Caspian Journal of Mathematical Sciences(CJMS)
University of Mazandaran, Iran
http://www.cjms.umz.ac.ir
ISSN: 1735-0611

CJMS. **1**(1)(2012), 39-46

COMPLEXITION AND SOLITARY WAVE SOLUTIONS OF THE (2+1)-DIMENSIONAL DISPERSIVE LONG WAVE EQUATIONS

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ABSTRACT. In this paper, the coupled dispersive (2+1)-dimensional long wave equation is studied. The traveling wave hypothesis yields complexiton solutions. Subsequently, the wave equation is studied with power law nonlinearity where the ansatz method is applied to yield solitary wave solutions. The constraint conditions for the existence of solitons naturally fall out of the derivation of the soliton solution.

Classification AMS: 37K10, 35Q51,35Q55

1. INTRODUCTION

The study of nonlinear evolution equations (NLEEs) has been going on for the past few decades [1-10]. There are several theoretical physicists that are studying these equations to obtain closed form exact and physically relevant meaningful solutions. Success has been overwhelming. These variety of solutions have helped the physicists, engineers, biologists and applied mathematicians beyond measure. Besides the usual solitons and solitary waves, some of the additional solutions that are obtained lately are shock waves, peakons, stumpons, cuspons, complexitons, Gaussons, cnoidal waves, snoidal waves and several others.
This paper will obtain complexiton solution and solitary wave solution of the (2+1)-dimensional dispersive long wave equations. There are several integrability techniques that have been developed in the past few decades to extract these solutions from the governing NLEEs. Some of these commonly used techniques are Adomian decomposition method, G'/G method [5], exp-function method [5], Fan's F-expansion method, simplest equation method, semi-inverse variational method [2] just to name a few. This paper will adopt the traveling wave hypothesis method and the ansatz approach to obtain solitary waves and complexiton solutions to the (2+1)-dimensional nonlinear dispersive wave equations.

Received: 15 Nov 2011
Accepted: 15 Jan 2012
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2. Mathematical Analysis

This section will carry out the analysis to derive the soliton and complexiton solutions of the equation of study. The analysis will be in the following two sub-sections. In the first case, the regular dispersive long wave equations will be studied, while in the second section, the power law nonlinearity will be addressed. The traveling wave hypothesis will be used to integrate in the first case and the ansatz method will be exploited later to solve the wave equation with power law nonlinearity.

2.1. Complexiton Solution.

The (2+1)-dimensional dispersive long wave equations, in shallow water, are given by [1]

$$u_{yt} + v_{xx} + \frac{1}{2}\left(u^2\right)_{xy} = 0, \tag{2.1}$$

$$v_t + \left(uv + u + u_{xy}\right)_x = 0, \tag{2.2}$$

where u and v are functions of the spatial variables x, y and the temporal variable t, and subscripts denote partial derivatives.

By introducing a variable ξ such that [1]

$$\xi = k_1 x + k_2 y + k_3 t + \xi_o, \tag{2.3}$$

with k_1, k_2, k_3, and ξ_o are nonzero real constants, the system (2.1) and (2.2) is reduced to the ordinary differential equations in the form

$$k_2 k_3 u'' + k_1^2 v'' + \frac{1}{2} k_1 k_2 \left(u^2\right)'' = 0, \tag{2.4}$$

$$k_3 v' + k_1 \left(uv + u + k_1 k_2 u''\right)' = 0, \tag{2.5}$$

Integrating (2.4) once gives

$$k_2 k_3 u' + k_1^2 v' + \frac{1}{2} k_1 k_2 \left(u^2\right)' = C, \tag{2.6}$$

where C is an integration constant. Integrating Eq. (2.6) once again, we obtain

$$k_2 k_3 u + k_1^2 v + \frac{1}{2} k_1 k_2 u^2 = C_1, \tag{2.7}$$

where C_1 is again an integration constant. From Eq. (2.7), we find that

$$v\left(\xi\right) = \frac{C_1}{k_1^2} - \frac{k_2 k_3}{k_1^2} u - \frac{k_2}{2k_1} u^2. \tag{2.8}$$

