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Some Fixed Point Theorems for Generalized Contractions in Metric Spaces with a Graph

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Abstract. Jachymski [Proc. Amer. Math. Soc., 136 (2008), 1359-1373] gave modified version of a Banach fixed point theorem on a metric space endowed with a graph. In the present paper, (G, φ)-graphic contractions have been defined by using a comparison function and studied the existence of fixed points. Also, Hardy-Rogers G-contractions have been introduced and some fixed point theorems have been proved. Some examples are presented to support the results proved herein. Our results generalize and extend various comparable results in the existing literature.

Keywords: Connected graph, Fixed point, $\varphi-$contraction, Hardy-Rogers contraction.

2010 Mathematics subject classification: 47H10, 54H25.

1. Introduction

Fixed point theory for nonlinear mappings is an important subject of nonlinear functional analysis. One of the basic and the most widely applied fixed point theorem in all of analysis is "Banach (or Banach-Caccioppoli) Contraction principle" due to Banach [21]. Banach contraction principle [21] is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. Due to its applications in

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mathematics and other related disciplines, Banach contraction principle has been generalized in many directions.

Existence of fixed points in ordered metric spaces has been discussed by Ran and Reurings [2]. Jachymski [12] investigated a new approach in metric fixed point theory by replacing an order structure with graph structure on a metric space. In this way, the results proved in ordered metric spaces are generalized (see for detail [12] and the reference therein). For further work in this direction, we refer to (see, e.g., [3, 4, 5, 7, 8, 11, 16, 17, 18, 20]).

The main purpose of this paper is to prove some fixed point theorems for G-graphic contraction by using a comparison function and also study existence and uniqueness of fixed points for Hardy-Rogers G-contraction in complete metric spaces with a graph.

2. Basic Facts and Definitions

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integer numbers, respectively. A mapping f from a matric space (X, d) into (X, d) is called a weakly Picard operator (WPO) if $\lim_{n\to\infty} f^n x = z$ for all $x \in X$, and z is a fixed point of f. Moreover, z is a unique fixed point of f and we say that f is a Picard operator (PO).

Let (X, d) be a metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. Assume that G has no parallel edges, so one can identify G with the pair $(V(G), E(G))$.

The conversion of a graph G is denoted by G^{-1} and which is a graph obtained from G by reversing the direction of edges. Hence

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

By $\tilde{G}$, we denote the undirected graph obtained from G by omitting the direction of edges. Indeed; it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric, under this convention, we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

For any $x, y \in V', (x, y) \in E'$ such that $V' \subseteq V(G), E' \subseteq E(G)$, then (V', E') is called a subgraph of G. If x and y are vertices in a graph G, then a path from x to y of length $N (N \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, 2, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. Some basic notations related to connectivity of graphs can be found in [19].

If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G) = [x]_G$ where $[x]_G$ denotes the equivalence class of relation $\Re$ defined on $V(G)$ by the rule;

$$y\Re z \quad \text{if there is a path in } G \text{ from } y \text{ to } z.$$

If $f : X \to X$ is an operator, then we denote by

$$F(f) = \{x \in X : x = fx\}$$

the set of all fixed points of f and also denote

$$X^f = \{x \in X : (x, fx) \in E(G) \quad \text{or} \quad (fx, x) \in E(G)\}$$

Definition 2.1. [12] A mapping $f : X \to X$ is a Banach $G-$contraction or simply G-contraction if the following conditions hold;

i. f preserves edges of G, that is,

$$(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$$

for all $x, y \in X$,

ii. f decreases weights of edges of G if there exists $\alpha \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow d(fx, fy) \leq \alpha d(x, y)$$

for all $x, y \in X$.

Definition 2.2. [8] The mapping $f : X \to X$ is a G-graphic contraction if the following conditions hold;

i. f preserves edges of G;

$$(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$$

for all $x, y \in X$,

ii. there exists $\alpha \in [0, 1)$ such that

$$d(fx, f^2x) \leq \alpha d(x, fx)$$

for all $x \in X^f$.

Now, we give some definitions related to types of continuity of mappings.

