a null graph and a graph whose edge set is empty is an empty graph. For a connected graph G, x is a cut vertex of G, if $G \setminus \{x\}$ is not connected [15].

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in \mathcal{L}$). A non-empty subset F of a lattice $\mathcal{L}$ is called a filter, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with $0, 1$, then $1 \in F$, $0 \in F$ if and only if $F = \mathcal{L}$ and $\{1\}$ is a filter of $\mathcal{L}$). A proper filter F of $\mathcal{L}$ is said to be maximal if G is a filter in $\mathcal{L}$ with $F \subsetneqq G$, then $G = \mathcal{L}$. The set of all maximal filters of $\mathcal{L}$ is denoted by $\text{Max}(\mathcal{L})$. A simple lattice (resp. a simple filter) is a lattice (resp. a filter) that has no filters besides the $\{1\}$ and itself. The set of all simple filters of $\mathcal{L}$ is denoted by $\mathcal{S}(\mathcal{L})$. For each lattice $\mathcal{L}$,

$\mathrm{Soc}(\mathcal{L}) = \bigwedge_{S\in\mathcal{S}(\mathcal{L})} S$. A filter G of a lattice $\mathcal{L}$ is called essential in $\mathcal{L}$ if $G \cap H \neq \{1\}$ for any filter $H \neq \{1\}$ of $\mathcal{L}$ [7].

Lemma 2.1. *Let $\mathcal{L}$ be a lattice* [8].

(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.

(2) If F_1, F_2 are filters of $\mathcal{L}$ and $a \in \mathcal{L}$, then $F_1 \vee F_2 = \{a_1 \vee a_2 : a_1 \in F_1, a_2 \in F_2\}$ and $a \vee F_1 = \{a \vee a_1 : a_1 \in F_1\}$ are filters of $\mathcal{L}$ and $F_1 \cap F_2 = F_1 \vee F_2 \subseteq F_1, F_2$.

(3) If $\mathcal{L}$ is distributive and F_1, F_2 are filters of $\mathcal{L}$, then $F_1 \wedge F_2 = \{a \wedge b : a \in F_1, b \in F_2\}$ is a filter of $\mathcal{L}$, $F_1, F_2 \subseteq F_1 \wedge F_2$ (for if $x \in F_1$, then $x = x \wedge 1 \in F_1 \wedge F_2$) and if $F_1 \subseteq F_2$, then $F_1 \wedge F_2 = F_2$.

The undefined terms related to lattice theory are taken from [7] and terms related to graph theory are taken from [15].

3. Basic properties of $\mathbb{SG}(\mathcal{L})$

In this section, we collect some basic properties concerning the join-simple filter graph $\mathbb{SG}(\mathcal{L})$ of $\mathcal{L}$. We remind the reader with the following definition.

Definition 3.1. *The join-simple filter graph $\mathbb{SG}(\mathcal{L})$ of $\mathcal{L}$ is a simple undirected graph whose vertices are all nontrivial filters of $\mathcal{L}$ (i.e. different from $\{1\}$ and $\mathcal{L}$) and any two distinct vertices F and G are adjacent if and only if $F \vee G = F \cap G$ is a simple filter of $\mathcal{L}$.*

Lemma 3.2. *The following hold in $\mathbb{SG}(\mathcal{L})$:*

(1) Every non-simple filter of $\mathcal{L}$ is adjacent to at least one of the simple filters of $\mathcal{L}$;

(2) If $\mathrm{Soc}(\mathcal{L}) \neq \mathcal{L}$, then every element of $\mathcal{S}(\mathcal{L})$ is adjacent to $\mathrm{Soc}(\mathcal{L})$.

Proof. It is straightforward. □

Proposition 3.3. *The following hold in $\mathbb{SG}(\mathcal{L})$:*

(1) The subgraph induced by the simple filters of $\mathcal{L}$ is empty.

(2) The subgraph induced by the non-simple filters of $\mathcal{L}$ is connected graph of diameter not bigger than 4.

Proof. (1) It is clear.

(2) Let F and G be distinct non-simple vertices of the graph $\mathbb{SG}(\mathcal{L})$. If F adjacent to G, then $F \backsim G$ is a path. So we may assume that $F \cap G$ is not simple. Then there exist $S, S' \in \mathcal{S}(\mathcal{L})$ such that $S \subsetneqq F$ and $S' \subsetneqq G$. If $S = S'$, then $F \backsim S \backsim G$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 2$. If $S \neq S'$, then $F \backsim S \backsim S \wedge S' \backsim S' \backsim G$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 4$. This completes the proof. □

Proposition 3.4. *The following hold in* $\mathbb{SG}(\mathcal{L})$*:*

(1) If $\mathbb{SG}(\mathcal{L})$ *has a non-simple universal vertex* M*, then* M *is a maximal filter;*

(2) If $\mathcal{S}(\mathcal{L}) = \{S\}$*, then* S *is a universal vertex;*

(3) If $\mathbb{SG}(\mathcal{L})$ *has a simple universal vertex* S*, then* $\mathcal{S}(\mathcal{L}) = \{S\}$*.*

Proof. (1) By [8, Lemma 2.1], there exists a maximal filter Q of $\mathcal{L}$ such that $M \subseteq Q$. If $M \neq Q$, then M is a universal vertex gives $M = M \cap Q$ is a simple filter which is impossible. Thus, $M = Q$.

