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# Spectrum preserving linear maps between Banach Algebras

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Abstract. In this paper we show that if $A$ is a unital Banach algebra, $B$ is a purely infinite $C^*$-algebra such that has a non-zero commutative maximal ideal and $\phi : A \to B$ is a unital surjective spectrum preserving linear map, then $\phi$ is a Jordan homomorphism.

Keywords: Banach Algebra, $C^*$-algebra, Jordan homomorphism, Linear Preserving.

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## 1. Introduction and Preliminaries

All algebras we deal with in this paper are complex and unital. The identity element of an algebra $A$ will be denoted $e$, or $e_A$, for distinction. For a given algebra $A$ and $a \in A$, $\sigma(a)$ and $r(a)$ will denote the *spectrum* and the *spectral radius* of $a$, respectively. Let $A$ and $B$ be two Banach algebras. A linear map $\phi : A \to B$ is said to be spectrum preserving if $\sigma(\phi(a)) = \sigma(a)$ for all $a \in A$. Furthermore, $\phi$ is said to be *unital* if $\phi(e_A) = e_B$ and it is called *invertibility preserving* if $\phi(a)$ is invertible in $B$ whenever $a$ is invertible in $A$. Now, if $A$ is a Banach algebra, the set of all non-zero complex homomorphisms of $A$ is a compact Hausdorff space in its usual (weak*) topology, the so-called Gelfand topology. This

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space will be called the maximal ideal space of $A$, and it will be denoted by $M_A$.

Spectrum preserving linear mappings were studied for the first time by G. Frobenius [8]. He proved that a surjective linear mapping $\varphi : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ which preserves the spectrum has one of the forms $\varphi(T) = ATA^{-1}$ or $\varphi(T) = A^tTA^{-1}$ for some invertible $A$. In [12] Jafarian and sourour proved that a surjective linear map preserving spectrum from $B(X)$ onto $B(Y)$ is either an isomorphism or an anti-isomorphism where $X$ and $Y$ are complex Banach spaces and $B(X)$ is the Banach algebra of all bounded linear operators acting on $X$. The following conjecture seems to be still open:

Any spectrum-preserving linear map from a unital Banach algebra onto a unital semi-simple (non-commutative) Banach algebra that preserves the unit is a Jordan homomorphism, (Kaplansky's conjecture).

The G-K-Z Theorem ([15], [14]) asserts that a unital linear functional defined on a Banach algebra is multiplicative if it is invertibility preserving and the theorem has inspired a number of papers on more general preserver problems. It is a straightforward conclusion of the G-K-Z Theorem that a unital and invertibility preserving linear map from a Banach algebra into a semi-simple commutative Banach algebra is a homomorphism. This conjecture is still unsolved. The most important partial results obtained this direction are [1], [2], [4], [6], [9], [10], [11], [12], [13], [19], [20], [21], [22].

Recently Aupetit [2] showed that a spectrum preserving surjective linear map from a von Neumann algebra onto another is a Jordan isomorphism.

A $C^*$-subalgebra $B$ of $A$ is hereditary if $0 \leq a \leq b, a \in A, b \in B$ implies $a \in B$. A projection in a $C^*$-algebra $A$ is called infinite if it is equivalent to a proper subprojection of itself. A $C^*$-algebra is purely infinite if every hereditary subalgebra contains an infinite projection.

In this paper we show that if $A$ is a unital Banach algebra, $B$ is a purely infinite $C^*$-algebra such that has a non-zero commutative maximal ideal and $\phi : A \to B$ is a unital surjective spectrum preserving linear map, then $\phi$ is a Jordan isomorphism (Theorem 2.2).

There are many results on the conjecture of Kaplansky. One of the most important results is [2, Theorem 1.3] of Aupetit. Among other theorems, Larwrence Harris proved the following.

**Theorem 1.1.** [9] *Let $A$ be a unital Banach algebra, $B$ be a unital semi-simple commutative Banach algebra and $\phi : A \to B$ is a unital invertibility preserving linear map. Then $\phi$ is a continuous multiplicative.*

**Theorem 1.2.** [17] *Let $A$ be a $C^*$-algebra, and $a \in A$. Then there is an irreducible representation $\pi$ of $A$ such that $\|a\| = \|\pi(a)\|$.*

*Remark* 1.3. By Theorem 5.1.6 (2) in [17], it follows that if $A$ is a non-commutative $C^*$-algebra, then irreducible representations $\pi$ in Theorem 1.2 has dimension greater than 1.