Definition 2.3. A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$f^{k_n}x \to y \quad \text{implies} \quad f(f^{k_n}x) \to fy \quad \text{as} \quad n \to \infty.$$

Definition 2.4. A mapping $f : X \to X$ is called orbitally G-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,

$$f^{k_n}x \to y, \quad (f^{k_n}x, f^{k_{n+1}}x) \in E(G) \quad \text{imply} \quad f(f^{k_n}x) \to fy \quad \text{as} \quad n \to \infty.$$

Now, we give a definition of a mapping which we use throughout the paper.

Definition 2.5. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function.

i. $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;
ii. $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$;
iii. $\sum\limits_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$.

If the conditions (i-ii) hold then φ is called a comparison function and the comparison function satisfies (iii) then φ is called a strong comparison function.

Remark 2.6. Any strong comparison function is a comparison function.

Remark 2.7. If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function, then $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0.

For a detailed study of $\varphi-$contractions we refer to [9], [10] and [22].

Definition 2.8. [5] Let (X, d) be a metric space and G a graph. The mapping $f : X \to X$ is called as $(G, \varphi)-$contraction if the following conditions hold;

i. f preserves edges of G;

$$(x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)$$

for all $x, y \in X$,

ii. there exist a comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(fx, fy) \leq \varphi(d(x, y))$$

for all $(x, y) \in E(G)$.

Definition 2.9. [20] The graph G is called a (C)-graph whenever for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 0$.

Definition 2.10. [6] Let (X, d) be a metric space. The operator $f : X \to X$ is called Hardy-Rogers contraction if there exist nonnegative numbers $\alpha, \beta, \gamma, \delta, \eta$ with $\alpha + \beta + \gamma + \delta + \eta < 1$, such that

$$d(fx, fy) \leq \alpha d(x, fx) + \beta d(y, fy) + \gamma d(x, fy) + \delta d(y, fx) + \eta d(x, y),$$

for all $x, y \in X$.

3. (G,φ)-Graphic Contraction and Fixed Point Theorems

In this section, we study the existence of fixed points in metric spaces with a graph by defining $(G,\varphi)-$ graphic contraction. Also, we will consider that the function φ is a strong comparison function.

Definition 3.1. Let (X,d) be a metric space and G a graph. The mapping $f : X \to X$ is called a $(G,\varphi)-$graphic contraction if the following conditions hold;

i. f preserves edges of G, i.e.,

$$(x,y)\in E(G) \Rightarrow (fx,fy)\in E(G)$$

for all $x,y\in X$,

ii. there exist a comparison function $\varphi:\mathbb{R}^{+}\to\mathbb{R}^{+}$ such that

$$d\left(fx,f^{2}x\right)\le\varphi\left(d\left(x,fx\right)\right)$$

for all $x\in X^{f}$.

Remark 3.2. If f is a $(G,\varphi)-$graphic contraction, then f is both a $\left(G^{-1},\varphi\right)-$graphic contraction and a $\left(\tilde{G},\varphi\right)-$graphic contraction.

Example 3.3. Any $G-$graphic contraction is a $(G,\varphi)-$graphic contraction, if the comparison function is given as $\varphi:\mathbb{R}^{+}\to\mathbb{R}^{+}, \quad \varphi(t)=\alpha t$.

The following example shows that $(G,\varphi)-$graphic contraction is an extension of $(G,\varphi)-$contraction given in [5].

Example 3.4. Let $X=[0,1]$ be endowed with the usual metric. Take

$$E\left(G\right)=\{(0,0)\}\cup\{(0,1)\}\cup\{(x,y)\in(0,1]\times(0,1]:\ x\ge y\},$$

and $f:X\to X$ as follows:

$$fx=\begin{cases}\frac{x}{4}, & if\ x\in(0,1);\\ \frac{1}{4}, & if\ x=0;\\ 1, & if\ x=1.\end{cases}$$

Then G is weakly connected and X^{f} is nonempty and f is a $(G,\varphi)-$graphic contraction with $\varphi\left(t\right)=\frac{3t}{4}$ which is not a $(G,\varphi)-$contraction.
Moreover; $F(f)=\{1\}$.