(2) If G is a nontrivial filter of $\mathcal{L}$, then $S \subseteq G$ gives $G \backsim S$, as needed.

(3) Apply Proposition 3.3(1). □

Proposition 3.5. *The following hold in* $\mathbb{SG}(\mathcal{L})$*:*

(1) The graph $\mathbb{SG}(\mathcal{L})$ *is an empty graph if and only if every nontrivial filter of* $\mathcal{L}$ *is simple;*

(2) The graph $\mathbb{SG}(\mathcal{L})$ *is a null graph if and only if* $\mathcal{L}$ *is a simple lattice.*

Proof. (1) If every nontrivial filter of $\mathcal{L}$ is simple, then $\mathbb{SG}(\mathcal{L})$ is a empty graph by Proposition 3.3(1). Conversely, assume that $\mathbb{SG}(\mathcal{L}$ is empty and let F be any vertex of the graph of $\mathbb{SG}(\mathcal{L}$ such that $F \notin \mathcal{S}(\mathcal{L})$. Then by Lemma 3.2(1), there exists a simple filter S of $\mathcal{L}$ such that F is adjacent to S which is impossible, as needed.

(2) The proof is clear. □

Proposition 3.6. *If* $\mathbb{SG}(\mathcal{L})$ *is a complete graph, then* $|\mathcal{S}(\mathcal{L})| = 1$*.*

Proof. It is clear by Proposition 3.3(1). □

The following example shows that, in general, the converse of Proposition 3.5 is not true.

Example 3.7. *Assume that* $\mathcal{L} = \{0, a, b, c, d, e, 1\}$ *is a lattice with the relations* $0 \leq e \leq a \leq b \leq c \leq 1$*,* $0 \leq e \leq a \leq d \leq c \leq 1$*,* $d \vee b = c$ *and* $d \wedge b = a$*. Set* $F = \{a, b, c, d, 1\}$*,* $G = \{1, c, d\}$ *and* $S = \{1, c\}$*. Clearly,* $\mathcal{S}(\mathcal{L}) = \{S\}$*. Then* $F \cap G = G$ *is not a simple filter gives* $\mathbb{SG}(\mathcal{L})$ *is not a complete graph.*

In the following theorem, we give a condition under which the graph $\mathbb{SG}(\mathcal{L})$ is complete.

Theorem 3.8. *Let* S *be an element of* $\mathcal{V}$ $(\mathbb{SG}(\mathcal{L}))$ *with degree* 1*. If* S *is simple in* $\mathcal{L}$ *which is not a maximal filter, then* $\mathbb{SG}(\mathcal{L}) \cong K_2$*, the complete graph with two vertices.*

Proof. By [8, Lemma 2.1], $S \subsetneqq M$ for some maximal filter M of $\mathcal{L}$; so $S \backsim M$. If M' is a filter $\mathcal{L}$ such that $S \subsetneqq M' \subsetneqq M$, then $S \backsim M'$ gives $\deg_{\mathbb{SG}(\mathcal{L})}(S) \geq 2$ which is a contradiction. Thus, $\mathcal{V}$ $(\mathbb{SG}(\mathcal{L})) = \{S, M\}$, as needed. □

Proposition 3.9. *The following hold in* $\mathbb{SG}(\mathcal{L})$*:*
(1) If $S \in \mathcal{S}(\mathcal{L})$ *and* $\mathrm{Soc}(\mathcal{L}) \subsetneqq K$*, then* $S \backsim K$*;*
(2) If $K \notin \mathcal{S}(\mathcal{L})$ *and* $K \subsetneqq H$*, then* K *is not adjacent to* H*.*

Proof. They are clear. □

Theorem 3.10. *The graph* $\mathbb{SG}(\mathcal{L})$ *is connected with* $\mathrm{diam}(\mathbb{SG}(\mathcal{L})) \leq 4$ *if and only if either the meet of any two distinct simple filters of* $\mathcal{L}$ *is not* $\mathcal{L}$ *or* $|\mathcal{S}(\mathcal{L})| = 1$*.*

Proof. Suppose that $\mathcal{S}(\mathcal{L}) = \{S\}$ and let F and G be distinct non-simple vertices of the graph $\mathbb{SG}(\mathcal{L})$. Then $F \backsim S \backsim G$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 2$. So suppose that $|\mathcal{S}(\mathcal{L})| \neq 1$ and the meet of any two distinct simple filters of $\mathcal{L}$ is not $\mathcal{L}$. Consider two distinct vertices K and H of $\mathbb{SG}(\mathcal{L})$. If K adjacent to H, then $K \backsim H$ is a path. So we may assume that $K \cap H$ is not a simple filter of $\mathcal{L}$. Then either $S \subsetneqq K \cap H$ for some simple filter S of $\mathcal{L}$ or $K \cap H = \{1\}$. If $S \subsetneqq K \cap H$ (so $S \subsetneqq K$ and $S \subsetneqq H$), then $K \backsim S \backsim H$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(K, H) = 2$. If $K \cap H = \{1\}$, we split the proof into three cases:

Case 1. $K, H \in \mathcal{S}(\mathcal{L})$. Then $K \backsim K \wedge H \backsim H$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(K, H) = 2$.