**Theorem 1.4.** [16] *Let $\phi : A \to B$ be a unital surjective spectrally bounded operator from a unital $C^*$-algebra $A$ onto a unital semisimple Banach algebra $B$. If $A$ is a purely infinite $C^*$-algebra of real rank zero, then $\phi$ is a Jordan homomorphism.*

## 2. Main results

Recall that all algebras we deal with have an identity element. Moreover, note that by an ideal we always mean a 2-sided ideal.

*Remark* 2.1. We recall that if $A, B$ and $D$ are $C^*$-algebras, and if homomorphisms $\varphi : A \longrightarrow D$ and $\psi : B \longrightarrow D$ are given, then the $C^*$-algebra $A \oplus_D B$ is defined as

$$A \oplus_D B = \{(a, b) \in A \oplus_D B : \varphi(a) = \psi(b)\}.$$

Let $A$ be a $C^*$-algebra, by [18, Lemmas 10 and 11] $A$ has a unique maximal commutative ideal $I$ ($I$ may be obtained as the intersection of the kernels of all irreducible representations of $A$ of dimension greater than 1) and a closed ideal $J$ such that $I \cap J = \{0\}$ and $A/J$ is commutative, furthermore, $A \cong A/J \oplus_{A/(I+J)} A/I$ by $*$-isomorphism $\varphi : A \longrightarrow A/J \oplus_{A/(I+J)} A/I$ such that $\varphi(a) = (a + J, a + I)$.

**Theorem 2.2.** *Let $A$ be a unital Banach algebra and $B$ be a purely infinite $C^*$-algebra such that it has a non-zero commutative maximal ideal. Suppose that $\phi : A \to B$ is a unital surjective spectrum preserving linear map, then $\phi$ is a Jordan homomorphism.*

**Proof**. By [18, Lemmas 10 and 11] $B$ has a unique maximal commutative ideal $I$ and a closed ideal $J$ with the properties $I \cap J = 0$, $B/J$ is commutative and $B \cong B/J \oplus_{B/(I+J)} B/I$. Define $\phi_1 : A \longrightarrow B/J$ and $\phi_2 : A \longrightarrow B/I$ by $\phi_1(a) = \phi(a) + J$ and $\phi_2(a) = \phi(a) + I$ for every $a \in A$.

We can show that $\phi_1$ and $\phi_2$ are well-defined and non-zero unital linear maps. For any $a \in A$, if $\phi(a)$ is invertible then $\phi_1(a)$ and $\phi_2(a)$ are invertible. Hence $\phi_1$ and $\phi_2$ preserve invertibility. Therefore, $\phi_1$ is continuous homomorphism by Theorem 1.1. (Note that $B/J$ is commutative $C^*$-algebra). Since $I$ contains every commutative ideal by the hypothesis $I$ is a commutative maximal ideal in $B$ (see proof of Lemma 10 in [18]). It is clear that $B/I$ is purely infinite simple $C^*$-algebra, so $B/I$ has real rank zero by [7, Theorem V.7.4].

We prove that $\phi_2$ is injective. To prove this, suppose $a \in A$ such that $\phi_2(a) = 0$, so $\phi(a) \in I$, and Theorem 1.2, Remarks 1.3 and 2.1

imply that $\|\phi(a)\| = \|\pi(\phi(a))\| = 0$ for some irreducible representation with dimension greater than 1. On the other hand $\phi$ is continuous and injective (see [2] and [3]), so $\phi_2$ is injective. Also, by Theorem 1.5, $\phi_2^{-1}$ is Jordan isomorphism and hence $\phi_2$ is Jordan isomorphism.

Now, we show that $\phi$ is a Jordan homomorphism. We have $\phi_1(a) = \phi(a) + J$ and $\phi_2(a) = \phi(a) + I$ for all $a \in A$. Hence for every $a \in A$

$$\phi_1(a)^2 = \phi(a)^2 + J,\ \phi_2(a)^2 = \phi(a)^2 + I. \tag{1}$$

Also we have

$$\phi_1(a^2) = \phi(a^2) + J,\ \phi_2(a^2) = \phi(a^2) + I. \tag{2}$$

Since $\phi_1$ and $\phi_2$ are Jordan homomorphism, (1) and (2) imply that $\phi(a)^2 - \phi(a^2) \in J$ and $\phi(a)^2 - \phi(a^2) \in I$. But $I \cap J = 0$. Therefore, $\phi(a^2) = \phi(a)^2$ for all $a \in A$, that is, $\phi$ is a Jordan homomorphism. This completes the proof. □

