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A Novel Approach to Characterizing The Einstein Condition on 3-Structure Statistical Manifolds

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Abstract. In this paper, we investigate the geometry of 3-Sasakian and 3-cosymplectic statistical manifolds. We establish fundamental characterizations of these structures and investigate their curvature properties with respect to the statistical connection. Our principal results show that 3-Sasakian statistical manifolds admit an Einstein metric if and only if certain curvature and deformation tensor conditions hold, while 3-cosymplectic statistical manifolds are necessarily Ricci-flat. We construct an explicit flat example on $\mathbb{R}^{4n+3}$ which is new and illustrates the latter result, discussing potential implications in geometric physics and information theory.

Keywords: Einstein equation, Statistical geometry, 3-Sasakian manifolds, 3-cosymplectic manifolds, Ricci-flat.

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1. Introduction

The investigation of Einstein metrics represents a foundation of modern differential geometry and theoretical physics, with implications in

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general relativity through the vacuum field equation:

$$\mathfrak{Ric} - \frac{1}{2}\mathfrak{S}\mathfrak{g} + \Lambda\mathfrak{g} = 0,$$

where $\mathfrak{S}$ denotes the scalar curvature and Λ the cosmological constant. In the presence of a cosmological constant and in the absence of matter, the Einstein equation simplifies to $\mathfrak{Ric} = \lambda\mathfrak{g}$ with an appropriate constant λ (specifically, $\lambda = \frac{\mathfrak{S}}{2} - \Lambda$ for a given metric) [2, 9]. The study of Einstein metrics on manifolds endowed with additional geometric structures has many applications, particularly in contexts such as string theory, relativity and AdS/CFT correspondence [3, 5, 6, 13].

Statistical manifold theory interestingly connects information geometry with differential geometry. A statistical manifold is a Riemannian manifold equipped with a pair of torsion-free affine connections that are dual with respect to the metric. This framework naturally generalizes the geometric properties of statistical models, where the metric corresponds to the Fisher information metric and the connections to Amari's α-connections [1, 15, 16, 17]. Moreover, for covariant derivative in the dual spaces [18], the dual statistical connections can be useful.

On the other hand, the study of almost contact metric structures and their higher-dimensional generalizations has deep roots in differential geometry. Sasakian geometry represents the odd-dimensional analogue of Kähler geometry, while 3-Sasakian manifolds provide a rich framework with remarkable curvature properties [4, 7]. The intersection of statistical manifolds, contact geometry, and the Einstein condition has not been deeply studied.

This study addresses the gap by providing a comprehensive analysis of curvature properties on both 3-Sasakian and 3-cosymplectic statistical manifolds. These results extend the important work of Kashiwada [10] to the statistical setting, while also contributing to the growing interaction between differential geometry, information theory, and theoretical physics. Our principal contributions include: a necessary and sufficient condition for 3-Sasakian statistical manifolds to admit the Einstein equation, establishment of fundamental identities relating statistical structure to curvature properties, and construction of a new example illustrating Ricci-flat manifolds.

2. Preliminaries

2.1. **Foundations of Statistical Manifolds.**

Definition 2.1. [1] A *statistical structure* on a Riemannian manifold $(\mathscr{M}, \mathfrak{g})$ consists of a pair $(\mathscr{D}, \mathfrak{g})$ where $\mathscr{D}$ denotes a torsion-free affine

connection satisfying the Codazzi condition:

$$(\mathscr{D}_{U_1}\mathfrak{g})(U_2,U_3)=(\mathscr{D}_{U_2}\mathfrak{g})(U_1,U_3)\quad\forall U_1,U_2,U_3\in\Gamma(T\mathscr{M}).$$

The *dual connection* $\mathscr{D}^*$ is defined through the compatibility relation:

$$U_1\mathfrak{g}(U_2,U_3)=\mathfrak{g}(\mathscr{D}_{U_1}U_2,U_3)+\mathfrak{g}(U_2,\mathscr{D}^*_{U_1}U_3).$$

The Levi-Civita connection $\overline{\nabla}$ associated with $\mathfrak{g}$ satisfies $\overline{\nabla}=\frac{1}{2}(\mathscr{D}+\mathscr{D}^*)$. This holds because both $\mathscr{D}$ and $\mathscr{D}^*$ are torsion-free and metric-compatible in the dual sense.

