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On a $p(x)$-Kirchhoff equation via variational methods

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Abstract. This paper is concerned with the existence of two nontrivial weak solutions for a $p(x)$-Kirchhoff type problem of the following form

$$\begin{cases} -M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Delta_{p(x)}u = \lambda(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle and the theory of the variable exponent Sobolev spaces.

Keywords: Generalized Lebesgue-Sobolev spaces, Nonlocal condition, Mountain pass theorem, Ekeland's variational principle.

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1. Introduction

In this paper, we study the following problem

$$\begin{cases} -M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Delta_{p(x)}u = \lambda(x)|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x)$, $q(x) \in C(\overline{\Omega})$, $\inf_{\overline{\Omega}} p(x) > 1$ and $\inf_{\overline{\Omega}} q(x) > 1$, $M(t)$ is a continuous real-valued function.

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The operator $-\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). An essential difference between them is that the p-Laplacian operator is $(p-1)$-homogeneous, that is, $\triangle_p(\lambda u) = \lambda^{p-1}\triangle_p u$ for every $\lambda > 0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [27], electrorheological fluids [1] or image restoration [5].

Problem (1.1) is called nonlocal because of the presence of the term M, which implies that the equation in (1.1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particulary interesting. Nonlocal differential equations are also called Kirichhoff-type equations because Kirichhoff [23] has investigated an equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of Eq. (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise identity. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [24] has proposed an abstract framework for the Kirchhoff-type equations. After the work of Lions [24], various equations of Kirchhoff-type have been studied extensively, see e.g. [3]-[11]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [6, 7, 10, 11, 25], [19]-[22]) and $p(x)$-Laplacian (see [8, 9, 18]).

2. Notations and preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [12]-[15].

Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\}$$

$$p^+ = \max\{p(x);\ x \in \overline{\Omega}\}, \quad p^- = \min\{p(x);\ x \in \overline{\Omega}\};$$

$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)} dx \le 1\Big\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the natural norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u(x)|_{L^{p(x)}(\Omega)} + |\nabla u(x)|_{L^{p(x)}(\Omega)},$$

or equivalently

$$\|u\| = \inf\Big\{\mu > 0;\ \int_\Omega \frac{|\nabla u(x)|^{p(x)} + |u|^{p(x)}}{\mu^{p(x)}} dx \le 1\Big\}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. For $u \in W_0^{1,p(x)}(\Omega)$, we define an equivalent norm

$$\|u\| = |\nabla u(x)|_{L^{p(x)}(\Omega)},$$

since Poincaré inequality holds, i.e., there exists a positive constant C such that

$$|u|_{p(x)} \le C|\nabla u(x)|_{p(x)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$, see [17].

Proposition 2.1 (See [13, 15]). *The space $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 (See [13, 15]). **(i)** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}$$

(ii) *If $p_1(x),\ p_2(x) \in C_+\overline{\Omega},\ p_1(x) \le p_2(x),\ \forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.3 (See [16]). *Set* $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}dx$, *then for* $u,\ u_k \in W^{1,p(x)}(\Omega)$; *we have*

(1)$\|u\| < 1(respectively = 1; > 1) \Longleftrightarrow \rho(u) < 1(respectively = 1; > 1)$;

(2) $for\ u \neq 0,\ \|u\| = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(3) $if\ \|u\| > 1, then\ \ \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(4) $if\ \|u\| < 1, then\ \ \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(5)$\|u_k\| \to 0(respectively \to \infty) \Longleftrightarrow \rho(u_k) \to 0(respectively \to \infty)$.

Let us define, for every $x \in \Omega$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N. \end{cases}$$

Proposition 2.4 (See [15]). *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$, *then there is a continuous (compact) embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In this paper, we denote by $X = W_0^{1,p(x)}(\Omega)$; $X^* = (W_0^{1,p(x)}(\Omega))^*$, the dual space and $\langle .,. \rangle$, the dual pair.

Lemma 2.5 (See [17]). *Denote*

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad \forall u \in X,$$

then $J(u) \in C^1(X, R)$ *and the derivative operator* J' *of* J *is*

$$\langle J'(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall u, v \in X,$$

and we have

(1) J *is a convex functional* ,

(2) $J' : X \to X^*$ *is a bounded homeomorphism and strictly monotone operator,*

(3)J' *is a mapping of type* (S_+), *namely* $u_n \rightharpoonup u$ *and* $\limsup_{n\to+\infty} J'(u_n)(u_n - u) \leq 0$, *implies* $u_n \to u$.

Hereafter, $\lambda(x)$, $q(x)$ and $M(t)$ are always supposed to verify

(**M1**) there exists a positive constant m_0 such that $M(t) \geq m_0$,

(**M2**) there exists $\mu \in (0,1)$ such that $\widehat{M}(t) \geq (1-\mu)M(t)t$,

(**Λ1**) $\lambda \in L^\infty(\Omega)$,

($\mathbf{\Lambda 2}$) there exists an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $\lambda(x) = 0$ for $x \in \overline{B_R(x_0) \backslash B_r(x_0)}$ while $\lambda(x) > 0$ for $x \in \Omega \backslash \overline{B_R(x_0) \setminus B_r(x_0)}$,

($\mathbf{Q1}$) $q \in C_+(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for any $x \in \overline{\Omega}$,

($\mathbf{Q2}$) either $\max_{\overline{B_r(x_0)}} q < p^- < \frac{p^-}{1-\mu} < p^+ < \frac{p^+}{1-\mu}$
$< \min_{\overline{\Omega \backslash B_R(x_0)}} q$,
or $\max_{\overline{\Omega \backslash B_R(x_0)}} q < p^- < \frac{p^-}{1-\mu} < p^+ < \frac{p^+}{1-\mu} < \min_{\overline{B_r(x_0)}} q$.