Substituting (2.8) into (2.5) yields

$$\left[\frac{C_1}{k_1} - \frac{k_2 k_3^2}{k_1^2} + k_1\right] u' - \frac{3k_2 k_3}{k_1} uu' - \frac{3k_2}{2} u^2 u' + k_1^2 k_2 u''' = 0, \tag{2.9}$$

which can be rewritten as

$$u''' - \frac{3}{2k_1^2} u^2 u' - \frac{3k_3}{k_1^3} uu' + \frac{1}{k_2 k_1^2}\left[\frac{C_1}{k_1} - \frac{k_2 k_3^2}{k_1^2} + k_1\right] u' = 0, \tag{2.10}$$

Equation (2.10) can be integrated with respect to ξ directly to yield

$$u'' - a_1 u + a_2 u^2 - a_3 u^3 = A. \tag{2.11}$$

in which A is an integration constant to be determined and

$$a_1 = \frac{1}{k_2 k_1^2}\left[\frac{k_2 k_3^2}{k_1^2} - \frac{C_1}{k_1} - k_1\right], \tag{2.12}$$

$$a_2 = -\frac{3k_3}{k_1^3}, \tag{2.13}$$

$$a_3 = \frac{3}{2k_1^2}\ . \tag{2.14}$$

In order to construct both bright and dark solitary wave solutions, we adopt the following form of solution ansatz

$$u(\xi) = \beta + \lambda \tanh(\eta\xi) + \rho \,\mathrm{sech}\,(\eta\xi)\,, \tag{2.15}$$

where η, λ, β and ρ are nonzero coefficients which will be determined as a function of the dependent model coefficients.

In the limit $\beta = \lambda = 0$, we obtain bright solitary wave solutions, but when $\rho = 0$ the solution (2.15) exactly transform to dark-type solutions. The presence of the parameters β, λ and ρ permits to the ansatz (2.15) to describe the features of both bright and dark solitary waves.

Substituting (2.15) into (2.11) and collecting coefficients of $\mathrm{sech}^k \mu\xi \tanh^m \mu\xi$ with $k = 0, 1, 2, 3$ and $m = 0, 1$, then setting each coefficients to zero, we obtain the following equations:

$$-a_1\beta + a_2\left(\lambda^2 + \beta^2\right) - a_3\beta\left(3\lambda^2 + \beta^2\right) - A = 0, \tag{2.16}$$

$$\rho\left\{\eta^2 - a_1 + 2\beta a_2 - 3a_3\left(\lambda^2 + \beta^2\right)\right\} = 0, \tag{2.17}$$

$$a_2\left(\rho^2 - \lambda^2\right) - 3\beta a_3\left(\rho^2 - \lambda^2\right) = 0, \tag{2.18}$$

$$-2\rho\eta^2 - a_3\rho\left(\rho^2 - 3\lambda^2\right) = 0, \tag{2.19}$$

$$2a_2\rho\lambda - 6a_3\beta\lambda\rho = 0, \tag{2.20}$$

$$-2\lambda\eta^2 - a_3\lambda\left(3\rho^2 - \lambda^2\right) = 0, \tag{2.21}$$

$$-a_1\lambda + 2\beta\lambda a_2 - a_3\lambda\left(3\beta^2 + \lambda^2\right) = 0\ . \tag{2.22}$$

From Eq. (2.20) or (2.22), one obtains

$$\beta = \frac{a_2}{3a_3}. \tag{2.23}$$

From Eqs. (2.21) and (2.23), one finds

$$\eta^2 = -\frac{a_3\left(\rho^2 - 3\lambda^2\right)}{2} \ \text{ and } \ \eta^2 = -\frac{a_3\left(3\rho^2 - \lambda^2\right)}{2}, \tag{2.24}$$

It follows directly that

$$\rho^2 = -\lambda^2. \tag{2.25}$$

From Eq. (2.19) and using (2.25), one gets

$$\lambda = \frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}, \tag{2.26}$$