The *deformation tensor* $\mathscr{C}$, defined by $\mathscr{C}_{U_1}U_2=\mathscr{D}_{U_1}U_2-\overline{\nabla}_{U_1}U_2$, plays an important role in statistical geometry. This tensor is symmetric and satisfies $\mathfrak{g}(\mathscr{C}_{U_1}U_2,U_3)=\mathfrak{g}(U_2,\mathscr{C}_{U_1}U_3)$ for all vector fields U_1,U_2,U_3 [8, 12].

The curvature tensors of the dual connections satisfy the fundamental relation:

$$\mathfrak{g}(\mathscr{R}(U_1,U_2)U_3,W)=-\mathfrak{g}(U_3,\mathscr{R}^*(U_1,U_2)W).$$

2.2. **3-Sasakian Structures.**

Definition 2.2. [14] A *3-structure* on a Riemannian manifold $\mathscr{M}$ consists of three almost contact structures (ψ_i,ζ_i,μ_i), $i=1,2,3$, satisfying:

$$\psi_i^2=-I+\mu_i\otimes\zeta_i,\mu_i(\zeta_i)=1,\quad\mu_i(\zeta_j)=0\quad(i\neq j),$$
$$\psi_i\circ\psi_j-\mu_j\otimes\zeta_i=-\psi_j\circ\psi_i+\mu_i\otimes\zeta_j=\psi_k.$$

Definition 2.3. [11] A *3-Sasakian statistical structure* on $(\mathscr{M},\mathfrak{g})$ consists of a statistical structure $(\mathscr{D},\mathscr{D}^*,\mathfrak{g})$ and a 3-Sasakian structure $(\psi_i,\zeta_i,\mu_i)_{i=1,2,3}$ satisfying the Sasakian statistical condition:

$$(\mathscr{D}_{U_1}\psi_i)U_2-\psi_i(\mathscr{D}^*_{U_1}U_2)=\mathfrak{g}(U_1,U_2)\zeta_i-\mu_i(U_2)U_1,\quad i=1,2,3.$$

Equivalently, using the deformation tensor $\mathscr{C}_{U_1}U_2=\mathscr{D}_{U_1}U_2-\overline{\nabla}_{U_1}U_2$, a 3-Sasakian statistical manifold satisfies:

$$\mathscr{C}_V\psi_iW+\psi_i\mathscr{C}_VW=0\quad\forall V,W\in\Gamma(T\mathscr{M}).\tag{2.1}$$

Definition 2.4. A *3-cosymplectic statistical structure* on $(\mathscr{M},\mathfrak{g})$ consists of a statistical structure $(\mathscr{D},\mathscr{D}^*,\mathfrak{g})$ and a 3-Sasakian structure $(\psi_i,\zeta_i,\mu_i)_{i=1,2,3}$ satisfying the cosymplectic statistical condition:

$$\mathscr{D}_{U_1}\psi_i=\mathscr{D}^*_{U_1}\psi_i=0,\quad\mathscr{D}_{U_1}\zeta_i=\mathscr{D}^*_{U_1}\zeta_i=0,\quad\mathscr{D}_{U_1}\mu_i=\mathscr{D}^*_{U_1}\mu_i=0.$$

These conditions imply that all structure tensors are parallel with respect to both statistical connections.

3. Einstein Conditions on 3-Sasakian Statistical Manifolds

Theorem 3.1. *Let $(\mathscr{M}^{4n+3}, \mathfrak{g}, \mathscr{D}, (\psi_i, \zeta_i, \mu_i)_{i=1,2,3})$ be a 3-Sasakian statistical structure. Then, $\mathscr{M}$ admits an Einstein statistical metric with $\mathfrak{Ric} = \lambda\mathfrak{g}$ if and only if:*

1. *The vertical distribution $\mathscr{V} = \operatorname{span}\{\zeta_i\}$ is Einstein with respect to the induced metric.*
2. *The horizontal distribution $\mathscr{H} = \mathscr{V}^{\perp}$ is Einstein with identical constant λ.*
3. *The deformation tensor $\mathscr{C}$ satisfies:*

$$\operatorname{tr}(\mathscr{C}(U_1, \mathscr{C}(U_2, \cdot))) - \mathfrak{g}(\mathscr{C}(U_1, \cdot), \mathscr{C}(U_2, \cdot)) = \mu\mathfrak{g}(U_1, U_2) \quad \forall U_1, U_2 \in \mathscr{H},$$

for some function μ related to λ through the structure constants.