Case 2. If exactly one of H and K is simple, then without loss of generality, assume that $K \in \mathcal{S}(\mathcal{L})$ and $H \notin \mathcal{S}(\mathcal{L})$. By lemma 3.2(1), there is a simple filter S of $\mathcal{L}$ such that $S \subsetneqq H$ which implies that $K \backsim S \wedge K \backsim S \backsim H$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(K, H) = 4$.

Case 3. $K, H \notin \mathcal{S}(\mathcal{L})$. Then there exist $S, S' \in \mathcal{S}(\mathcal{L})$ such that $S \subsetneqq K$ and $S' \subsetneqq H$. If $S = S'$, then $K \backsim S \backsim H$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(K, H) = 2$. If $S \neq S'$, then $K \backsim S \backsim S \wedge S' \backsim S' \backsim H$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(K, H) = 4$. Hence, we infer that $\mathbb{SG}(\mathcal{L})$ is connected with $\mathrm{diam}(\mathbb{SG}(\mathcal{L})) \leq 4$.

Conversely, assume that $\mathbb{SG}(\mathcal{L})$ is connected. If $|\mathcal{S}(\mathcal{L})| = 1$, then we are done. So we may assume that $|\mathcal{S}(\mathcal{L})| \neq 1$. On the contrary, assume that there are two simple filters S_1 and S_2 of $\mathcal{L}$ such that $S_1 \wedge S_2 = \mathcal{L}$. We claim that S_2 is a maximal filter of $\mathcal{L}$. Assume to the contrary, $S_2 \subsetneqq H \subsetneqq \mathcal{L}$ for some filter H of $\mathcal{L}$. Then $\mathcal{L} = S_1 \wedge S_2 \subseteq S_1 \wedge H$ gives $\mathcal{L} = S_1 \wedge H$ and $S_1 \cap H = \{1\}$ (otherwise, $S_1 \subseteq H$ and so $\mathcal{L} = H$, a contradiction). Now, we have $S_2 = S_2 \wedge \{1\} = S_2 \wedge (S_1 \vee H) = (S_2 \wedge S_1) \vee (S_2 \wedge H) = H$ which is a contradiction. So S_2 is a maximal filter. Similarly, S_1 is a maximal filter. By the hypothesis, $S_1 \backsim F$ for some vertex F of $\mathbb{SG}(\mathcal{L})$. Then $S_1 \cap F = S_1$ which gives $S_1 \subsetneqq F$. Since S_1 is maximal, we infer that $S_1 = F$. This shows that S_1 is an isolated vertex, a contradiction, as required. □

Following example confirms the Theorem 3.10.

Example 3.11. *Let* $X = \{a, b, c\}$. *Then* $\mathcal{L} = \{T : T \subseteq X\}$ *forms a distributive lattice under set inclusion, greatest element* X *and least element* $\emptyset$*; (note that if* $A, B \in \mathcal{L}$*, then* $A \wedge B = A \cap B$ *and* $A \vee B = A \cup B$*). Now, the set of all filters of* $\mathcal{L}$ *is* $\{F_1, F_2, F_3, F_4, F_5, F_6\}$*, where*

$$F_1 = \{X, \{a, b\}\},\ F_2 = \{X, \{a, c\}\},\ F_3 = \{X, \{b, c\}\},$$

$$F_4 = \{X, \{a, b\}, \{a, c\}, \{a\}\},\ F_5 = \{X, \{a, b\}, \{b, c\}, \{b\}\},$$

$$F_6 = \{X, \{a, c\}, \{b, c\}, \{c\}\}.$$

It is clear that $\mathbb{SG}(\mathcal{L})$ *is connected with* $\operatorname{diam}(\mathbb{SG}(\mathcal{L})) \leq 4$ *and the meet of any two distinct simple filters of* $\mathcal{L}$ *is not* $\mathcal{L}$.

Theorem 3.12. *Suppose that* $\mathbb{SG}(\mathcal{L})$ *is a graph which contains a cycle and let* $\operatorname{Soc}(\mathcal{L}) \neq \mathcal{L}$. *Then* $\operatorname{gr}(\mathbb{SG}(\mathcal{L})) = 3, 4$.