Proof. It is obvious that G is weakly connected and $X^{f}\neq\emptyset$. It can be easily seen that f is a (G,φ)-graphic contraction. Take $d(f1,f\frac{1}{2})\le\varphi(d(1,\frac{1}{2})\Rightarrow\frac{7}{8}\le\frac{3}{8}$, which is a contradiction. Thereby, f is not (G,φ)-contraction. □

Lemma 3.5. *Let (X,d) be a metric space endowed with a graph G. Let $f : X \to X$ be a $(G,\varphi)-$graphic contraction. If $x \in X^f$ then, there exists $r(x) \geq 0$ such that*

$$d\left(f^n x, f^{n+1}x\right) \leq \varphi^n\left(r\left(x\right)\right),$$

for all $n \in \mathbb{N}$, where $r(x) = d(x, fx)$.

Proof. Take $x \in X^f$, that is, $(x, fx) \in E(G)$ or $(fx, x) \in E(G)$. If $(x, fx) \in E(G)$, then by induction we have $\left(f^n x, f^{n+1}x\right) \in E(G)$ for each $n \in \mathbb{N}$. Thus

$$\begin{aligned} d\left(f^n x, f^{n+1}x\right) &\leq \varphi\left(d\left(f^{n-1}x, f^n x\right)\right) \leq \varphi^2\left(d\left(f^{n-2}x, f^{n-1}x\right)\right) \\ &\vdots \\ &\leq \varphi^n\left(d\left(x, fx\right)\right) = \varphi^n\left(r\left(x\right)\right). \end{aligned}$$

If $(fx, x) \in E(G)$, again by induction, we have that $\left(f^{n+1}x, f^n x\right) \in E(G)$ for each $n \in \mathbb{N}$. Hence

$$\begin{aligned} d\left(f^n x, f^{n+1}x\right) &\leq \varphi\left(d\left(f^{n-1}x, f^n x\right)\right) \leq \varphi^2\left(d\left(f^{n-2}x, f^{n-1}x\right)\right) \\ &\vdots \\ &\leq \varphi^n\left(d\left(x, fx\right)\right) = \varphi^n\left(r\left(x\right)\right). \end{aligned}$$

□

Lemma 3.6. *Let (X,d) be a complete metric space endowed with a graph G. Assume that $f : X \to X$ is a $(G,\varphi)-$graphic contraction. Then, for each $x \in X^f$, there exists $x^* \in X$ such that the sequence $(f^n x)_{n\in\mathbb{N}}$ converges x^* as $n \to \infty$.*

Proof. Let $n, m \in \mathbb{N}$, $m > n$, using property of φ and lemma 3.5, we have

$$\begin{aligned} d\left(f^n x, f^{n+m}x\right) &\leq d\left(f^n x, f^{n+1}x\right) + d\left(f^{n+1}x, f^{n+2}x\right) \\ &+ ... + d\left(f^{n+m-1}x, f^{n+m}x\right) \\ &\leq \varphi^n\left(r\left(x\right)\right) + \varphi^{n+1}\left(r\left(x\right)\right) + ... + \varphi^{n+m-1}\left(r\left(x\right)\right) \\ &= \sum_{j=1}^{m} \varphi^{n+j-1}\left(r\left(x\right)\right) < \infty \qquad (n, m \to \infty). \end{aligned}$$

Thus, the sequence $(f^n x)$ is a Cauchy sequence. Because (X,d) is a complete metric space, $(f^n x)_{n\in\mathbb{N}}$ is convergence sequence and say limit is $x^* \in X$. □

In the following example shows that above the lemma does not satisfy unless the function φ is not strong comparison.

Example 3.7. Recall that $\varphi(t) = \frac{t}{t+1}$, $t \geq 0$ is a comparison function but not a strong comparison function. If we use $\varphi(t) = \frac{t}{t+1}$, $t \geq 0$ in the previous lemma, we have

$$\sum_{n=1}^{\infty} \varphi^n\left(d\left(x, fx\right)\right) = \sum_{n=1}^{\infty} \frac{d\left(x, fx\right)}{nd\left(x, fx\right)+1}$$

diverges if $d(x, fx) > 0$.

Thus, this shows that it is necessary to use a strong comparison function.