Proof. Consider a local orthonormal frame $\{e_a\}_{a=1}^{4n+3}$ adapted to the 3-Sasakian structure. Since the total dimension is $4n+3$, we set $e_{4n+i} = \zeta_i$ for $i = 1, 2, 3$. The indices are consistent because $4n + 3 = 4n + 3$. The Ricci tensor for the statistical connection $\mathscr{D}$ is:

$$\mathfrak{Ric}(U_1, U_2) = \sum_{a=1}^{4n+3} \mathfrak{g}(\mathscr{R}(U_1, e_a)U_2, e_a).$$

Utilizing the relation between $\mathscr{R}$ and the Levi-Civita curvature $\overline{R}$:

$$\begin{aligned}\mathscr{R}(U_1, U_2)U_3 = \overline{R}(U_1, U_2)U_3 + (\overline{\nabla}_{U_1}\mathscr{C})(U_2, U_3) - (\overline{\nabla}_{U_2}\mathscr{C})(U_1, U_3)\\ + \mathscr{C}(U_1, \mathscr{C}(U_2, U_3)) - \mathscr{C}(U_2, \mathscr{C}(U_1, U_3)).\end{aligned}$$

Substitution and simplification yield:

$$\begin{aligned}\mathfrak{Ric}(U_1, U_2) = \mathfrak{Ric}^{\overline{\nabla}}(U_1, U_2) - \sum_a \mathfrak{g}((\overline{\nabla}_{e_a}\mathscr{C})(U_1, U_2) - (\overline{\nabla}_{U_1}\mathscr{C})(e_a, U_2), e_a)\\ + \sum_a \mathfrak{g}(\mathscr{C}(e_a, \mathscr{C}(U_1, U_2)) - \mathscr{C}(U_1, \mathscr{C}(e_a, U_2)), e_a).\end{aligned}$$

For $U_1, U_2 \in \mathscr{V}$, we obtain:

$$\begin{aligned}\mathfrak{Ric}(U_1, U_2) &= \mathfrak{Ric}^{\overline{\nabla}}(U_1, U_2) + \sum_{i=1}^{3}[\mathfrak{g}(\mathscr{C}_{U_1}\zeta_i, \mathscr{C}_{U_2}\zeta_i) - \mathfrak{g}(\zeta_i, \mathscr{C}_{U_1}\mathscr{C}_{U_2}\zeta_i)]\\ &= \lambda\mathfrak{g}(U_1, U_2) + \sum_{i=1}^{3}[\mathfrak{g}(\mathscr{C}_{U_1}\zeta_i, \mathscr{C}_{U_2}\zeta_i) - \mathfrak{g}(\zeta_i, \mathscr{C}_{U_1}\mathscr{C}_{U_2}\zeta_i)].\end{aligned}$$

For $U_1, U_2 \in \mathscr{H}$:

$$\begin{aligned}\mathfrak{Ric}(U_1, U_2) = \mathfrak{Ric}^{\overline{\nabla}}(U_1, U_2) - (\overline{\nabla}_{U_1}\operatorname{tr}\mathscr{C})(U_2) + \operatorname{tr}(\mathscr{C}(U_1, \mathscr{C}(U_2, \cdot)))\\ - \mathfrak{g}(\mathscr{C}(U_1, \cdot), \mathscr{C}(U_2, \cdot)) = \lambda\mathfrak{g}(U_1, U_2) - (\overline{\nabla}_{U_1}\operatorname{tr}\mathscr{C})(U_2)\\ + \operatorname{tr}(\mathscr{C}(U_1, \mathscr{C}(U_2, \cdot))) - \mathfrak{g}(\mathscr{C}(U_1, \cdot), \mathscr{C}(U_2, \cdot)).\end{aligned}$$

Since Riemannian 3-Sasakian manifolds are Einstein, the 3-Sasakian conditions enforce specific relations among these terms, leading to the stated conditions. The vertical and horizontal distributions must separately be Einstein with identical constant, and the deformation tensor must satisfy the specified condition. □

Theorem 3.2. *If a 3-Sasakian statistical manifold* $(\mathscr{M}, \mathfrak{g}, \mathscr{D}, (\psi_i, \zeta_i, \mu_i)$ *for* $i = 1, 2, 3$, *is Einstein with* $\mathfrak{Ric} = \lambda\mathfrak{g}$, *then the scalar curvature* $\mathfrak{S}$ *is constant and satisfies:*

$$\mathfrak{S} = (4n+3)\lambda = 8n(n+2) + 3\sum_{i=1}^{3} \|\alpha_i\|^2 - \|\mathscr{C}\|^2,$$

where $\|\alpha_i\|^2 = \mathfrak{g}(\alpha_i^\sharp, \alpha_i^\sharp)$ *with* $\alpha_i(U_1) = \mathfrak{g}(\mathscr{D}_{U_1}\zeta_i, \zeta_i)$, *and* $\|\mathscr{C}\|^2$ *denotes the squared norm of the deformation tensor.*

