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A UNIQUE CONTINUOUS SOLUTION FOR THE BAGLEY-TORVIK EQUATION

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ABSTRACT. In this paper the Bagley-Torvik equation as a prototype fractional differential equation with two derivatives is investigated by means of homotopy perturbation method. The results reveal that the present method is very effective and accurate.

Keywords: Caputo fractional derivative; homotopy perturbation technique; Bagley-Torvik fractional differential equations.

1. INTRODUCTION

The application of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many famous mathematicians. For an interesting history and more scientific applications of fractional calculus, see [1].

In this paper, we consider the following of initial value problem

$$\begin{aligned} MD^2_{*t}\xi(t) &+ 2S\sqrt{\mu\rho}\, D^{3/2}_{*t}\xi(t) + K\xi(t) = \vartheta(t),\ 0 \le t \le T, \\ \xi(0) &= 1, \quad \xi'(0) = 1. \end{aligned} \tag{1.1}$$

This problem described the motion of a large plate of the surface S and mass M in a Newotonian fluid viscosity μ and density ρ. The plate is hanging on a massless spring of stiffness K. The function $f(t)$ represents the loading force. The Eq (1.1) is called Bagley-Torvik equation [2].

finding approximate or exact solutions of fractional differential equations is an important task. Except in a limited number of these equations, we have difficulty in finding their analytical solutions. During the last decade, a promising analytic technique is called the homotopy perturbation technique [3], has successfully been applied to solve many types of linear and nonlinear functional equations.

In this paper a unique continuous solution for the Bagley-Torvik equation is obtained by homotopy perturbation technique. Some theorems about uniqueness and existence are presented.

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2. PRELIMINARIES AND NOTATION

Definition 2.1. A real function $f(x),\ x > 0$, is said to be in the space $C_\alpha, \alpha \in \Re$ if there exists a real number $p\,(> \alpha)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$. Clearly, $C_\alpha \subset C_\beta$ if $\beta \le \alpha$.

Definition 2.2. A function $f(x),\ x > 0$, is said to be in the space $C_\alpha^m, m \in \mathcal{N} \bigcup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 2.3. The (left sided) Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in C_\alpha$, $\alpha \ge -1$, is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau,\ \alpha > 0,\ t > 0, \tag{2.1}$$

$$I_t^0 f(t) = f(t), \tag{2.2}$$

$$I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x,s)}{(t-s)^{1-\alpha}} ds,\ \alpha > 0,\ t > 0, \tag{2.3}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.4. The (left sided) Riemann-Liouville fractional derivative of f, $f \in C_{-1}^m$, $m \in \mathcal{N} \bigcup \{0\}$, of order α is defined as

$$D_t^\alpha f(t) = \frac{d^m}{dt^m} I_t^{m-\alpha} f(t),\ m-1 < \alpha \le m,\ m \in \mathcal{N}. \tag{2.4}$$

Definition 2.5. The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in \mathcal{N} \bigcup \{0\}$, is defined as

$$D_{*t}^\alpha f(t) = \begin{cases} \left[I_t^{m-\alpha} f^{(m)}(t) \right] & m-1 < \alpha < m,\ m \in \mathcal{N}, \\ \\ \dfrac{d^m}{dt^m} f(t) & \alpha = m, \end{cases} \tag{2.5}$$

$$D_{*t}^\alpha f(x,t) = I_t^{m-\alpha} \frac{\partial^m f(x,t)}{\partial t^m},\ m-1 < \alpha < m, \tag{2.6}$$

$$D_{*t}^\alpha D^m f(t) = D_{*t}^{\alpha+m} f(t),\ m = 0, 1, \cdots,\ n-1 < \alpha < n, \tag{2.7}$$

$$I^\alpha D_{*t}^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!},\ m-1 < \alpha \le m,\ m \in \mathcal{N}. \tag{2.8}$$

Theorem 2.1 If f is continuous, then $I^\alpha D^\alpha f = f, 0 < \alpha < 1$, and, if f and f' are continuous, then $D^\alpha f$ exists and, is integrable for $0 < \alpha < 1$.

3. THE BAGLEY-TORVIK EQUATION AS A SYSTEM OF DIFFERENTIAL EQUATION OF FRACTIONAL ORDER

A Bagley-Torvik equation can be presented in the following form[4, 5, 6]

$$\begin{aligned} MD_{*t}^{2}\xi(t) &+ D_{*t}^{3/2}\xi(t)+\xi(t)=t+1,\ 0\le t\le T, \\ \xi(0) &= 1,\ \xi'(0)=1. \end{aligned} \tag{3.1}$$

where $D_{*t}^{\alpha_i}$ is used to represent the Caputo fractional derivative of order α_i.

Theorem 3.1. The Eq. (3.1) is equivalent to the system of equations in the following form

$$\begin{aligned} D_{*t}^{3/2}\xi_1(t) &= \xi_2(t), \\ D_{*t}^{1/2}\xi_2(t) &= -\xi_2(t)-\xi_1(t)+1+t, \\ \xi_1(0) &= \xi_1'(0)=1, \\ \xi_2(0) &= 0. \end{aligned} \tag{3.2}$$

Proof. See [6].