Lemma 3.8. *Let (X, d) be a complete metric space endowed with a graph G, $f : X \to X$ is a $(G, \varphi)-$graphic contraction for which there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing x_0. Then $[x_0]_{\tilde{G}}$ is $f-$invariant and $f\big|_{[x]\,\tilde{G}}$ is a $\left(\tilde{G}_{x_0}, \varphi\right)$-graphic contraction.*

Proof. The proof can be obtained by using a similar method to that used in [5]. □

Theorem 3.9. *Let (X, d) be a complete metric space and G be a directed graph. Let the triple (X, d, G) has the following condition;*

$$\begin{aligned}
&\text{for any } (x_n)_{n\in\mathbb{N}} \text{ in } X, \text{ if } x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \\
&\text{(or respectively } (x_{n+1}, x_n) \in E(G)) \text{ for all } n \in \mathbb{N}, \\
&\text{then there is a subsequence} (x_{k_n})_{n\in\mathbb{N}} \text{ with } (x_{k_n}, x) \in E(G) \\
&\text{(or respectively } (x, x_{k_n}) \in E(G)) \text{ for all } n \in \mathbb{N}.
\end{aligned} \tag{3.1}$$

Let $f : X \to X$ be a $(G, \varphi)-$graphic contraction which is orbitally $G-$continuous. Then the following statements hold:

i. $F(f) \neq \emptyset$ iff $X^f \neq \emptyset$.
ii. *If $X^f \neq \emptyset$ and G is weakly connected, then f is a WPO.*
iii. *For any $x \in X^f$, we have that $f\big|_{[x]\,\tilde{G}}$ is a WPO.*

Proof. We begin with the statement *(iii)*. Let $x \in X^f$. Hence, there exists $r(x) \geq 0$ such that

$$d\left(f^n x, f^{n+1} x\right) \leq \alpha^n r(x)$$

for all $n \in \mathbb{N}$. So, there exists $x^* \in X$ such that $\lim_{n\to\infty} f^n x = x^*$. Since $x \in X^f$, then $f^n x \in X^f$ for every $n \in \mathbb{N}$. Now assume that $(x, fx) \in E(G)$. (This can be done if $(fx, x) \in E(G)$.) By using 3.1, a subsequence $\left(f^{k_n} x\right)_{n\in\mathbb{N}}$ of $(f^n x)_{n\in\mathbb{N}}$ such that $\left(f^{k_n} x, x^*\right) \in E(G)$ for each $n \in \mathbb{N}$.

A path in G can be formed by using the points $x, fx, ..., f^{k_1}x, x^*$ and hence $x^* \in [x]_{\tilde{G}}$. Since f is orbitally $G-$continuous, we obtain that x^* is a fixed point for $f\left|_{[x]\ \tilde{G}}\right.$.

To prove *(i)*, using *(iii)* we have $F(f) \neq \emptyset$ if $X^f \neq \emptyset$. Suppose that $F(f) \neq \emptyset$. By using the assumption that $\Delta \subseteq E(G)$, we immediately obtain that $X^f \neq \emptyset$. Hence *(i)* holds.

To prove *(ii)*, let $x \in X^f$. If we use weak connectivity of G, we have that $X = [x]_{\tilde{G}}$ and by applying *(iii)*, we obtain the desired result. □

The next example shows that for any $(G, \varphi)-$graphic contraction $f : X \to X$, being orbitally G-continuous, is a necessary condition to be a WPO.

Example 3.10. Let $X = [0, 1]$ be endowed with the usual metric. Consider

$$E(G) = \{(0,0)\} \cup \{(0,x) : x \geq 1/2\} \cup \{(x,y) : x, y \in (0,1]\},$$

and $f : X \to X$,

$$fx = \begin{cases} \frac{x}{2}, & if\ x \in (0,1]; \\ \frac{1}{2}, & if\ x = 0. \end{cases}$$

Then G is weakly connected, X^f is nonempty and f is a $(G, \varphi)-$graphic contraction with $\varphi(t) = \frac{t}{2}$, but is not orbitally $G-$continuous. Thus f has not a fixed point.

The example which is given below satisfies all conditions and statements (i-iii) of Theorem 3.9.

