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Ginsburg-Pitaevski-Gross differential equation with the Rosen-Morse and modified Woods-Saxon potentials

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Abstract. In this paper, we consider non-linear Ginsburg-Pitaevski-Gross equation with the Rosen-Morse and modified Woods-Saxon potentials which is corresponding to the quantum vortices and has important applications in turbulence theory. We use the Runge-Kutta-Fehlberg approximation method to solve the resulting non-linear equation.

Keywords: Quantum Vortices, Non-Linear Differential Equation, Wave Function.

2000 Mathematics subject classification: 34A34, 34A45.

1. Introduction

Traveled light through the atmosphere affected by a number of phenomena such as scattering, absorption and "turbulence" [**?**, **?**, **?**]. Turbulence has been investigated not only in applied sciences but also in basic science, such as physics and mathematics. Turbulence is a complicated dynamical phenomenon which based on strong nonlinearity. This phenomenon is far from an equilibrium state, and may be understand in context of vortices. However, vortices are not well-defined in a classical turbulence, therefore quantum turbulence will be more convenient

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[?, ?, ?, ?]. Comparing quantum turbulence and classical turbulence reveals definite differences, which demonstrates the importance of studying quantum turbulence. Turbulence in a classical viscous fluid admitted vortices, but these vortices are unstable. Moreover, the circulation is not conserved and is not identical for each vortex. Quantum turbulence consists of vortices that have the same conserved circulation. Thus, quantum turbulence is an easier system to study than classical turbulence and has a much simpler model of turbulence than classical turbulence [?]. In that case the turbulence in superconductors has been studied experimentally [?]. The hydrodynamics of superfluid helium is well described by the two-fluid model, where the system consists of a viscous superfluid and a viscous normal fluid with two independent velocity fields. The mixing ratio of the two fluids depends on temperature. By decreasing the temperature below the critical point, the ratio of the superfluid component increases, and the entire fluid becomes a superfluid in low temperature limit. A vortex in this system with circular quantization is called a quantized vortex. Any rotational motion of a superfluid is admitted only by quantized vortices. A quantized vortex is a topological defect characteristic of a BoseEinstein condensate [?] and is different from a vortex in a classical viscous fluid. Thermal counterflow of superfluid turbulence has been studied experimentally, where the normal fluid and superfluid flow assumed in opposite directions. The flow is driven by an injected heat current, and it was found that the superflow becomes dissipative when the relative velocity between the two fluids exceeds a critical value [?]. One can understand vortex dynamics observations quantitatively. Because the dynamics of quantized vortices is nonlinear and non-local. Superfluid turbulence is often called quantum turbulence, which indeed studies quantized vortices. In physics, a quantum vortex is a topological defect exhibited in superfluids and superconductors. The existence of quantum vortices was predicted in 1947 in connection with superfluid helium. It is also pointed out that quantum vortices describe the circulation of superfluid and conjectured that their excitations are responsible for superfluid phase transitions. These ideas were further developed by the Ref. [?] were applied to describe the magnetic phase diagram of type-II superconductors by the Ref. [?]. Quantum vortices are observed experimentally in Type-II superconductors, liquid helium, and atomic gases. In a superfluid, a quantum vortex carries quantized angular momentum, thus allowing the superfluid to rotate; in a superconductor, the vortex carries quantized magnetic flux. Turbulence phenomenon also affect on the laser beam. The most important effects of atmospheric turbulence on the laser beam are phase-front

distortion, beam broadening, beam wander and redistribution of the intensity within the beam. The temporary redistribution of the intensity, known as scintillation, results from the chaotic flow changes of air and from thermal gradients within the optical path caused by the variation in air temperature and density. Information about the turbulence profile is crucial to assist the tomographic process in wide field adaptive optics (AO) system. Information about the turbulent layers could be used to reduce the impact of the delays present in AO systems [**?**].
In this paper, we consider quantum vortices [**?**] and corresponding non-linear differential equation. In section 3 we assume constant potential and obtain analytical expression of the wave function. In section 4 we consider Rosen-Morse potential and in section 5 we consider modified Woods-Saxon potential. Both cases solved numerically. Finally in section 6 we give conclusion and summarized our results.