Consequently, we find from Eqs. (2.25) and (2.26) that

$$\rho = \pm i\frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}, \tag{2.27}$$

where i is the imaginary unit. By inserting Eq. (2.26) into Eq. (2.24), one obtains

$$\eta = \sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}, \tag{2.28}$$

Additionally, we find after inserting (2.23) and (2.26) into (2.16) that

$$A = \frac{a_2\left(2a_2^2 - 9a_1a_3\right)}{27a_3^2}. \tag{2.29}$$

By substituting Eqs. (2.23)-(2.28) into Eq. (2.15), one obtains the following solutions:

$$\begin{aligned} u(\xi) \;=\; & \frac{a_2}{3a_3} + \frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}\tanh\left[\sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}\xi\right] \\ & \pm i\frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}\operatorname{sech}\left[\sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}\xi\right], \end{aligned} \tag{2.30}$$

Substituting Eq. (2.30) into Eq. (2.8) gives

$$\begin{aligned} v(\xi) \;=\; & \frac{C_1}{k_1^2} - \frac{k_2k_3}{k_1^2}\left\{\frac{a_2}{3a_3} + \frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}\tanh\left[\sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}\xi\right]\right. \\ & \left.\pm i\frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}\operatorname{sech}\left[\sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}\xi\right]\right\} \\ & -\frac{k_2}{2k_1}\left\{\frac{a_2}{3a_3} + \frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}\tanh\left[\sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}\xi\right]\right. \\ & \left.\pm i\frac{\sqrt{3\left(a_2^2 - 3a_1a_3\right)}}{3a_3}\operatorname{sech}\left[\sqrt{\frac{2\left(a_2^2 - 3a_1a_3\right)}{3a_3}}\xi\right]\right\}^2 \end{aligned} \tag{2.31}$$

Thus, the solutions (2.30) and (2.31) are the complexiton solutions for the (2+1)-dimensional dispersive long wave equations (2.1) and (2.2) which exist provided that $a_3\left(a_2^2 - 3a_1a_3\right) > 0$ as seen from (28). In these solutions, $\xi = k_1x + k_2y + k_3t + \xi_o$ with k_1, k_2, k_3 and ξ_o are arbitrary nonzero constants. It is remarkable that when the variable ξ approaches infinity, the solitary wave solutions (2.30) and (2.31) do not approach zero. However, in the opposite case where

$$a_3\left(a_2^2-3a_1a_3\right)<0, \tag{2.32}$$

the complexiton solutions respectively transforms to complex singular periodic solutions as seen below.

$$\begin{aligned} u(\xi) &= \frac{a_2}{3a_3}+\frac{\sqrt{3\left(a_2^2-3a_1a_3\right)}}{3a_3}\tan\left[\sqrt{\frac{2\left(a_2^2-3a_1a_3\right)}{3a_3}}\xi\right] \\ &\pm i\frac{\sqrt{3\left(a_2^2-3a_1a_3\right)}}{3a_3}\sec\left[\sqrt{\frac{2\left(a_2^2-3a_1a_3\right)}{3a_3}}\xi\right] \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} v\left(\xi\right) &= \frac{C_1}{k_1^2}-\frac{k_2k_3}{k_1^2}\left\{\frac{a_2}{3a_3}+\frac{\sqrt{3\left(a_2^2-3a_1a_3\right)}}{3a_3}\tan\left[\sqrt{\frac{2\left(a_2^2-3a_1a_3\right)}{3a_3}}\xi\right]\right. \\ &\left.\pm i\frac{\sqrt{3\left(a_2^2-3a_1a_3\right)}}{3a_3}\sec\left[\sqrt{\frac{2\left(a_2^2-3a_1a_3\right)}{3a_3}}\xi\right]\right\} \\ &-\frac{k_2}{2k_1}\left\{\frac{a_2}{3a_3}+\frac{\sqrt{3\left(a_2^2-3a_1a_3\right)}}{3a_3}\tan\left[\sqrt{\frac{2\left(a_2^2-3a_1a_3\right)}{3a_3}}\xi\right]\right. \\ &\left.\pm i\frac{\sqrt{3\left(a_2^2-3a_1a_3\right)}}{3a_3}\sec\left[\sqrt{\frac{2\left(a_2^2-3a_1a_3\right)}{3a_3}}\xi\right]\right\}^2 \end{aligned} \tag{2.34}$$