Proof. Suppose that $\operatorname{Soc}(\mathcal{L}) \neq \mathcal{L}$ and let $F \backsim G$. By Proposition 3.3(1), either $K \notin \mathcal{S}(\mathcal{L})$ or $G \notin \mathcal{S}(\mathcal{L})$. If $F, G \notin \mathcal{S}(\mathcal{L})$, then $K \backsim K \cap G \backsim G \backsim K$ is a cycle (so $\operatorname{gr}(\mathbb{SG}(\mathcal{L})) = 3$). Suppose that one of K or G is a simple filter. We can assume that $K \in \mathcal{S}(\mathcal{L})$ and $G \notin \mathcal{S}(\mathcal{L})$. By Lemma 3.2(1), there is a simple filter S of $\mathcal{L}$ such that $S \subseteq G$. Hence we obtain the cycle $K \backsim G \backsim S \backsim \operatorname{Soc}(\mathcal{L}) \backsim K$ which implies that $\operatorname{gr}(\mathbb{SG}(\mathcal{L})) = 4$, as needed. □

Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2), \cdots, (\mathcal{L}_n, \leq_n)$ are lattices ($n \geq 2$) and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

Theorem 3.13. *Let* $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ *be a decomposable lattice, where* $\mathcal{L}_i$ *is a simple lattice for* $i \in \{1, 2, \cdots, n\}$. *Then* $\operatorname{diam}(\mathbb{SG}(\mathcal{L})) \leq 2$.

Proof. By [9, Lemma 3.1], any filter F of $\mathcal{L}$ is of the form $F = F_1 \times \cdots \times F_n$, where $F_i = \mathcal{L}_i$ or $F_i = \{1\}$. Then it is clear that $\mathcal{S}(\mathcal{L}) = \{S_1, \cdots, S_n\}$, where $S_k = \{1\} \times \cdots \{1\} \times \mathcal{L}_k \times \{1\} \times \cdots \times \{1\}$ for $k \in \{1, 2, \cdots, n\}$. Let F and G be two vertices of $\mathbb{SG}(\mathcal{L})$ such that they are not adjacent. If F and G both contain same simple filter, then $\operatorname{d}(F, G) = 2$. Otherwise, there are $S_i, S_j \in \mathcal{S}(\mathcal{L})$ such that $S_i \subseteq F$, $S_i \nsubseteq G$, $S_j \subseteq G$ and $S_j \nsubseteq F$. We may assume that $i < j$. Consider the filter

$$H = \{1\} \times \cdots \{1\} \times \mathcal{L}_i \times \{1\} \times \cdots \times \{1\} \times \mathcal{L}_j \times \{1\} \times \cdots \times \{1\}.$$

Since $S_j \nsubseteq F$ and $S_i \nsubseteq G$, we conclude that $H \cap F = S_i$ and $H \cap G = S_j$. This shows that $F \backsim H \backsim G$ is a path in $\mathbb{SG}(\mathcal{L})$ with $\mathrm{d}(F, G) = 2$. Thus, $\mathrm{diam}(\mathbb{SG}(\mathcal{L})) \leq 2$. □

Theorem 3.14. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ be a decomposable lattice, where $\mathcal{L}_i$ is a simple lattice for $i \in \{1, 2, \cdots, n\}$. Then $\mathrm{gr}(\mathbb{SG}(\mathcal{L})) = 3$.*

Proof. Let $\mathcal{S}(\mathcal{L}) = \{S_1, \cdots, S_n\}$, where $S_k = \{1\} \times \cdots \{1\} \times \mathcal{L}_k \times \{1\} \times \cdots \times \{1\}$ for $k \in \{1, 2, \cdots, n\}$. Now we consider the filters $F = \mathcal{L}_1 \times \mathcal{L}_2 \times \{1\} \times \cdots \times \{1\}$, $G = \mathcal{L}_1 \times \{1\} \times \mathcal{L}_3 \times \{1\} \times \cdots \times \{1\}$ and $H = \{1\} \times \mathcal{L}_2 \times \mathcal{L}_3 \times \{1\} \times \cdots \times \{1\}$. Since $F \cap G = S_1$, $F \cap H = S_2$ and $G \cap H = S_3$, we get the cycle $F \backsim G \backsim H \backsim F$. This shows that $\mathrm{gr}(\mathbb{SG}(\mathcal{L})) = 3$. □

Proposition 3.15. *If $\mathcal{V}$ $(\mathbb{SG}(\mathcal{L}))$ is a totally ordered set with set inclusion, then $\mathbb{SG}(\mathcal{L})$ is a star graph.*

Proof. By the hypothesis, there exists $S \in \mathcal{S}(\mathcal{L})$ such that $S \backsim G$ for every $G \in \mathcal{V}$ $(\mathbb{SG}(\mathcal{L}))$. If $F \neq S$ and $G \neq S$ are two distinct vertices of $\mathbb{SG}(\mathcal{L})$, then either $F \subseteq G$ or $G \subseteq F$. For both cases, F and G are not adjacent vertices. Thus $\mathbb{SG}(\mathcal{L})$ is a star graph with center S. □

The following theorem provides a useful tool for solving certain optimization problems related to graph coloring. Among the classical coloring problems is the bipartite graph problem, which examines whether the vertices of a simple undirected graph can be colored with two colors so that every pair of adjacent vertices is assigned distinct colors. It is well known that such a coloring exists precisely when the graph is bipartite.

Theorem 3.16. *If $\mathrm{Soc}(\mathcal{L}) \neq \mathcal{L}$, then the following assertions are equivalent:*

(1) Every vertex of $\mathbb{SG}(\mathcal{L})$ is either simple or essential;

(2) $\mathbb{SG}(\mathcal{L})$ is a complete bipartite graph.

