Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **4**(1)(2015), 87-94

An Analysis on The Lotka-Volterra Food Chain Model: Stability

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Abstract. The food chain refers to a natural system by which energy is transmitted from one organism to another. In fact, a food chain consists of producers, consumers and decomposition. Presence of complex food web increase the stability of the ecosystem. Classical food chain theory arises from Lotka-Volterra model. In the present paper, the dynamics behavior of three level food chain is studied. A system of 3 nonlinear ODEs for interaction modeling of three-species food chain where intraspcies competition exists indeed is studied. The first population is the prey for the second which is prey for the third one. It is clear that it is the top of food pyramid. The techniques of linearization and first integral are employed.

Keywords: Lotka-Volterra Model, Food Chain, Competition, Linearization, Predator-Prey.

2000 Mathematics subject classification: 34A34, 78A70, 93B18, 92D25.

1. Introduction

In the 1920's, the Italian mathematician Vito-Volterra[1] proposed differential equation model to describe the population dynamics of two interaction species a predator and its prey. The same equations were studied by Alfred-Lotka to describe a hypothetical chemical reaction independently [2].

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Received: 09 July 2014
Revised: 17 October 2014
Accepted: 29 October 2014

The ecosystem that we wish to model is a nonlinear three-species food chain where the lowest-level prey x is preyed upon a mid-level species y, which in turn is preyed upon the top level predator z. The three species food chain model is an extension of the general two species model given by Lotka and Volterra. Some related topics may be found in [3,4,5,6] and references therein. Having existed competition within each species and assumption that interaction terms be constant coefficients, the system (1.1) will be obtained.

$$\begin{cases} x_1' = x_1(r_1 - a_{11}x_1 - a_{12}x_2) \\ x_2' = x_2(-r_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3) \\ x_3' = x_3(-r_3 + a_{32}x_2 - a_{33}x_3). \end{cases} \tag{1.1}$$

Here all parameters r_i and a_{ij} are real positive constants [7]. Having considered $i, j = 1, 2, 3$; the parameter r_i represents the natural growth rate of the i^{th} population, and coefficients a_{ij} describes the effect of j^{th} upon the i^{th} population, which is positive provided it enhances and negative if it inhibits the growth.

The functions $x(t), y(t)$ and $z(t)$ represent the population densities of first, second and third species respectively. Let us restrict our attention to nonnegative octant: $\{(x, y, z) \in R^3 | x, y, z \in R^+\}$.

2. Mathematical Model of Three Species Food Chain

Now, for the sake of convenience, we denote system (1.1) by the following system:

$$\begin{cases} x' = x(a - bx - cy) \\ y' = y(-d + ex - fy - gz) \\ z' = z(-h + iy - jz). \end{cases} \tag{2.1}$$

Following model is studied in [8]:

$$\begin{cases} x' = x(a - by) \\ y' = y(-c + dx - ez) \\ z' = z(-h + iy). \end{cases} \tag{2.2}$$

We extend the same study to (2.1). In the absence of the top predator (i.e. $z = 0$ in (2.1)), the model simply reduces to the Lotka-Volterra predator-prey model with interaspecies competition [9]:

$$\begin{cases} x' = x(a - bx - cy) \\ y' = y(-d + ex - fy). \end{cases}$$

In the analysis of system of differential equations

$$\begin{cases} x_i' = f_i(t, x_1, x_2, x_3, ..., x_n) \\ x_i(a) = a_i. \end{cases} \tag{2.3}$$

where $t \in [a,b]$, $x_i \in C^1[a,b]$ and $f_i \in C[[a,b],R^n]$; for $i = 1,2,3,...,n$. Indeed, it is system of nonlinear differential equations which is an initial value problem.

It is useful to consider solution that don't have change with time, that is $x_i^{'} = 0$ for all $i = 1,2,3,...,n$. Such solution are called equilibria. An equilibrium is called stable provided solution starting close enough to the equilibrium remains close to this point and it is called asymptotically stable provided solution starting close enough to equilibrium, tend to that equilibrium.

If the system of differential equations (2.3) can be linearized, that is, provided for all $i = 1,2,3,...,n$; f_i has continuous partial in x_i for all $i = 1,2,3,...,n$;

Then the stability of an equilibrium often can be determined by the stability of this point in the following associated linearized system:

$$X^{'} = JX. \tag{2.4}$$

where $X = (x_1, x_2, x_3, ..., x_n) \in R^n$, $J = (\frac{\partial f_i}{\partial x_j})$ is Jacobian matrix and all partial derivatives are evaluated at $X_0 = (x_1(0), x_2(0), x_3(0), , ..., x_n(0))$, for $i,j = 1,2,3,...,n$.

The behavior of the linearized system at X_0 is determined by the eigenvalues of the Jacobian matrix, evaluated at X_0.

A surface S is invariant with respected to a system of differential equations provided every solution that starts in S doesn't escape S. The property of invariant coordinates planes matches biological consideration since some species is exist, it will not reappear. The invariant surface S is usually given as the level set of a function $G(x,y,z)$ which is known as first integral.