Example 3.11. Let $X = [0, 1]$ be endowed with the usual metric. Consider

$$E(G) = \{(0,0)\} \cup \{(0,x) : x \geq 1/2\} \cup \{(x,y) : x, y \in (0,1]\},$$

and $f : X \to X$,

$$fx = \begin{cases} \frac{x}{2}, & if\ x \in (0,1); \\ 0, & if\ x = 0; \\ 1, & if\ x = 1. \end{cases}$$

Then G is weakly connected, X^f is nonempty and f is a $(G, \varphi)-$graphic contraction with $\varphi(t) = \frac{t}{2}$ and also, f is orbitally $G-$continuous. Moreover; $F(f) = \{0, 1\}$.

4. Hardy-Rogers G-Contraction and Fixed Point Theorems

Throughout this section, we assume that $X_f = \{x \in X : (x, fx) \in E(G)$.

Definition 4.1. The mapping $f : X \to X$ is a Hardy-Rogers G-contraction if the following conditions hold:

i. f preserves edges of G, that is,

$$(x,y) \in E(G) \Rightarrow (fx, fy) \in E(G),$$

ii. there exist $\alpha, \beta, \gamma, \delta, \eta$ nonnegative real numbers and $\alpha+\beta+\gamma+\delta+\eta<1$ such that

$$d(fx,fy) \leq \alpha d(x,fx) + \beta d(y,fy) + \gamma d(x,fy) + \delta d(y,fx) + \eta d(x,y),$$

for all $(x,y) \in E(G)$.

If f is a Hardy-Rogers G-contraction, then f is both a Hardy-Rogers G^{-1}-contraction and a Hardy-Rogers $\tilde{G}$-contraction.

Also; any G-contraction is a Hardy-Rogers G-contraction where $\alpha = \beta = \gamma = \delta = 0$.

Example 4.2. Let $X = \{0,1,2,3\}$ and $d(x,y) = |x-y|$ for all $x, y \in X$. Define the operator $f : X \to X$ as;

$$fx = \begin{cases} 0, \text{ if } x \in \{0,1\} \\ 1, \text{ if } x \in \{2,3\} \end{cases} .$$

f is a Hardy-Rogers G-contraction with constants $\alpha = \beta = \gamma = \eta = \frac{1}{5}$ and $\delta = 0$, where $E(G) = \{(0,1); (0,2); (2,3); (0,0); (1,1); (2,2); (3,3)\}$, but it is not a Hardy-Rogers contraction

$$d(f1, f2) \leq \alpha d(1, f1) + \beta d(2, f2) + \gamma d(1, f2)$$
$$+ \delta d(2, f1) + \eta d(1,2),$$

it is a contradiction since $1 \leq \frac{3}{5}$.

Lemma 4.3. *Let (X,d) be a metric space endowed with a graph G. Let $f : X \to X$ be a Hardy-Rogers G-contraction with $\alpha, \beta, \gamma, \delta, \eta$ nonnegative real numbers and $\alpha+\beta+\gamma+\delta+\eta<1$. If $x \in X_f$ then there exists $r(x) \geq 0$ such that*

$$d\left(f^n x, f^{n+1} x\right) \leq \lambda^n r(x)$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{\alpha+\gamma+\eta}{1-\beta-\gamma} < 1$.

Proof. Take $x \in X_f$, then $(x, fx) \in E(G)$, then by induction we get $\left(f^n x, f^{n+1} x\right) \in E(G)$ for each $n \in \mathbb{N}$. Therefore,

$$
\begin{aligned}
d\left(f^n x, f^{n+1} x\right) \leq\; & \alpha d\left(f^{n-1} x, f^n x\right)+\beta d\left(f^n x, f^{n+1} x\right)+\gamma d\left(f^{n-1} x, f^{n+1} x\right) \\
& +\delta d\left(f^n x, f^n x\right)+\eta d\left(f^{n-1} x, f^n x\right) \\
\leq\; & \alpha d\left(f^{n-1} x, f^n x\right)+\beta d\left(f^n x, f^{n+1} x\right)+\gamma d\left(f^{n-1} x, f^n x\right) \\
& +\gamma d\left(f^n x, f^{n+1} x\right)+\eta d\left(f^{n-1} x, f^n x\right)
\end{aligned}
$$

$$
d\left(f^n x, f^{n+1} x\right) \leq \lambda d\left(f^{n-1} x, f^n x\right)
$$

where $\lambda=\frac{\alpha+\gamma+\eta}{1-\beta-\gamma}<1$. Hence, we obtain that

$$
d\left(f^n x, f^{n+1} x\right) \leq \lambda d\left(f^{n-1} x, f^n x\right) \leq \ldots \leq \lambda^n d(x, f x)=\lambda^n r(x) .
$$