Proof. (1) ⇒ (2) Let $V_1 = \mathcal{S}(\mathcal{L})$ and V_2 be the set of all essential filters of $\mathcal{L}$. If $S, S' \in V_1$, then S and S' are not adjacent by Proposition 3.3(1). If $F, G \in V_2$, then $F \cap G$ is essential in $\mathcal{L}$ gives $\mathrm{Soc}(\mathcal{L}) \subsetneqq F \cap G$, which implies that any two vertices of V_2 are not adjacent. Moreover, every vertex in V_1 is adjacent to each vertex in V_2 by Proposition 3.9. Therefore, $\mathbb{SG}(\mathcal{L})$ is a complete bipartite graph.

(2) ⇒ (1) Suppose that V_1 and V_2 are parts of $\mathbb{SG}(\mathcal{L})$ and let S be a simple filter of $\mathcal{L}$. Without loss of generality, let $S \in V_1$. If $S \neq S' \in \mathcal{S}(\mathcal{L})$ with $S' \notin V_1$, then $S' \in V_2$ gives $S \cap S'$ is simple which is impossible. Thus $\mathcal{S}(\mathcal{L}) \subseteq V_1$. If $G \in V_1$ with $G \notin \mathcal{S}(\mathcal{L})$, then there exists a simple filter S of $\mathcal{L}$ (so $S \in V_1$) such that $G \backsim S$ by Lemma 3.2(1),

a contradiction. Therefore, $V_1 = \mathcal{S}(\mathcal{L})$. Suppose that $\mathrm{Soc}(\mathcal{L}) \subseteq G$ for some vertex G and let $F \neq \{1\}$ be any filter of $\mathcal{L}$. If F is simple, then $F \cap G = F \neq \{1\}$. If F is not simple, then there exists a simple filter T of $\mathcal{L}$ such that $T \subseteq F \cap G$ which implies that G is an essential filter of $\mathcal{L}$; so $H = \{G \in \mathcal{V}\ (\mathbb{SG}(\mathcal{L}) : \mathrm{Soc}(\mathcal{L}) \subseteq G\}$ is the set of all essential filters of $\mathcal{L}$. An easy inspection will show that $V_2 = H$, as required. □

The following example shows that the condition $\mathrm{Soc}(\mathcal{L}) \neq \mathcal{L}$ is not superficial in Theorem 3.16.

Example 3.17. *Let $\mathcal{L}$ be a lattice as in Example 3.11. Then* $\mathrm{Soc}(\mathcal{L}) = \mathcal{L}$ *and every vertex of* $\mathbb{SG}(\mathcal{L})$ *is either simple or essential. However* $\mathbb{SG}(\mathcal{L})$ *is not a complete bipartite graph (see Figure 1).*

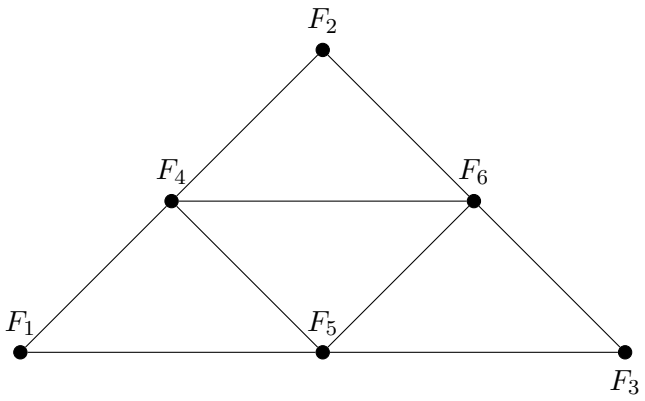


FIGURE 1. $\mathbb{MG}(\mathcal{L})$

Theorem 3.18. *Suppose that the meet of any two distinct simple filters of $\mathcal{L}$ is not $\mathcal{L}$ and let C be a cut vertex of $\mathbb{SG}(\mathcal{L})$. Then there exist $S, S' \in \mathcal{S}(\mathcal{L})$ such that $C = S \wedge S'$.*

Proof. If $C \in \mathcal{S}(\mathcal{L})$, then $C = C \wedge C$. So we may assume that $C \notin \mathcal{S}(\mathcal{L})$. Let F and G be two vertices of $\mathbb{SG}(\mathcal{L})$ such that $F \in V_1$ and $G \in V_2$, where V_1 and V_2 are the distinct components $\mathbb{SG}(\mathcal{L}) \setminus \{C\}$. We split the proof into four cases:

Case 1. $F, G \in \mathcal{S}(\mathcal{L})$. Since $F \backsim F \wedge G \backsim G$ is a path in $\mathbb{SG}(\mathcal{L})$ and C is a cut vertex, we conclude that $K = F \wedge G$.

Case 2. $F \in \mathcal{S}(\mathcal{L})$ and $G \notin \mathcal{S}(\mathcal{L})$. By the hypothesis, $S \subsetneqq G$ for some simple filter S of $\mathcal{L}$ by Lemma 3.2(1); so $S \in V_2$. Since $F \backsim F \wedge S \backsim S$ is a path in $\mathbb{SG}(\mathcal{L})$, $F \in V_1$ and C is a cut vertex, we conclude that $C = F \wedge S$.