3. Analysis of The Model

We first show that each coordinate plane is invariant with respected to the system (2.1). We need the following theorem appears in most advanced text on differential equations [10].

Theorem 3.1. *Let S be a smooth surface without boundary in R^3 and*

$$\begin{cases} x^{'} = f(x,y,z) \\ y^{'} = g(x,y,z) \\ z^{'} = h(x,y,z). \end{cases} \tag{3.1}$$

wher f,g and h are continuously differentiable. Suppose that $\boldsymbol{n}$ is a normal vector to the surface S at the point (x,y,z) and for all $(x,y,z) \in S$, we have that $\boldsymbol{n}. < x,y,z >= 0$.

Then S is invariant for system (3.1).

Now consider system (2.1). Let S be the plane $y = 0$, note that the vector $< 0, 1, 0 >$ is always normal to S, and that at the point $(x, 0, z)$ of S we have

$$< x^{'}, y^{'}, z^{'} >=< x(a - bx), 0, z(-h - jz) >$$
$$\Rightarrow< 0, 1, 0 > . < x(a - bx), 0, z(-h - jz) >= 0.$$

Having considered S be the plane $x = 0$,

we see that vector $< 1, 0, 0 >$ is always normal to S and the point $(x, 0, z)$ of S. We obtain

$$< x^{'}, y^{'}, z^{'} >=< 0, y(-d - fy - gz), z(-h + iy - jz) >$$
$$\Rightarrow< 1, 0, 0 > . < 0, y(-d - fy - gz), z(-h + iy - jz) >= 0.$$

Finally, let S be the plane $z = 0$, vector $< 0, 0, 1 >$ is always normal to S and the point $(x, 0, z)$ of S. It is easy to see that

$$< x^{'}, y^{'}, z^{'} >=< x(a - bx - cy), y(-d + ex - fy), 0 >$$
$$\Rightarrow< 0, 0, 1 > . < x(a - bx - cy), y(-d + ex - fy), 0 >= 0.$$

Now, we solve each of the three corresponding planar (two variable) system in the respective coordinate planes. The case of $z = 0$ has been analyzed in [11].

Theorem 3.2. *For system (2.2) with making assumption that $y = 0$ the following statements hold:*

i) $x(t) = \frac{a}{b}$ *is asymptotically stable.*
ii) $z(t) \to 0$ *as* $t \to \infty$.
iii) $z = \frac{h}{(\frac{a-bx}{kx})^{(\frac{h}{a})} - j}$ *where k is constant real number.*

Proof. ***i***)

For a trajectory starting on the plane $y = 0$, system (2.2) reduces to

$$\begin{cases} x^{'} = x(a - bx) \\ y^{'} = 0 \\ z^{'} = z(-h - jz). \end{cases} \tag{3.2}$$

First equation in the system (2.2) is logistic equation.

$$x^{'} = x(a - bx). \tag{3.3}$$

Having assumed $k = b$ and $M = \frac{a}{b}$, one can see easily the above equation is equivalent with the following equation:

$$x^{'} = kx(M - x). \tag{3.4}$$

The solutions of the above equation are given by

$$x(t) = \frac{Mc_1}{c_1 + (M - c_1)e^{-kMt}}. \tag{3.5}$$

where c_1 is constant integral. Provided making initial value population $x(0) = x_0$ for equation (3.4), we obtain

$$x(t) = \frac{Mx_0}{x_0 + (M - x_0)e^{-kMt}}. \tag{3.6}$$

Equation (3.6) implies that provided initial value population x be less than carrying capacity of ecosystem (M), the variable x grows exponentially to parameter M as $t \to \infty$. Then $x(t) = M$ is asymptotically stable for equation (3.4).

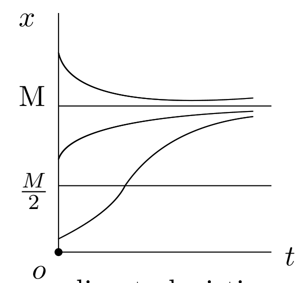

Fig 1: Solution curves, corresponding to logistic equation(3.4)

ii)

Second equation system (2.2) is a extinction equation.

$$z^{'} = z(-h - jz). \tag{3.7}$$

Being negative the right side of equation (3.7) implies that $z(t) \to 0$ as $t \to \infty$.

iii)

Notwithstanding the unbounded growth of x, this behavior fits with what we expected biological in the absence of mid-level species y. That is, x is free from predation and z is without a source of food. The trajectory in xz-plane can be directly computed from the separable equations.

$$\begin{aligned} \tfrac{dz}{dx} = \tfrac{dz}{dt}\tfrac{dt}{dx} &= \tfrac{z(-h-jz)}{x(a-bx)}. \\ \Rightarrow \tfrac{-1}{z(h+jz)}dz &= \tfrac{1}{x(a-bx)}dx. \\ \Rightarrow \int \tfrac{-1}{z(h+jz)}dz &= \int \tfrac{1}{x(a-bx)}dx + c_2. \end{aligned}$$