2.2. **Power Law Nonlinearity.** In this section the two-dimensional generalization of the regular dispersive long wave equations with power law nonlinearity and perturbation terms will be studied. A problem of our interest consists in solving the following family of the (2+1)-dimensional dispersive long wave equations:

$$u_{yt}+av_{xx}+b\left(u^{2n}\right)_{xy}=ku+duv, \tag{2.35}$$

$$v_t+c\left(uv+u+u_{xy}\right)_x=0, \tag{2.36}$$

where a, b, c, d and k are nonzero real constants, while the parameter n indicates the power law nonlinearity parameter. On setting $a=c=n=1$, $k=d=0$ and $b=1/2$, the system of coupled equations (2.35) and (2.36) reduces to the model equations (2.1)-(2.2). Here in (2.35), the first term is the evolution term, while the second and third terms respectively represent the dispersion and the power law nonlinearity terms. The first perturbation term on the right-hand side of Eq. (2.35) is the linear damping term, while the second term represents the nonlinear term.

In order to obtain the bright soliton solutions to (2.35) and (2.36), the solitary wave ansatz is assumed as [3-7]

$$u(x,y,t)=\frac{A_1}{\cosh^{p_1}\tau} \tag{2.37}$$

and

$$v(x,y,t)=\frac{A_2}{\cosh^{p_2}\tau} \tag{2.38}$$

where

$$\tau=B_1x+B_2y-vt, \tag{2.39}$$

Here, in (2.37)-(2.39), A_1 and A_2 are the amplitudes of the u-soliton and v-soliton respectively, v represents the velocity of the solitons, while B_1 and B_2 are the inverse widths in the x-direction and the y-direction, respectively. The exponents p_1 and p_2 are unknown at this point and their values will be determined as a function of n. Thus from (2.37) and (2.38), we obtain

$$u_{yt}=-\frac{p_1^2A_1B_2v}{\cosh^{p_1}\tau}+\frac{p_1(p_1+1)A_1B_2v}{\cosh^{p_1+2}\tau} \tag{2.40}$$

$$v_{xx}=\frac{p_2^2A_2B_1^2}{\cosh^{p_2}\tau}-\frac{p_2(p_2+1)A_2B_1^2}{\cosh^{p_2+2}\tau} \tag{2.41}$$

$$\left(u^{2n}\right)_{xy}=\frac{4n^2p_1^2A_1^{2n}B_1B_2}{\cosh^{2np_1}\tau}-\frac{2np_1(2np_1+1)A_1^{2n}B_1B_2}{\cosh^{2(np_1+1)}\tau} \tag{2.42}$$

$$v_t=\frac{p_2vA_2\tanh\tau}{\cosh^{p_2}\tau} \tag{2.43}$$

$$\begin{aligned}(uv+u+u_{xy})_x \;=\;& -\frac{A_1A_2\,(p_1+p_2)\,B_1\tanh\tau}{\cosh^{p_1+p_2}\tau}-\frac{p_1A_1B_1\,(p_1^2B_1B_2+1)\tanh\tau}{\cosh^{p_1}\tau}\\ &+\frac{p_1(p_1+1)(p_1+2)A_1B_1^2B_2\tanh\tau}{\cosh^{p_1+2}\tau}\end{aligned} \tag{2.44}$$