**Corollary 2.3.** *Let $A$ be a unital Banach algebra and $B$ be a purely infinite $C^*$-algebra such that it has a non-zero commutative ideal. Suppose that $\phi : A \to B$ is a surjective spectrum preserving linear map, so $\phi$ is a Jordan homomorphism multiplied by an invertible element.*

**Proof**. For $b \in B$, denote $L_b$ the linear map from $B$ into itself defined by multiplying by $b$ from the left hand, that is, $L_b(x) = bx$ for every $x \in B$. Let $\psi = L_{\phi(e)^{-1}} \circ \phi$, then $\psi(e) = e$. As a preserver, $\psi$ has the same property as $\phi$ has. It is easy to check $\psi$ preserves invertibility. Now by Theorem 2.2, $\psi$ is a Jordan homomorphism and $\phi = L_{\phi(e)} \circ \psi$. This completes the proof. □

## References

[1] J. Alaminos, M. Bresar, P.Semrl, and A. R. Villena, *A note on spectrum-preserving maps*, Journal of Mathematical Analysis and Applications, **387**, 15 (2012), 595-603

[2] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras*, J. London. Math. Soc. **62**(2000), 917–924.

[3] B. Aupetit, *A primer on spectral theory*, Springer, New York, 1991.

[4] B. Aupetit and H. du Toit Mouton, *Spectrum preserving linear mappings in Banach algebras*, Studia Math. **109**(1994), 91–100.

[5] N. Boudi and H. Youness, *Linear maps preserving generalized invertibility on commutative Banach algebras*, Rocky mountain journal of mathematics, **Vol. 42, No. 4**(2012), 1073–1120.

[6] M. Burgos, A. C. Mrquez-Garca and A. Morales-Campoy, *Approximate Preservers on Banach Algebras and C*-Algebras*, Hindawi Publishing Corporation Abstract and Applied Analysis. **V**olume 2013, Article ID 757646, 12 pages.

[7] K. R. Davidson, *$C^*$-algebras by example*, Amer. Math. Soc. 1991.

[8] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen I*, Sitzungberichte Königlich Preussischen Akademie Wissenschaften Berlin **(**1897), 994–1015.

[9] L.A. Harris, *Invertibility preserving linear maps of Banach algebras*, Israel Mathematical conference proceedings. **Vol. 15**, (2001).

[10] O. Hatori, T. Miura and H. Takagi, *Polynomially spectrum-preserving maps between commutative Banach algebras*, arXiv:0904.2322v1 [math.FA] 15 Apr 2009.

[11] A.A. Jafarian, *A survey of invertibility and spectrum preserving linear maps*, Bull. Iran. Math. Soc. **35**(2009), 1–10.

[12] A.A. Jafarian and A.R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. **66**(1986), 255–266.

[13] Jian Lian CUI, *A Characterization of Homomorphisms Between Banach Algebras*, Acta Mathematica Sinica, English Series, **Vol. 20, No. 4**(2004), 761–768.

[14] J.P. Kahane and W. Zelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. **29**(1968), 339–343.

[15] A.M. Gleason, *A characterization of maximal ideals*, J. Analyse Math. **19**(1967), 171–172.

[16] M. Mathieu, *Spectrally Bounded Operators on Simple $C^*$-algebras, II*, Irish Math. Soc. Bulletin. **54** (2004), 33-40.

[17] G. J. Murphy. Dales, *$C^*$-algebra and operator theory*, London, 1990.

[18] N. C. Phillips and A. Weaver, *Modules with norms which take values in a $C^*$-algebra*, Pacific. J. Math. **185**, No. 1, (1998), 163–181.

[19] P. Semrl, *Two chractreizations of automorphisims on $B(X)$*, Studia Math. **105**(1993), 143–149.

[20] A.R. Sourour, *Invertibility preserving linear maps on $L(X)$* , Trans. Amer. Math. Soc. **348**(1996), 13–30.

[21] A. Taghavi and R. Hosseinzadeh, *Maps preserving the spectrum of certain products of spectrum*, arXiv:1309.4032v1 [math.FA] 16 Sep 2013.

[22] A. Tajmouati and A. Bakkali , *Linear Maps Preserving the Kato Spectrum*, Int. Journal of Math. Analysis, **6**, No. 6,(2012), 253 – 263.

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