Now, we denote the left and right sides of recent equation by I_1 and I_2 respectively. Using the method of fractal partial, one can find the last integrals as following form:

$$\begin{cases} I_1 = \frac{1}{h}\int \frac{dz}{z} - \frac{j}{h}\int \frac{dz}{h+jz} = \frac{1}{h}\ln(\frac{z}{h+jz}) \\ I_2 = \frac{1}{a}\int \frac{dx}{x} - \frac{b}{a}\int \frac{dx}{a-bx} = \frac{1}{a}\ln(\frac{x}{x-bx}). \end{cases}$$

$$\Rightarrow \frac{1}{h}\ln(\frac{z}{h+jz}) = \frac{1}{a}\ln(\frac{kx}{x-bx}).$$

where k is the integration constant.

After simplification, we obtain:

$$z = \frac{h(\frac{kx}{a-bx})^{(\frac{h}{a})}}{1 - j(\frac{kx}{a-bx})^{(\frac{h}{a})}}.$$

where a, b, k, h and j are constant real numbers.

As regarding $\frac{kx}{a-bx} \neq 0$, we obtain

$$z = \frac{h}{(\frac{a-bx}{kx})^{(\frac{h}{a})} - j}. \tag{3.8}$$

This concludes the proof of the theorem. □

For solution starting in the plane $x = 0$, one see that system (2.1) reduces to

$$\begin{cases} y^{'} = y(-d - fy - gz) \\ z^{'} = z(-h + iy - jz). \end{cases} \tag{3.9}$$

Which is system of nonlinear differential equations describes a special class of Lotka-Volterra predator- prey model. Because of denoting

$$\begin{cases} M(y,z) = y(-d - fy - gz) \\ N(y,z) = z(-h + iy - jz). \end{cases}$$

The Kolmogorov equations (3.9) will be obtained:

$$\begin{cases} y^{'} = M(y,z) \\ z^{'} = N(y,z). \end{cases} \tag{3.10}$$

Having given attention to systems (3.9) and (3.10), we see that

$$\begin{cases} \frac{\partial M}{\partial z} = -g < 0 \\ \frac{\partial N}{\partial y} = i > 0. \end{cases} \tag{3.11}$$

And hence, the above inequalities prove that system (3.9) is predator-prey model.

The first equation in (3.9) implies that $y(t) \to 0$ as $t \to 0$ and the second equation in (3.9) implies, in absence y , $z(t) \to 0$ as $t \to \infty$ and it enhances as y increases as $t \to \infty$.

Theorem 3.3. *For system (2.1) with making assumption that* $x = 0$ *the following statements hold:*

i) *Origin is stable point.*

ii) *There is no any equilibrium point in* R^2_+.

Proof. ***i***)

Consider $x = 0$. Then the system (2.1) leads into the system (3.9). Having linearized system (3.9) at origin , one can find the solution behavior around this equilibrium point.

It's Jacobian matrix is given by

$$J = \begin{bmatrix} d - 2fy - gz & -gz \\ iz & -h + iy - 2jz \end{bmatrix}. \tag{3.12}$$

After evaluating the Jacobian matrix J at origin and denoting with J_0, we obtain

$$J_0 = \begin{bmatrix} -d & 0 \\ 0 & -h \end{bmatrix}. \tag{3.13}$$

Simply examine eigenvalues of J_0 gives us information about the dynamics near the equilibrium points of original system. Because of being negative both of it's eigenvalues ($\lambda_1 = -d$ and $\lambda_2 = -h$), it is easy to conduct that origin is stable point for system(3.9).

ii)

Let $A = (\overline{y}, \overline{z})$ be equilibrium point in R^2_+ for system (3.8). Hence the point A should satisfy in the following equations system:

$$\begin{cases} fy + gz = -d \\ iy - jz = h. \end{cases}$$

The solution of recent equations system is given by

$$\begin{cases} \overline{y} = \frac{dj - hg}{-fj - gi} \\ \\ \overline{z} = \frac{fh + id}{-fj - gi}. \end{cases}$$

On one hand $\overline{z} = -\frac{fh+id}{fg+gi}$ implies that $\overline{z} < 0$, on another hand $\overline{z} > 0$ because $A \in R^2_+$.

This leads to a contradiction and so, the proof of theorem is done. □

4. Conclusion

By using differential equations, scientists are able to analyze almost all of the ecological models. In the above models, the equilibrium points explain the equilibrium populations. It is obvious that the stability and asymptotically stability are main concepts in this area. In the present work, we focused on food chain models having three species. Indeed, by adding some conditions on existing parameters we determined the stability and asymptotically stability which are analyzed in theorems 3.2 and 3.3.

Acknowledgment

Author would like to express his thanks to department of mathematics university of Neyshabur, Iran.

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