Case 3. $F \notin \mathcal{S}(\mathcal{L})$ and $G \in \mathcal{S}(\mathcal{L})$. By an argument like that in Case 2, $C = G \wedge S$ for some simple filter S of $\mathcal{L}$ with $S \subsetneq F$.

Case 4. $F, G \notin \mathcal{S}(\mathcal{L})$. By assumption, $S \subsetneq F$ and $T \subsetneq G$ for some simple filters S and T of $\mathcal{L}$ by Lemma 3.2(1); so $S \in V_1$ and $T \in V_2$. Since $S \backsim T \wedge S \backsim T$ is a path in $\mathbb{SG}(\mathcal{L})$ and C is a cut vertex, we infer that $C = S \wedge T$. □

The following example verifies Theorem 3.18.

Example 3.19. *Let $\mathcal{L} = \{0, a, b, c, d, 1\}$ be a lattice with $0 \leq a \leq c \leq 1$, $0 \leq b \leq d \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$, $a \wedge b = 0$, $c \wedge d = b$ and $c \vee d = 1$. Then $\mathcal{S}(\mathcal{L}) = \{F_1, F_2\}$, where $F_1 = \{1, c\}$ and $F_2 = \{1, d\}$. An inspection shows that $F_3 = \{1, d, b, c\}$ is a cut vertex and $F_3 = F_1 \wedge F_2$.*

4. Clique number, domination number and planarity of $\mathbb{SG}(\mathcal{L})$

We continue this section with the investigation of the stability of join-simple filter graphs in various lattice-theoretic constructions. Let us begin this section with the following proposition:

Proposition 4.1. *Let $F, G \notin \mathcal{S}(\mathcal{L})$ such that they are adjacent. Then there is a unique simple filter S of $\mathcal{L}$ such that $S \in N(F) \cap N(G)$.*

Proof. Since $F \cap G \in \mathcal{S}(\mathcal{L})$, we conclude that $F \cap G$ is adjacent to both F and G which implies that $F \cap G \in N(F)$ and $F \cap G \in N(G)$. Let $S \in \mathcal{S}(\mathcal{L})$ such that $S \in N(F) \cap N(G)$. It suffices to show that $S = F \cap G$. On the contrary, assume that $S \neq F \cap G$. At first, we show that $S \in N(F) \cap N(G)$ if and only if $S \in N(F \cap G)$. If $S \backsim F \cap G$, then $S \cap (F \cap G)$ is a simple filter of $\mathcal{L}$ gives $S \cap F \neq \{1\}$ and $S \cap G \neq \{1\}$. Since $S \cap F, S \cap G \subseteq S$, we conclude that $S \cap F = S = S \cap G$ and so $S \backsim F$ and $S \backsim G$. Conversely, assume that $S \backsim F$ and $S \backsim G$. Then $S \cap F$ and $S \cap G$ are simple filters gives $S \cap F = S = S \cap G$ which implies that $(F \cap G) \cap S = S$; so $S \backsim F \cap G$. Since $S \cap (F \cap G)$ is a simple filter, we infer that $S \cap (F \cap G) \neq \{1\}$ which implies that $S \subseteq F \cap G$. Therefore, $S = F \cap G$, as $F \cap G$ is a simple filter, a contradiction. Thus $S = F \cap G$. □

Proposition 4.2. *Let $\mathcal{C}$ be a clique in $\mathbb{SG}(\mathcal{L})$. Then $\mathcal{C}$ is contained in the subgraph induced by $\{F \in \mathcal{V}(\mathbb{SG}(\mathcal{L})) : S \subseteq F\}$ for some simple filter S of $\mathcal{L}$.*

Proof. By Proposition 3.3(1), $\mathcal{C}$ has at most one simple filter. If $C \cap \mathcal{S}(\mathcal{L}) = \{S'\}$, then $S' \subseteq F$ for every $F \in \mathcal{C}$ and so $\mathcal{C} \subseteq \{F \in \mathcal{V}(\mathbb{SG}(\mathcal{L})) : S' \subseteq F\}$. So we may assume that $\mathcal{C} \cap \mathcal{S}(\mathcal{L}) = \emptyset$. The adjacency of every two vertices of $\mathcal{C}$ and Proposition 4.1 shows that there exists

a unique simple filter S of $\mathcal{L}$ for which $\mathcal{C}$ is a subgraph of the graph induced by $\{F \in \mathcal{V}\ (\mathbb{SG}(\mathcal{L})) : S \subseteq F\}$. □

In the following results we show that domination numbers are really of interest in indecomposable lattices.

Theorem 4.3. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ be a decomposable lattice. Then the following hold:*
(1) If $|\mathcal{S}(\mathcal{L}_i)| = 1$ for $i \in \{1, 2\}$, then $\gamma(\mathbb{SG}(\mathcal{L})) = 2$;
(2) If $|\mathcal{S}(\mathcal{L}_1)| = 1$ and $\mathcal{L}_2$ is a simple lattice, then $\gamma(\mathbb{SG}(\mathcal{L})) = 1$
(3) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are simple lattices, then $\gamma(\mathbb{SG}(\mathcal{L})) = 2$.