Substituting (2.40)-(2.44) into (2.35) and (2.36) respectively yields

$$\begin{aligned}&-\frac{p_1^2A_1B_2v}{\cosh^{p_1}\tau}+\frac{p_1(p_1+1)A_1B_2v}{\cosh^{p_1+2}\tau}+\frac{ap_2^2A_2B_1^2}{\cosh^{p_2}\tau}-\frac{ap_2(p_2+1)A_2B_1^2}{\cosh^{p_2+2}\tau}\\ &+\frac{4bn^2p_1^2A_1^{2n}B_1B_2}{\cosh^{2np_1}\tau}-\frac{2bnp_1(2np_1+1)A_1^{2n}B_1B_2}{\cosh^{2(np_1+1)}\tau}=\frac{kA_1}{\cosh^{p_1}\tau}+\frac{dA_1A_2}{\cosh^{p_1+p_2}\tau},\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}&\frac{p_2vA_2\tanh\tau}{\cosh^{p_2}\tau}-\frac{cA_1A_2\,(p_1+p_2)\,B_1\tanh\tau}{\cosh^{p_1+p_2}\tau}-\frac{cp_1A_1B_1\,(p_1^2B_1B_2+1)\tanh\tau}{\cosh^{p_1}\tau}\\ &+\frac{cp_1(p_1+1)(p_1+2)A_1B_1^2B_2\tanh\tau}{\cosh^{p_1+2}\tau}=0,\end{aligned} \tag{2.46}$$

From (2.45), equating the exponents p_1+2 and p_1+p_2 gives

$$p_1+2=p_1+p_2 \tag{2.47}$$

so that

$$p_2=2 \tag{2.48}$$

Now from (2.46), equating the exponents $2\,(np_1+1)$ and p_2+2 gives

$$2\,(np_1+1)=p_2+2 \tag{2.49}$$

and therefore

$$p_1=\frac{1}{n} \tag{2.50}$$

which is also obtained by equating the exponents $2np_1$ and p_2 in (2.45). Again from (2.45), setting the coefficients of the linearly independent functions $1/\cosh^{p_1+j}\tau$ to zero, where $j=0,2$, gives

$$v=-\frac{n^2k}{B_2} \tag{2.51}$$

$$v=\frac{n^2dA_2}{(n+1)B_2} \tag{2.52}$$

and

$$aA_2B_1+bA_1^{2n}B_2=0 \tag{2.53}$$

Equating the two values of v from (2.53) and (2.54) leads to the following expression for the amplitude A_2:

$$A_2=-\frac{k(n+1)}{d} \tag{2.54}$$

Next, from (2.46), setting the coefficients of the linearly independent functions $\tanh\tau/\cosh^{p_1+j}\tau$ to zero, where $j=0,2$, gives

$$v=\frac{cA_1B_1\left(B_1B_2+n^2\right)}{2A_2n^3} \tag{2.55}$$

and

$$A_2=\frac{(n+1)B_1B_2}{n^2} \tag{2.56}$$

Now, substituting (2.56) into (2.53) gives the following expression for the amplitude A_1:

$$A_1=\left[-\frac{a(n+1)B_1^2}{bn^2}\right]^{\frac{1}{2n}} \tag{2.57}$$

which shows that the u-soliton will exists for

$$ab<0 \tag{2.58}$$

if n is an even integer. However, if n is an odd integer there is no such restriction but the soliton will be pointing downwards.

Hence, finally, the 1-soliton solutions of the the (2+1) dimensional dispersive long wave equations are given by

$$u(x,y,t)=\frac{A_1}{\cosh^{\frac{1}{n}}[B_1x+B_2y-vt]} \tag{2.59}$$

and

$$v(x,y,t)=\frac{A_2}{\cosh^2[B_1x+B_2y-vt]} \tag{2.60}$$

where the amplitudes A_1 and A_2 are given by the expressions (2.57) and (2.56) respectively, the velocity of the solitons is given by (2.51) or (2.52) or (2.55), while the inverse widths B_1 and B_2 are connected by (2.53).

3. CONCLUSIONS

This paper studied the (2+1)-dimensional dispersive long wave equations. The traveling wave hypothesis approach, coupled with ansatz method was used to obtain the complexiton solution of this equation. Subsequently, this equation was generalized to power law nonlinearity where the pure ansatz approach was applied to retrieve solitary wave solutions together with the technical conditions, also known as constraints, for the existence of such solitary waves. These results are going to be extremely useful in carrying out further studies in this area. For example, later on the cnoidal wave solutions will be obtained along with the numerical analysis of these solitary waves. The time-dependent coefficients, both deterministic and stochastic, will be later added and the solitary wave solutions will be obtained in those cases as well. Those results will be reported in future.

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