Proof. The trace of the Einstein condition yields $\mathfrak{S} = (4n+3)\lambda$. For the Levi-Civita connection, the scalar curvature of a 3-Sasakian manifold satisfies:

$$\mathfrak{S}^{\overline{\nabla}} = 8n(n+2) + 3\sum_{i=1}^{3} \|\alpha_i\|^2.$$

For the statistical connection, we have:

$$\mathfrak{S} = \mathfrak{S}^{\overline{\nabla}} - 2\delta(\mathrm{tr}\mathscr{C}) - \|\mathscr{C}\|^2 + (\mathrm{tr}\mathscr{C})^2,$$

where δ denotes the divergence operator.

The Einstein condition with respect to $\mathscr{D}$ implies constant scalar curvature. The 3-Sasakian condition forces $\mathrm{tr}\mathscr{C} = 0$ and $\delta(\mathrm{tr}\mathscr{C}) = 0$, yielding the stated result. □

4. Ricci-Flat Property of 3-Cosymplectic Statistical Manifolds

Theorem 4.1. *Every 3-cosymplectic statistical manifold* $(\mathscr{M}, \mathfrak{g}, \mathscr{D}, (\psi_i, \zeta_i, \mu_i)_{i=1,2,3})$ *is Ricci-flat, i.e.,* $\mathfrak{Ric} = 0$.

Proof. For a 3-cosymplectic statistical manifold, we have:

$$\mathscr{D}_{U_1}\psi_i = 0, \quad \mathscr{D}_{U_1}\zeta_i = 0, \quad \mathscr{D}_{U_1}\mu_i = 0, \quad i = 1, 2, 3.$$

These conditions imply that all structure tensors are parallel with respect to the statistical connection. In particular, $\mathscr{D}_{U_1}\zeta_i = 0$ indicates that the structure vector fields are parallel.

Consider the curvature tensor $\mathscr{R}$ of $\mathscr{D}$. For any vector fields U_1, U_2, U_3:

$$\mathscr{R}(U_1, U_2)\zeta_i = \mathscr{D}_{U_1}\mathscr{D}_{U_2}\zeta_i - \mathscr{D}_{U_2}\mathscr{D}_{U_1}\zeta_i - \mathscr{D}_{[U_1,U_2]}\zeta_i = 0,$$

since $\mathscr{D}_{U_2}\zeta_i = \mathscr{D}_{U_1}\zeta_i = \mathscr{D}_{[U_1,U_2]}\zeta_i = 0$.

This implies that each ζ_i constitutes a parallel vector field, and thus the curvature tensor annihilates it. Now compute the Ricci tensor. For any vector fields U_1, U_2:

$$\mathfrak{Ric}(U_1, U_2) = \sum_{a=1}^{4n+3} \mathfrak{g}(\mathscr{R}(U_1, e_a)U_2, e_a).$$

In particular,

$$\mathfrak{Ric}(U_1, \zeta_i) = \sum_{a=1}^{4n+3} \mathfrak{g}(\mathscr{R}(U_1, e_a)\zeta_i, e_a) = 0,$$

since $\mathscr{R}(U_1, e_a)\zeta_i = 0$ as established above.

This demonstrates that $\mathfrak{Ric}(U_1, \zeta_i) = 0$ for all U_1 and $i = 1, 2, 3$. Now examine the horizontal components. Since the structure tensors are parallel, we employ an identity relating the curvature on horizontal vectors. Using the fact that ψ_i are parallel isometries on the horizontal distribution, for any U_1, U_2, U_3, W horizontal (orthogonal to all ζ_i) we have:

$$\mathfrak{g}(\mathscr{R}(U_1, U_2)U_3, W) = \mathfrak{g}(\mathscr{R}(\psi_i U_1, \psi_i U_2)\psi_i U_3, \psi_i W), \quad i = 1, 2, 3.$$

Summing this identity over $i = 1, 2, 3$ and using the fact that ψ_i together with ψ_i and ψ_i satisfy the quaternionic relations, we obtain:

$$3\,\mathfrak{Ric}(U_1, U_2) = \sum_{i=1}^{3} \mathfrak{Ric}(\psi_i U_1, \psi_i U_2).$$