4. STANDARD HPM

Here, for convenience of the reader, we will present a review of the standard HPM [3]. To achieve our goal, we consider the nonlinear differential equation

$$L(u)+N(u)=f(r),\ r\in\Omega, \tag{4.1}$$

with boundary conditions

$$B(u,\frac{\partial u}{\partial n})=0,\ r\in\Gamma, \tag{4.2}$$

where L is a linear operator, while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function.

The He's homotopy perturbation method defines the homotopy $v(r,p):\Omega\times[0,1]\rightarrow\Re$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)]=0, \tag{4.3}$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0, \tag{4.4}$$

where $r\in\Omega$ and $p\in[0,1]$ is an impeding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eqs. (4.3) and (4.4), we have

$$H(v,0) = L(v)-L(u_0)=0, \tag{4.5}$$

$$H(v,1) = L(v)+N(v)-f(r)=0, \tag{4.6}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this called deformation, $L(v)-L(u_0)$ and $L(v)+N(v)-f(r)$ are homotopic. The basic assumption is that the solution of Eqs. (4.3) and (4.4) can be expressed as a power series in p

$$v=v_0+pv_1+p^2v_2+\cdots\ . \tag{4.7}$$

The approximate solution of Eq. (4.1), therefore, can be readily obtained

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots\ . \tag{4.8}$$

5. APPLICATIONS

Consider the following initial value problem for the inhomogeneous Bagley-Torvik equation [2]

$$\begin{aligned} MD^2_{*t}\xi(t) &+ 2S\sqrt{\mu\rho}\, D^{3/2}_{*t}\xi(t) + K\xi(t) = \vartheta(t),\ 0 \le t \le T, \\ \xi(0) &= 1, \quad \xi'(0) = 1. \end{aligned} \tag{5.1}$$

In order to make comparison with the numerical solution of [2] we choose $M = 2S\sqrt{\mu\rho} = K = 1, T = 5$ and $\vartheta(t) = K(t+1)$. By the same manipulation as Section 5 we set

$$\begin{aligned} D^{3/2}_{*t}\xi_1(t) &= \xi_2(t), \\ D^{1/2}_{*t}\xi_2(t) &= -\xi_2(t) - \xi_1(t) + 1 + t, \\ \xi_1(0) &= \xi_1'(0) = 1, \\ \xi_2(0) &= 0. \end{aligned} \tag{5.2}$$

Theorem 5.1. In equation $D^{\alpha_j}_{*}\xi(t) = f(t, \xi(t), D^{\alpha_i}_{*}\xi(t))$ if f is a continuous function which satisfy in a uniform Lipschitz condition, then this equation subject to the given initial conditions has a unique continuous solution on interval $[0, T]$.
Proof. See [7, 8].

Since ξ_i is continuous on the compact set $[0, T]$, it is uniformly continuous there. Whence absolute maximum theorem implies that

$$\left|\xi_i(t) - \xi_i(\tilde{t})\right| \le |\xi_i(t)| + \left|\xi_i(\tilde{t})\right| \le \delta_i + \tilde{\delta}_i = \delta,\ t, \tilde{t} \in [0, T]. \tag{5.3}$$

Thus, by the Archimedean property of real number we will have

$$\exists\, L > 0,\ s.t.\ \delta \le L\left|t - \tilde{t}\right|, \tag{5.4}$$

Consequently, we have

$$\begin{aligned} \xi_{1,0}(t) &= 1 + t, \\ \xi_{1,m+1}(t) &= I^{1.5}\xi_{2,m},\ m = 0, 1, \cdots, \\ \xi_{2,0}(t) &= 0, \\ \xi_{2,1}(t) &= 0, \\ \xi_{2,m+1}(t) &= -I^{0.5}(\xi_{1,m}(t) + \xi_{2,m}(t)),\ m = 1, 2, \cdots. \end{aligned} \tag{5.5}$$

Thus, we obtain

$$\begin{cases} \xi_{1,m+1}(t) = 0, \\ \xi_{2,m+1}(t) = 0,\ m = 0, 1, \cdots. \end{cases} \tag{5.6}$$

Hence, $\xi_1(t) = 1 + t$ and $\xi_2(t) = 0$. So, $\xi(t) = 1 + t$ is the solution of Eq. (5.1).

Remark 5.2. We have

$$\begin{cases} D^2_{*t}(t+1) = 0, \\ \\ D^{1.5}_{*t}(t+1) = I^{0.5}_t \frac{d^2}{dt^2}(t+1) = I^{0.5}_t(0) = 0. \end{cases} \tag{5.7}$$

Hence, it is easily verified that $t+1$ is the exact solution of Eq. (5.1).
Table 5.1 shows the resulting error at $t=5$ obtained by numerical method in [5] and compared with the solution obtained by the proposed scheme.

Table 5.1. The resulting errors

Error at $t=5$ by proposed approach	Error at $t=5$ by [5] (step size)
0	-0.15131473519232 (0.5000)
0	-0.04684102179946 (0.2500)
0	-0.01602947553912 (0.1250)
0	-0.00562770408881 (0.0625)

6. DISCUSSION AND CONCLUSION

In this paper, HPM has been successfully applied to compute the approximate solution of the Bagley-Torvik equation with coordinate derivatives of non-integer order α. These derivatives were defined in the form of Caputo sense. The convergence and stability of HPM solution, as applied to this type of equations have been thoroughly investigated. Results show that, our approach provides the solutions in terms of convergent series with easily computable components in a direct way, without using linearization, perturbation or restrictive assumption.

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