2. Quantum vortices

In this work we interest to quantum vortices in superfluid. In that case we deals with an equation describing the static and dynamic behavior of the condensate wave function which is the non-linear Schrödinger equation, or GinsburgPitaevskiGross equation [**?**, **?**],

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+(U_0|\phi|^2+V_0|\psi|^2-\mu m+V)\psi, \tag{2.1}$$

where ψ is the condensate wave function of particle and ϕ is the wave function of electron. Also $V_0=4\pi d\hbar^2/m$ and $U_0=2\pi l\hbar^2/\mu$ are measures of the repulsive interatomic forces in the fluid, where l is the boson-impurity scattering length, and d is the boson diameter. Moreover μ is the chemical potential. We assume $U_0\ll V_0$ and set $\hbar=1$ for simplicity. A single-quantum rectilinear vortex along $r=0$ in cylindrical polar coordinates is described by the following function,

$$\psi=f(r)e^{i\theta}, \tag{2.2}$$

where $f(0)=0$ and $f(\infty)=f_0$. Therefore, non-linear differential equation (**??**) reduced to the following equation,

$$f''-8\pi df^3+2m(\mu m-V(r))f=0, \tag{2.3}$$

where prime denote derivative with respect to r. In this paper we would like to solve the equation (**??**) for various famous exponential potentials. First of all we consider constant potential. Then, we examine Rosen-Morse and modified Wood-Saxon potentials and also a general exponential form of potential.

3. Constant potential

In the simplest case we assume,

$$V = E_0, \tag{3.1}$$

where E_0 is a constant. This situation is special form of exponential function Ee^r when $r \ll 1$. In that case the equation (**??**) has the following solution,

$$f(r) = B[jacobiSN(B\bar{r}, D)], \tag{3.2}$$

where,

$$\bar{r} = c_1 + r\sqrt{2\mu m^2 - 2mE_0 - 4\pi d}, \tag{3.3}$$

and,

$$\begin{aligned} B &= c_2\sqrt{\frac{m(\mu m - E_0)}{\mu m^2 + 2\pi d c_2^2 - mE_0 - 2\pi d}}, \\ D &= c_2\frac{\sqrt{c_2^2\pi d(2\mu m^2 - 2mE_0 - 4\pi d)}}{2\pi d + mE_0 - \mu m^2}, \end{aligned} \tag{3.4}$$

where c_1 and c_2 are integration constants. In order to solve equations, we use the Runge-Kutta-Fehlberg approximation method. In the Fig. 1 we can see that the wave function is periodic. We can fit the Fig. 1 and find that $f(r)$ may have the following behavior,

$$f(r) \approx 0.025\sin(16r). \tag{3.5}$$

4. Rosen-Morse potential

The Rosen-Morse potential [**?**, **?**] plays an important role in Atomic, Chemical and Molecular Physics, since it can be used to describe molecular vibrations and to obtain the energy spectra of linear and nonlinear systems. This potential is very useful for describing interatomic interaction of the linear molecules. The Rosen-Morse potential may be written in the following form

$$V(r) = -V_1 \sec h^2(\alpha r) + V_2\tanh(\alpha r), \tag{4.1}$$

where V_1 and V_2 are the depth of the potential and α is the range of the potential.
Numerically, we find behavior of the wave function in the Fig. 2, which shows periodic feature.

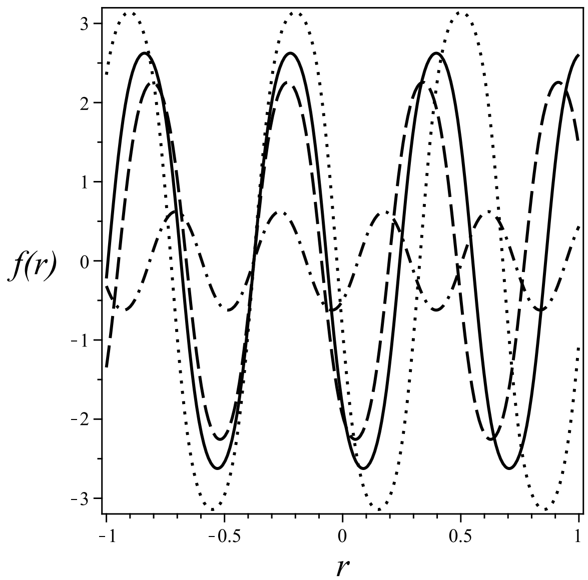

FIGURE 1. Wave function in terms of r with constant potential for $E_0 = 5$.

5. Modified Woods-Saxon potential

An interesting potential introduced by Woods and saxon to study elastic scattering [**?**]. The Woods-Saxon potential plays an important role in microscopic physics, since it can be used to describe the interaction of a nucleon with the heavy nucleus. This potential is utilized to represent the mean field which is felt by valance electron in Helium model [**?**]. The modified Woods-Saxon potential may be written as the following form,

$$V(r) = \frac{\nu}{1+e^{\beta r}} + \frac{\tau}{(1+e^{\beta r})^2} + \lambda \coth(\beta r) + \lambda \coth^2(\beta r), \qquad (5.1)$$

where ν, τ, λ and β are arbitrary constants. Numerically, we find behavior of the wave function in the plots of Fig. 3, which shows periodic feature for $\beta \geq 10$, however for $\beta < 10$ the modified Woods-Saxon potential has asymptotic behavior.