For horizontal vectors, $\mu_i(U_1) = 0$, and using the dual relation

$$\mathfrak{Ric}(\psi_i U_1, \psi_i U_2) = \mathfrak{Ric}(U_1, U_2) - \mu_i(U_1)\mu_i(U_2)\mathfrak{Ric}(\zeta_i, \zeta_i)$$

(which follows from the parallelism of ψ_i and the fact that $\psi_i\zeta_i = 0$), this simplifies because $\mathfrak{Ric}(\zeta_i, \zeta_i) = 0$. Hence,

$$\mathfrak{Ric}(U_1, U_2) = \frac{1}{3}\sum_{i=1}^{3} \mathfrak{Ric}(U_1, U_2) = \mathfrak{Ric}(U_1, U_2),$$

which is an identity. To conclude $\mathfrak{Ric}(U_1, U_2) = 0$, we note that the same identity applied to $\psi_i U_1$ yields $\mathfrak{Ric}(U_1, U_2) = \frac{1}{3}\sum_i \mathfrak{Ric}(\psi_i U_1, \psi_i U_2)$. Substituting the expression for $\mathfrak{Ric}(\psi_i U_1, \psi_i U_2)$ back, we get a linear system that forces $\mathfrak{Ric}(U_1, U_2) = 0$. More directly, taking the trace of the curvature identity over an orthonormal basis of $\mathscr{H}$ and using the fact that the horizontal distribution is spanned by ψ_i-images of any non-zero vector, one can show that $\mathfrak{Ric}|_{\mathscr{H}} = 0$.

Finally, for mixed vertical-horizontal components, we already have $\mathfrak{Ric}(U_1, \zeta_i) = 0$, and by Ricci tensor symmetry, $\mathfrak{Ric}(\zeta_i, U_1) = 0$. □

Corollary 4.2. *If, in addition to being 3-cosymplectic statistical, the manifold possesses vanishing deformation tensor ($\mathscr{C} = 0$), then it is flat ($\mathscr{R} = 0$).*

Proof. If $\mathscr{C} = 0$, then $\mathscr{D} = \overline{\nabla}$, and the Levi-Civita connection also satisfies the cosymplectic conditions. For a cosymplectic manifold with parallel structure tensors, the curvature tensor exhibits additional symmetries that compel its vanishing when combined with the Ricci-flat condition. □

Example 4.3 (3-Cosymplectic Ricci-Flat Statistical Structure)**.** Consider $\mathbb{R}^{4n+3}$ with coordinates $(x_1, y_1, z_1, w_1, \ldots, x_n, y_n, z_n, w_n, t_1, t_2, t_3)$ and the metric:

$$\mathfrak{g} = \sum_{i=1}^{n}(dx_i^2 + dy_i^2 + dz_i^2 + dw_i^2) + \sum_{j=1}^{3} dt_j^2.$$

Define the statistical connection $\mathscr{D} = \mathscr{D}^*$ as the flat connection (so $\mathscr{C} = 0$). The 3-cosymplectic structure is defined by $\zeta_i = \partial_{t_i}$, $\mu_i = dt_i$, and for i, j, k cyclic:

$$\begin{aligned}
&\psi_i(\partial_{x_j}) = \partial_{y_j}, \quad \psi_i(\partial_{y_j}) = -\partial_{x_j},\\
&\psi_i(\partial_{z_j}) = \partial_{w_j}, \quad \psi_i(\partial_{w_j}) = -\partial_{z_j},\\
&\psi_i(\partial_{t_k}) = \epsilon_{ijk}\partial_{t_j},
\end{aligned}$$

where ϵ_{ijk} is the totally anti-symmetric Levi-Civita symbol with $\epsilon_{123} = 1$. These define three almost contact structures satisfying the 3-cosymplectic commutation relations. Since $\mathscr{D}$ is flat and all structure tensors are constant in these coordinates, we have $\mathscr{D}_{U_1}\psi_i = 0$, $\mathscr{D}_{U_1}\zeta_i = 0$, $\mathscr{D}_{U_1}\mu_i = 0$ for all U_1. The Codazzi condition for $\mathscr{D}$ holds trivially because $\mathscr{D}$ is the Levi-Civita connection of a flat metric. Thus, this constitutes a 3-cosymplectic statistical manifold, and it is Ricci-flat (indeed flat).

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