Proof. (1) By [9, Lemma 3.1], any filter F of $\mathcal{L}$ is of the form $F = F_1 \times F_2$, where F_i is a filter of $\mathcal{L}_i$ for $i \in \{1, 2\}$. If $\mathcal{S}(\mathcal{L}_1) = \{S_1\}$ and $\mathcal{S}(\mathcal{L}_2) = \{S_2\}$, then it is clear that $\mathcal{S}(\mathcal{L}) = \{S_1', S_2'\}$, where $S_1' = S_1 \times \{1\}$ and $S_2' = \{1\} \times S_2$. Let $F = F_1 \times F_2$ be a nontrivial filter of $\mathcal{L}$. Since $F \cap S_1' = (F_1 \cap S_1) \times \{1\}$ and $F \cap S_2' = \{1\} \times (F_2 \cap S_2)$, we conclude that any vertex of the graph $\mathbb{SG}(\mathcal{L})$ is adjacent to at least one of the elements of the set $\{S_1', S_2'\}$. This shows that $\gamma(\mathbb{SG}(\mathcal{L})) = 2$.

(2) Clearly, if $\mathcal{S}(\mathcal{L}_1) = \{S_1\}$, then $\mathcal{S}(\mathcal{L}) = \{S_1 \times \{1\}\}$. So any vertex of the graph $\mathbb{SG}(\mathcal{L})$ is adjacent to $S_1 \times \{1\}$, and so $\gamma(\mathbb{SG}(\mathcal{L})) = 1$.

(3) Clearly, $\mathcal{S}(\mathcal{L}) = \{\mathcal{L}_1 \times \{1\}, \{1\} \times \mathcal{L}_2\}$, and so $\gamma(\mathbb{SG}(\mathcal{L})) = 2$. □

Theorem 4.4. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ ($n \geq 3$) be a decomposable lattice, where $\mathcal{L}_i$ is a simple lattice for $i \in \{1, 2, \cdots, n\}$. Then $\gamma(\mathbb{SG}(\mathcal{L})) \leq n$.*

Proof. By [9, Lemma 3.1], any filter F of $\mathcal{L}$ is of the form $F = F_1 \times \cdots \times F_n$, where $F_i = \mathcal{L}_i$ or $F_i = \{1\}$. Then it is clear that $\mathcal{S}(\mathcal{L}) = \{S_1, \cdots, S_n\}$, where $S_k = \{1\} \times \cdots \{1\} \times \mathcal{L}_k \times \{1\} \times \cdots \times \{1\}$ for $k \in \{1, 2, \cdots, n\}$. Then the set $\mathcal{S}(\mathcal{L})$ dominates all the vertices of the graph $\mathbb{SG}(\mathcal{L})$; hence $\gamma(\mathbb{SG}(\mathcal{L})) \leq n$. □

The following example shows that in general Theorem 4.4 is not true in the case $\gamma(\mathbb{SG}(\mathcal{L})) = n$.

Example 4.5. *Let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3$ be a decomposable lattice, where $\mathcal{L}_i$ is a simple lattice for $i \in \{1, 2, 3\}$. Then we have $\mathcal{V}$ ($\mathbb{SG}(\mathcal{L})$) = $\{\mathcal{L}_1 \times \{1\} \times \{1\}, \mathcal{L}_1 \times \mathcal{L}_2 \times \{1\}, \{1\} \times \mathcal{L}_2 \times \{1\}, \{1\} \times \mathcal{L}_2 \times \mathcal{L}_3, \mathcal{L}_1 \times \{1\} \times \mathcal{L}_3, \{1\} \times \{1\} \times \mathcal{L}_3\}$. Set $D = \{\mathcal{L}_1 \times \mathcal{L}_2 \times \{1\}, \{1\} \times \mathcal{L}_2 \times \mathcal{L}_3\}$. Then the set D dominates all the vertices of the graph $\mathbb{SG}(\mathcal{L})$; hence $\gamma(\mathbb{SG}(\mathcal{L})) = 2 \neq 3$.*

In the following theorem, we give a condition under which we characterize the lattices for which the join-simple filter graphs are split and planar. The concept of clique number is also investigated.

Theorem 4.6. *Suppose that* $\mathbb{SG}(\mathcal{L})$ *is not empty and let* $\mathcal{V}\ (\mathbb{SG}(\mathcal{L})) = \mathcal{S}(\mathcal{L}) \cup \mathrm{Max}(\mathcal{L})$. *Then the following hold:*

(1) The subgraph induced by the maximal filters of $\mathcal{L}$ *is a complete graph;*

(2) $\mathbb{SG}(\mathcal{L})$ *is a split graph;*

(3) If $|\mathrm{Max}(\mathcal{L})| \leq 3$, *then* $\mathbb{SG}(\mathcal{L})$ *is a planar graph;*

(4) $|\mathrm{Max}(\mathcal{L})| \leq \omega(\mathbb{SG}(\mathcal{L}))$.

