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On the Szeged and Eccentric connectivity indices of non-commutative graph of finite groups

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Abstract. Let G be a non-abelian group. The non-commuting graph Γ_G of G is defined as the graph whose vertex set is the non-central elements of G and two vertices are joined if and only if they do not commute.

In this paper we study some properties of Γ_G and introduce n-regular AC-groups. Also we then obtain a formula for Szeged index of Γ_G in terms of n, $|Z(G)|$ and $|G|$. Moreover, we determine eccentric connectivity index of Γ_G for every non-abelian finite group G in terms of the number of conjugacy classes $k(G)$ and the size of the group G.

Keywords: non-commuting graph, eccentric connectivity index, Szeged index.

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1. Introduction

Let Γ be an undirected connected graph without loops or multiple edges. We denote the vertex and the edge sets of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. For two vertices x and y of $V(\Gamma)$ the distance $d(x, y)$ is defined as the length of any shortest path connecting u and v in Γ and $deg(v)$ denotes the degree of vertex v, i.e., the number of its neighbors in Γ.

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For an edge $e(=uv)$ of Γ, let $n_u(e)$ denote the set of vertices of Γ lying closer to u than to v and $n_v(e)$ the set of vertices of Γ lying closer to v than to u.
The sets $n_u(e)$ and $n_v(e)$ play an important role in metric graph theory. For more information on research in this direction see [5, 8, 9]). Ivan Gutman [7] defined the Szeged index, $Sz(\Gamma)$, of a graph Γ as:

$$Sz(\Gamma) = \sum_{uv=e\in E(\Gamma)} |n_u(e)| \cdot |n_v(e)|.$$

For a given u of $V(\Gamma)$ its eccentricity the topological distance i.e., the number of edges on the shortest path, joining the two vertices of Γ. Since Γ is connected, $d(x, y)$ exists for all $x, y \in V(\Gamma)$. The eccentricity of a vertex v in $V(\Gamma)$, denoted by $\varepsilon(v)$, is defined to be

$$\varepsilon(v) = \max\{d(v, w)|w \in V(\Gamma)\}$$

Sharma, Goswami and Madan [13] introduced a distance-based molecular structure descriptor, they named " eccentric connectivity index" and which is defined as

$$\xi(\Gamma) = \sum_{v\in V(\Gamma)} deg(v) \cdot \varepsilon(v),$$

For further information see references [6, 14].

Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate with G a graph Γ_G as follows: Take $G\backslash Z(G)$ as vertices of Γ_G and join two distinct vertices x and y whenever $xy \neq yx$. Graph Γ_G is called the non-commutative graph of G and many of graph theoretical properties of Γ_G have been studied in [1, 2, 10]. Recently, it has been shown (see [3]), that if G is a finite non-abelian group, then Wiener index of Γ_G would be as:

$$W(\Gamma_G) = \frac{(|G| - |Z(G)|)(|G| - 2|Z(G)| - 2) + |G|(k(G) - |Z(G)|)}{2},$$

where $k(G)$ is the number of conjugacy classes of G.

In this paper we study some metric properties of Γ_G and introduce *n-regular AC-groups* and obtain a formula of Szeged index of Γ_G in terms n, $|Z(G)|$ and $|G|$. Also, for a finite non-abelian group G, we compute the eccentric connectivity index of Γ_G in terms of $k(G)$ and $|Z(G)|$. The main results are

Theorem 1.1. *Let G be a finite n-regular AC- group. Then*

$$Sz(\Gamma_G) = \frac{1}{2}(|G| - n)(|G| - |Z(G)|)(n - |Z(G)|)^2.$$

Theorem 1.2. *Let G be a finite non-abelian group. Then*

$$\xi(\Gamma_G) = 2|G|.(|G| - k(G)).$$

2. Szeged index of non-commuting graphs of certain groups

In this section we first introduce groups that have regular non-commuting graph and then introduce AC-groups. Then we obtain Szeged index of non-commuting graphs of some regular AC-groups.

We begin with some basic metric properties of the non-commuting graph of group G. Here, a significant result that we seriously need will be presented without proof of [1].