6. Conclusion

In this work, we considered quantum vortices which appeared in quantum turbulence and are useful to study superfluid. We calculated wave

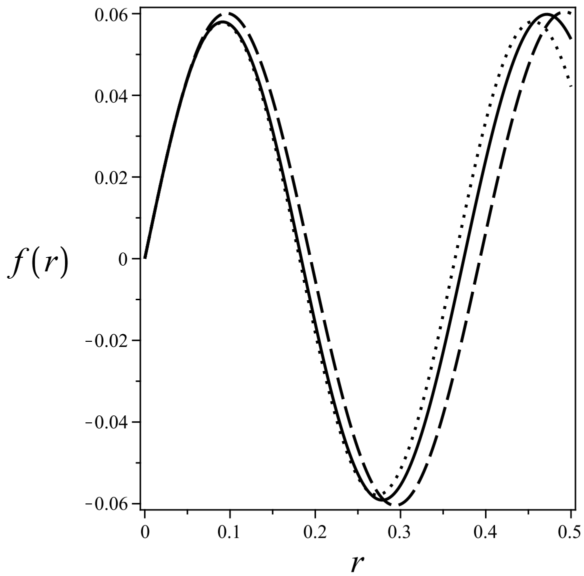

FIGURE 2. Wave function in terms of r with Rosen-Morse potential for $\alpha = 1$.

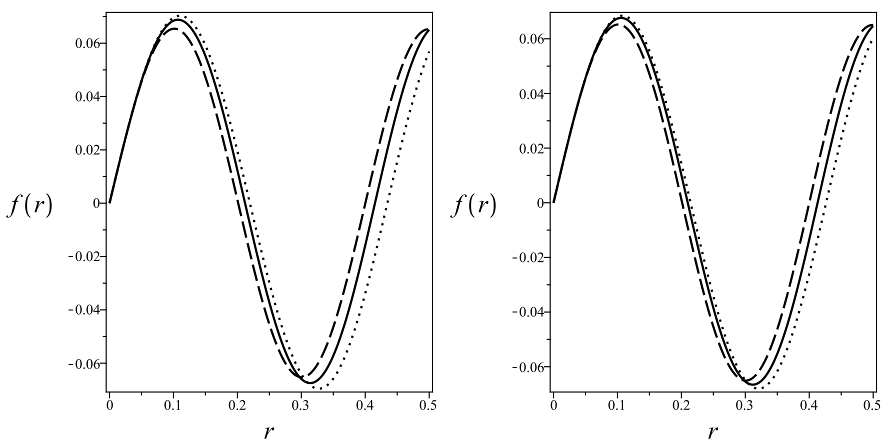

FIGURE 3. Wave function in terms of r with modified Woods-Saxon potential. Left: $\beta < 10$. Right: $\beta \geq 10$

function from non-linear Schrödinger equation known as GinsburgPitaevskiGross equation with various famous potentials such as Rosen-Morse and modified Woods-Saxon potentials. We found that the wave function has periodic behavior for Rosen-Morse potential and also for modified Woods-Saxon potential with $\beta \geq 10$. These are reasonable for nature of quantum vortex.

We compare all solution in a single plot to find differences of various models (Fig. 4). It is shown that results for modified Woods-Saxon potential with $\beta \geq 10$ coincide with the constant potential. Also it is found that Rosen-Morse potential has smallest period.

There are still many interesting potentials which may be used in the non-linear Schrödinger equation such as Dirac-Morse, Dirac-Rosen-Morse, Dirac-Eckart and Dirac-Scarf potentials [**?**, **?**, **?**, **?**].

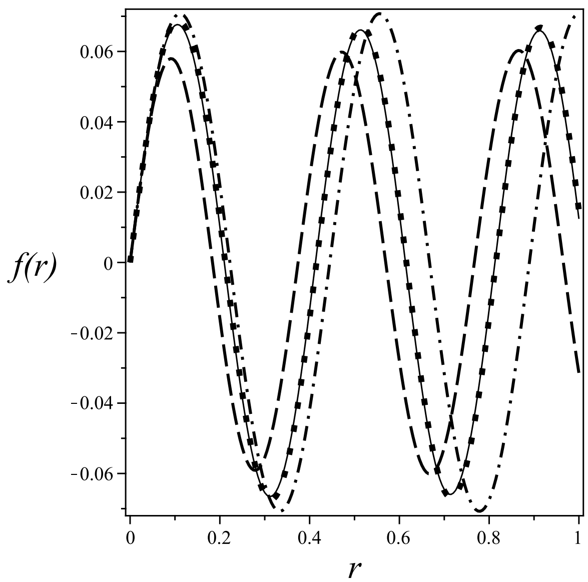

Figure 4. Wave function in terms of r with various potentials. Constant potential (solid line), Rosen-Morse potential (dashed line), Modified Woods-Saxon potential for $\beta \geq 10$ (dotted line) and $\beta < 10$ (dash-dotted line).

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