Proof. (1) It is enough to show if $F, G \in \mathrm{Max}(\mathcal{L})$ with $F \neq G$, then $F \backsim G$. At first, we claim that $F \cap G \neq \{1\}$. On the contrary, let $F \cap G = \{1\}$. Since $F \subsetneqq F \wedge G \subseteq \mathcal{L}$, we infer that $F \wedge G = \mathcal{L}$. Let $\{1\} \neq K \subseteq F$ for some filter K of $\mathcal{L}$. Then $K \cap G = \{1\}$ and $K \wedge G = \mathcal{L}$. Now, let $x \in F$. Then $x = x \wedge 1 \in F \wedge G = \mathcal{L} = K \wedge G$ gives $x \in K$; hence $F = K$. This shows that F is simple. Similarly, G is simple. It follows that $\mathcal{V}\ (\mathbb{SG}(\mathcal{L})) = \mathcal{S}(\mathcal{L})$ and so $\mathbb{SG}(\mathcal{L})$ is empty by Proposition 3.3(1) which is impossible. Therefore, $F \cap G \neq \{1\}$. It is clear that $F \cap G \notin \mathrm{Max}(\mathcal{L})$; so $F \cap G \in \mathcal{S}(\mathcal{L})$, i.e. any two maximal filters are adjacent.

(2) Consider the subgraph induced by $\mathrm{Max}(\mathcal{L})$ of $\mathbb{SG}(\mathcal{L})$. Let $F, G \in \mathrm{Max}(\mathcal{L})$ with $F \neq G$. Then $F \cap G \in \mathcal{S}(\mathcal{L})$ by (1) which implies that the subgraph induced by $\mathrm{Max}(\mathcal{L})$ is complete. Moreover, by Proposition 3.3(1), the subgraph induced by $\mathcal{S}(\mathcal{L})$ is empty. Thus, $\mathbb{SG}(\mathcal{L})$ is a split graph.

(3) Recall that $\mathbb{SG}(\mathcal{L})$ is a split graph by (2). Since $|\mathrm{Max}(\mathcal{L})| \leq 3$, we conclude that any subgraph induced by five vertices is not complete; so K_5 is not a subgraph of $\mathbb{SG}(\mathcal{L})$. Now, we show that $K_{3,3}$ is not a subgraph of $\mathbb{SG}(\mathcal{L})$. On the contrary, assume that $K_{3,3}$ is a subgraph of $\mathbb{SG}(\mathcal{L})$ with partite sets $|V_1| = 3$ and $|V_2| = 3$. Clearly, either $V_1 \subseteq \mathcal{S}(\mathcal{L})$ or $V_2 \subseteq \mathcal{S}(\mathcal{L})$. If $V_1 \subseteq \mathcal{S}(\mathcal{L})$, then $V_2 \subseteq \mathrm{Max}(\mathcal{L})$ which is impossible by (1). Therefore, $\mathbb{SG}(\mathcal{L})$ is a planar graph.

(4) Since any two maximal distinct filters of $\mathcal{L}$ are adjacent by (1), we infer that the subgraph of $\mathbb{SG}(\mathcal{L})$ with the vertex set of $\mathrm{Max}(\mathcal{L})$ is a complete subgraph of $\mathbb{SG}(\mathcal{L})$. Therefore, $|\mathrm{Max}(\mathcal{L})| \leq \omega(\mathbb{SG}(\mathcal{L}))$. □

The following example shows that the equality does not hold necessarily in Theorem 4.6(4).

Example 4.7. *Let* $\mathcal{L} = \{0, a, b, c, 1\}$ *be a lattice with the relations* $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ *and* $a \wedge b = 0$. *An inspection will show that the nontrivial filters of* $\mathcal{L}$ *are* $F_1 = \{1, a, c\}$, $F_2 = \{1, b, c\}$ *and* $F_3 = \{1, c\}$. *Clearly,* $\mathrm{Max}(\mathcal{L}) = \{F_1, F_2\}$ *and* $\mathrm{C} = \{F_1, F_2, F_3\}$ *is a clique. Hence* $|\mathrm{Max}(\mathcal{L})| = 2 < \omega(\mathbb{SG}(\mathcal{L})) = 3$.

5. Conclusion

In this paper, we established several connections between the algebraic structure of a lattice and the graph-theoretic properties of its join-simple filter graph. Specifically, we characterized the connectedness, complete bipartiteness, and splitness of the join-simple filter graph in terms of the properties of filters in the underlying lattice.

We proved that the join-simple filter graph of a lattice is connected if and only if either the meet of any two distinct simple filters is not simple or the lattice contains a unique simple filter. Furthermore, we showed that the join-simple filter graph is complete bipartite precisely when every filter of the lattice is either simple or essential. As an immediate consequence, it follows that if every filter of a lattice is simple or essential, then its join-simple filter graph is a split graph.

These results provide a deeper understanding of the interplay between lattice-theoretic concepts and graph-theoretic structures, and demonstrate how algebraic properties of lattices can be reflected through the behavior of their associated join-simple filter graphs.

6. Acknowledgment

The authors express their deep gratitude to the referees for their constructive comments, careful reading and fruitful suggestions that improve the quality of the manuscript.

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