Lemma 2.1. *Let G be a non-abelian group. Then Γ_G is a connected graph of diameter 2 and girth 3.*

Proof. See [1]. □

By the above Lemma it is clear that:

Remark 2.2. Let G be a non-abelian group and $x \in G \backslash Z(G)$. Then

$$d(x, y) = \begin{cases} 1, & \text{if } y \in G \backslash C_G(x) \ ; \\ 2, & \text{if } y \in C_G(x) \backslash Z(G). \end{cases}$$

Lemma 2.3. *Let G be a non-abelian group. Then Γ_G is regular if and only if there exists natural number k, $|C_G(x)| = k$, where x is a non-central element.*

Proof. We know that if x is a non-central element, then by definition of non-commuting graph of a group, $deg(x) = |G \backslash C_G(x)|$. Now, Γ_G is a regular if and only if $deg(x) = deg(y)$, for all $x, y \in V(G)$. So Γ_G is regular if and only if $|C_G(x)| = |C_G(y)|$, for all $x, y \in V(G)$. □

A group G is called an AC-group if the centralizer of every non-central element of G is abelian.

Lemma 2.4. *The following conditions on a group G are equivalent.*
(a) *G is an AC-group.*
(b) *If $[x, y] = 1$, then $C_G(x) = C_G(y)$, where $x, y \in G \backslash Z(G)$.*
(c) *If $[x, y] = [x, z] = 1$, then $[y, z] = 1$, where $x \in G \backslash Z(G)$.*
(d) *If $x, y \in G \backslash Z(G)$ with distinct centralizers, then $C_G(x) \cap C_G(y) = Z(G)$.*

Proof. The proof is straightforward. See also [12], Lemma 3.2. □

Definition 2.5. A non-abelian group G is called regular AC- group if G is AC-group and Γ_G is a regular graph. Also a non-abelian group G is said to be n-regular AC- group if G is regular AC- group and $|C_G(x)| = n$, where $x \in V(G)$.

Before proving one of the main theorems, we need to present a key lemma of [3] without any proof.

Lemma 2.6. *Let G be a non-abelian group and $e = uv \in E(\Gamma_G)$. Then*

$$n_u(e) = ((C_G(v) \backslash C_G(u)) \backslash \{v\}) \cup \{u\}.$$

Proof. See [3]. □

Now we are ready to prove Theorem1.1.

Proof of Theroem 1.1

Proof. Since G is a n-regular AC- group so we have $|C_G(x)| = n$, where $x \in V(\Gamma_G)$. We know that $Sz(G) = \sum_{uv=e\in E(\Gamma)} |n_u(e)| \cdot |n_v(e)|$ and by Lemma 2.6, $n_u(e) = ((C_G(v)\backslash C_G(u))\backslash\{v\}) \cup \{u\}$. Now, $C_G(v)\backslash C_G(u) = C_G(v)\backslash(C_G(v)\cap C_G(u))$. Since G is AC-group, $C_G(v)\backslash C_G(u) = C_G(v)\backslash Z(G)$. Hence $|n_u(e)| = |C_G(v)\backslash Z(G)| = n - |Z(G)|$. Therefore

$$Sz(\Gamma_G) = \sum_{uv=e\in E(\Gamma)} (n - |Z(G)|)^2 = |E(\Gamma_G)|(n - |Z(G)|)^2.$$

On the other hand,

$$\begin{aligned} |E(\Gamma_G)| &= \frac{1}{2} \sum_{x\in V(\Gamma)} d(x) = \frac{1}{2} \sum_{x\in V(\Gamma_G)} |G\backslash C_G(x)| \\ &= \frac{1}{2} \sum_{x\in V(\Gamma_G)} (|G| - n) = \frac{1}{2}(|G| - n)(|G| - |Z(G)|). \end{aligned}$$

The proof is completed by replacing the above value of $|E(\Gamma_G)|$. □

As mentioned, in [3] for every non-abelian finite group G, the value of Wiener index $W(\Gamma_G)$ is determined. Here, we present a simple and different proof to obtain $W(\Gamma_G)$ when G is a finite n-regular AC- group.

Theorem 2.7. *Let G be a finite n-regular AC- group. Then*

$$W(\Gamma_G) = \frac{1}{2}(|G| - |Z(G)|)(|G| - 2|Z(G)| + n - 2).$$

Proof. Since G is an n-regular AC- group, $|C_G(x)| = n$, where $x \in V(\Gamma_G)$. We know that

$$W(\Gamma_G) = \frac{1}{2} \sum_{x\in V(\Gamma_G)} d(G, x)$$

and

$$d(G, x) = \sum_{y\in G\backslash C_G(x)} d(x, y) + \sum_{y\in C_G(x)\backslash Z(G)} d(x, y).$$

Now by Remark 2.2, $\sum_{y\in G\setminus C_G(x)} d(x,y) = |G\setminus C_G(x)|$ and

$$\sum_{y\in C_G(x)\setminus Z(G)} d(x,y) = 2(|C_G(x)| - |Z(G)| - 1).$$

Hence $d(G,x) = |G| - 2|Z(G)| + n - 2$. Therefore
$W(\Gamma_G) = \frac{1}{2}\sum_{x\in V(\Gamma_G)} |G| - 2|Z(G)| + n - 2$. Since $V(G) = G\setminus Z(G)$ we have
$W(\Gamma_G) = \frac{1}{2}(|G| - |Z(G)|)(|G| - 2|Z(G)| + n - 2)$, as desired. □

Example 2.8. Let G be a finite non abelian group of order p^3. Then $Sz(\Gamma_G) = \frac{1}{2}p^4(p-1)^3(p+1)$ and $W(\Gamma_G) = \frac{1}{2}p(p+1)^2(p-1)(p^2-2)$.