□

Lemma 4.4. *Let (X, d) be a complete metric space endowed with a graph G. Suppose that $f: X \rightarrow X$ is a Hardy-Rogers G-contraction with constant $\alpha, \beta, \gamma, \delta, \eta$ with $\alpha+\beta+\gamma+\delta+\eta<1$. Then for each $x \in X_f$, there exists $x^* \in X$ such that the sequence $\left(f^n x\right)_{n \in \mathbb{N}}$ converges to $x^* \in X$ as $n \rightarrow \infty$.*

Proof. If $x \in X_f$, then $fx \in [x]_{\tilde{G}}$ and $\left(f^n x, f^{n+1} x\right) \in E(G)$ for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$, $m>n$, using lemma 4.3, we have

$$
\begin{aligned}
d\left(f^n x, f^m x\right) & \leq d\left(f^n x, f^{n+1} x\right)+d\left(f^{n+1} x, f^{n+2} x\right)+\ldots+d\left(f^{m-1} x, f^m x\right) \\
& \leq \lambda^n r(x)+\lambda^{n+1} r(x)+\ldots+\lambda^{m-1} r(x) \\
& =\lambda^n\left[1+\lambda+\lambda^2+\ldots+\lambda^{m-n-1}\right] r(x) \\
& \leq \frac{\lambda^n}{1-\lambda} r(x) \rightarrow 0, \qquad (n \rightarrow \infty) .
\end{aligned}
$$

Thus, $\left(f^n x\right)_{n \in \mathbb{N}}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $\left(f^n x\right)_{n \in \mathbb{N}}$ converges to $x^* \in X$ as $n \rightarrow \infty$. □

Lemma 4.5. *Let (X, d) be a complete metric space endowed with a graph G. The self mapping f is a Hardy-Rogers G-contraction for which there exists $x_0 \in X$ such that $f x_0 \in \left[x_0\right]_{\tilde{G}}$. Then the set $\left[x_0\right]_{\tilde{G}}$ invariant with respect to f and $f\Big|_{\left[x_0\right]_{\tilde{G}}}$ is a Hardy-Rogers $\tilde{G}_{x_0}$-contraction, where $\tilde{G}_{x_0}$ is the component of $\tilde{G}$ containing x_0.*

Theorem 4.6. *Let (X, d) be a complete metric space endowed with a graph G and f be a self-map on X. Assume that the following assertions hold:*

i. *G is weakly connected and (C)-graph;*
ii. *f is a Hardy-Rogers $\tilde{G}$-contraction;*
iii. *X_f is nonempty.*

Then f is a PO.

Proof. Let $x \in X_f$, then $fx \in [x]_{\tilde{G}}$ and $\left(f^n x, f^{n+1} x\right) \in E(G)$ for all $n \in \mathbb{N}$. From lemma 4.4, we get $(f^n x)_{n\in\mathbb{N}}$ converges to $x^* \in X$.
Now we show that x^* is a fixed point of f. As $f^n x \to x^*$, $\left(f^n x, f^{n+1} x\right) \in E(G)$ for all $n \in \mathbb{N}$ and G is a (C)-graph, there exists a subsequence $\{f^{n_k} x\}$ of $\{f^n x\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for each $k \in \mathbb{N}$. Since $(T^{n_k} x, x^*) \in E\left(\tilde{G}\right)$ and f is a Hardy-Rogers $\tilde{G}$-contraction. Hence it follows that

$$d\left(f^{n_k+1} x, fx^*\right) \leq \alpha d\left(f^{n_k} x, f^{n_k+1} x\right) + \beta d(x^*, fx^*) + \gamma d(f^{n_k} x, fx^*)$$
$$+\delta d\left(x^*, f^{n_k+1} x\right) + \eta d(f^{n_k} x, x^*)$$

as $k \to \infty$,

$$d(x^*, fx^*) \leq (\beta + \gamma)\, d(x^*, fx^*),$$

since $(\beta + \gamma) < 1$, then $fx^* = x^*$. Thus, x^* is a fixed point of f.