Proof. Since G is non-abelian, $|Z(G)| = p$ and if $x \in V(\Gamma_G)$ then $|C_G(x)| = p^2$. Hence $C_G(x)$ is abelian and so G is p^2-regular AC- group. Therefore $Sz(\Gamma_G) = \frac{1}{2}(p^3-p^2)(p^3-p)(p^2-p)^2 = \frac{1}{2}p^4(p-1)^3(p+1)$ and $W(\Gamma_G) = \frac{1}{2}(p^3-p)(p^3-2p+p^2-2) = \frac{1}{2}p(p+1)^2(p-1)(p^2-2)$. □

3. Eccentric connectivity index of non-commutative graph of group

In this section we suppose that G is a finite non-abelian group. The main objective in this section is to present an explicit formula of $\xi(\Gamma_G)$ the eccentric connectivity index in terms of $|G|$ and the number of conjugacy classes $k(G)$. Now, it is time to prove of secondary the main theorems.

Proof of Theroem 1.2

Proof. Suppose that $u \in V(\Gamma_G)$. If $x \in G\setminus C_G(u)$, then $d(u,x) = 1$ and if $x \in (C_G(u)\setminus Z(G))\setminus\{u\}$ then $d(u,x) = 2$. It follows that $\varepsilon(u) = max\{d(u,x) \mid x \in V(G)\} = 2$. We know, using the definition of non-commuting graph, that $deg(u) = |G\setminus C_G(u)|$. Hence

$$\begin{aligned}\xi(\Gamma_G) &= \sum_{u\in V(G)} deg(u)\cdot\varepsilon(u) = 2\sum_{u\in V(G)} |G\setminus C_G(u)| \\ &= 2\Big(\sum_{u\in V(G)} |G| - \sum_{u\in V(G)} |C_G(u)|\Big) = 2|G||G\setminus Z(G)| - 2\sum_{u\in V(G)} |C_G(u)|. \qquad (*)\end{aligned}$$

On the other hand, we have $\sum_{u\in G} |C_G(u)| = k(G)\cdot|G|$. So $\sum_{u\in Z(G)} |C_G(u)| + \sum_{u\in G\setminus Z(G)} |C_G(u)| = k(G)\cdot|G|$ and so $|Z(G)||G| + \sum_{u\in G\setminus Z(G)} |C_G(u)| = k(G)\cdot|G|$. It follows that

$$\sum_{u\in V(G)} |C_G(u)| = k(G)|G| - |Z(G)|\cdot|G| = |G|(k(G) - |Z(G)|).$$

Now, by using $(*)$, we have

$$\xi(\Gamma_G) = 2|G|\cdot(|G|-|Z(G))-2|G|\cdot(k(G)-|Z(G)|) = 2|G|(|G|-k(G)).$$

□

The paper winds up with the application of the above theorem for projective special linear group.

Example 3.1. Let G be a projective special linear group $PSL(2,q)$, where q is a power of a prime p and $q \geq 4$. Then

$$\xi(\Gamma_G) = \begin{cases} 2q(q+1)^2(q^2-q-1) & \text{if } q \equiv 0 \ (mod\ 4), \\ \frac{1}{2}q(q-1)(q+1)(q^3-2q-5) & \text{if } q > 5 \text{ and } q \not\equiv 0 \ (mod\ 4). \end{cases}$$

Proof. We know that if $q \equiv 0$ $(mod\ 4)$, then $|G| = q(q-1)(q+1)$ and if $q \not\equiv 0$ $(mod\ 4)$, then $|G| = \frac{1}{2}q(q-1)(q+1)$. However, by [11, Theorems 5.5, 5.6 and 5.7],

$$k(G) = \begin{cases} q+1 & q \equiv 0 \ (\text{mod } 4); \\ \frac{q+5}{2} & q > 5 \text{ and } q \not\equiv 0 \ (\text{mod } 4). \end{cases}$$

Now, by using of Theorem 1.2 , the result follows. □

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