Next, we prove that x^* is a unique fixed point. Suppose that f has another fixed point $y^* \in X - \{x^*\}$. Since G is a (C)-graph, then there exists a subsequence $\{f^{n_k} x\}$ of $\{f^n x\}$ such that $(f^{n_k} x, x^*) \in E(G)$ and $(f^{n_k} x, y^*) \in E(G)$ for each $k \in \mathbb{N}$. Furthermore, G is weakly connected $(x^*, y^*) \in E\left(\tilde{G}\right)$, we have

$$d(x^*, y^*) = d(fx^*, fy^*) \leq \alpha d(x^*, fx^*) + \beta d(y^*, fy^*) + \gamma d(x^*, fy^*)$$
$$+\delta d(y^*, fx^*) + \eta d(x^*, y^*)$$

where from

$$d(x^*, y^*) \leq (\gamma + \delta + \eta)\, d(x^*, y^*),$$

due to $(\gamma + \delta + \eta) < 1$, which is a contradiction. Therefore, x^* is unique fixed point of f. □

In theorem 4.6, if we replace the condition that G is a (C)-graph with orbitally G-continuity of f, then we have the following theorem.

Theorem 4.7. *Let (X, d) be a complete metric space endowed with a graph G and f be a self-map on X. Assume that the following assertions hold:*

i. *G is weakly connected;*
ii. *f is a Hardy-Rogers $\tilde{G}$-contraction and orbitally G-continuous;*
iii. *X_f is nonempty.*

Then f is a PO.

Proof. Let $x \in X_f$, then from lemma 4.4, we get $(f^n x)_{n\in\mathbb{N}}$ converges to $x^* \in X$. Because $\left(f^n x, f^{n+1}x\right) \in E(G)$ for all $n \in \mathbb{N}$ and f is orbitally G-continuous, therefore $x^* = \lim_{n\to\infty} f(f^n x) = fx^*$. That is, $fx^* = x^*$. Assume that y^* is another fixed point of f. If we use the same technique as in the theorem 4.6, then we obtain that $y^* = x^*$. □

Example 4.8. Let $X = [0,1]$ and $d(x,y) = |x-y|$, for all $x, y \in X$. Consider

$$E(G) = \{(x,y) \ : \ x, y \in [0,1]\},$$

and $f : X \to X$ as follows:

$$fx = \frac{x}{4}, \quad x \in X.$$

Note that G is weakly connected and (C)-graph, X_f is nonempty and f is both a Hardy-Rogers $\tilde{G}$-contraction and orbitally G-continuous, where $\eta = \frac{1}{3}$, $\alpha = \beta = \frac{1}{4}$, $\gamma = \delta = 0$. Thus, all conditions of theorem 4.6 and theorem 4.7 are satisfied. Moreover, 0 is a unique fixed point of f.

Corollary 4.9. [Kannan type] *Let (X,d) be a complete metric space endowed with a directed graph G and f be a self-map on X. Assume that the following assertions hold:*

i. *G is weakly connected and (C)-graph;*
ii. *there exist α, β non-negative real numbers with $\alpha + \beta < 1$ such that*

$$d(fx, fy) \leq \alpha d(x, fx) + \beta d(y, fy),$$

for all $(x,y) \in E(\tilde{G})$;

iii. *X_f is nonempty.*

Then f is a PO.

Corollary 4.10. [Reich type] *Let (X,d) be a complete metric space endowed with a graph G and f be a self-map on X. Assume that the following assertions hold:*

i. *G is weakly connected and (C)-graph;*
ii. *there exist α, β, η, non-negative real numbers with $\alpha + \beta + \eta < 1$ such that*

$$d(fx, fy) \leq \eta d(x,y) + \alpha d(x, fx) + \beta d(y, fy),$$

for all $(x,y) \in E(\tilde{G})$;

iii. *X_f is nonempty.*

Then f is a PO.

Remark 4.11. In above corollaries, if we replace the condition that G is a (C)-graph with orbitally G-continuity of f, then we obtain same desire results.

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