The Euler-Lagrange functional associated to (1.1) is given by

$$I(u) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \Big) - \int_\Omega \frac{\lambda(x)}{q(x)} |u|^{q(x)} dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau) d\tau$. It is easy to verify that $I \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous with the derivative given by

$$\langle I'(u), v \rangle = M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \Big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v) dx \\ - \int_\Omega \lambda(x) |u|^{q(x)-1} uv dx,$$

for all $u,\ v \in X$. Thus, we notice that we can seek weak solutions of (1.1) as critical point of the energetic functional I.

Remark 2.6. From ($\mathbf{M1}$) and Lemma 2.5 we can easily see that ϕ', i.e.

$$\langle \phi'(u), v \rangle = M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \Big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx$$

is of (S_+) type.

Theorem 2.7. *Assume that conditions* $(\mathbf{\Lambda 1}) - (\mathbf{\Lambda 2})$, $(\mathbf{Q1}) - (\mathbf{Q2})$ *and* $(\mathbf{M1}) - (\mathbf{M2})$ *are fulfilled. Then there exists* $\lambda^* > 0$ *such that problem (1.1) has at least two positive non-trivial weak solutions, provided that* $|\lambda|_{L^\infty(\Omega)} < \lambda^*$.

3. Proof of the main result

In this section we discuss the existence of two non-trivial weak solutions of (1.1) by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekland's variational principle. For simplicity, we use $C,\ c_i,\ i = 1, 2, \ldots$ to denote the general positive constant (the exact value may change from line to line).

Let us state now the mountain pass theorem and Ekland's variational principle.

Definition 3.1. A functional I satisfies the Palais-Smale condition $(PS)_c$ on a Banach space X, if any sequence $(u_n) \subset X$ such that

$$I(u_n) \to c, \quad \|I'(u_n)\|_{X^*} \to 0$$

has a convergent subsequence.

Theorem 3.2. *(Mountain Pass Theorem, Ambrosetti and Rabinowitz [2]). Let X be a Banach space and let $I \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$\inf_{u \in X,\ \|u\|=r} I(u) > I(0) \geq I(e).$$

If I satisfies the $(PS)_c$ condition with

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \quad \text{where } \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$$

then c is a critical value of I.

Theorem 3.3. *(Ekeland Variational Principle [26]). Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ be bounded below, and let $\epsilon, \delta > 0$ be arbitrary. If*

$$I(v) \leq \inf_{u \in X} I(u) + \epsilon \quad \text{for a } v \in X,$$

then there exists $u_0 \in X$ such that

$$I(u_0) \leq \inf_{u \in X} I(u) + 2\epsilon, \quad \|u_0 - v\| \leq 2\delta, \quad \text{and } \|I'(u_0)\|_{X^*} < \frac{8\epsilon}{\delta}.$$

Corollary 3.4. *(See [26]) Let $I \in C^1(X, \mathbb{R})$ be bounded below. If I satisfies the $(PS)_c$ condition with $c := \inf_{u \in X} I(u)$, then every minimizing sequence (u_n) for I, i.e. $\lim_{n\to\infty} I(u_n) = \inf_{u \in X} I(u)$, contains a converging subsequence. In particular, there exists $u_0 \in X$ such that*

$$I(u_0) = \min_{u \in X} I(u).$$

We confine ourselves to the case where the former condition of (**Q2**) holds true. A similar proof can be made if the later condition holds true.

Lemma 3.5. *Let $q(x)$, $\lambda(x)$, and $M(t)$ be as in Theorem 2.7, then there exist $\rho > 0$ and $\delta > 0$ such that $I(u) \geq \delta > 0$ for any $u \in X$ with $||u|| = \rho$.*

Proof. Let us define $q_1 : \overline{B_r(x_0)} \to [1, \infty)$, $q_1(x) = q(x)$ for any $x \in \overline{B_r(x_0)}$ and $q_2 : \overline{\Omega \backslash B_R(x_0)} \to [1, \infty)$, $q_2(x) = q(x)$ for any $x \in \overline{\Omega \backslash B_R(x_0)}$.

We also introduce the notation

$$q_1^- = \min_{x\in \overline{B_r(x_0)}} q_1(x), \quad q_1^+ = \max_{x\in \overline{B_r(x_0)}} q_1(x),$$
$$q_2^- = \min_{x\in \overline{\Omega\setminus B_R(x_0)}} q_2(x), \quad q_2^+ = \max_{x\in \overline{\Omega\setminus B_R(x_0)}} q_2(x).$$

Then by relations (**Q1**) and (**Q2**) we have

$$1 \le q_1^- \le q_1^+ < p^- < \frac{p^-}{1-\mu} < p^+ < \frac{p^+}{1-\mu} < q_2^- \le q_2^+ < p^*(x),$$

for any $x \in X$. Thus, we have

$$X \hookrightarrow L^{q_i^\pm}(\Omega), \ \ i \in \{1,2\}.$$

So, there exists a positive constant C such that

$$\int_\Omega |u|^{q_i^\pm}\, dx \le C\|u\|^{q_i^\pm}, \quad \forall u \in X, \ i\in\{1,2\}.$$

It follows that there exist two positive constants c_1 and c_2 such that

$$\begin{aligned}\int_{B_r(x_0)} |u|^{q_1(x)} dx &\le \int_{B_r(x_0)} |u|^{q_1^-}\, dx + \int_{B_r(x_0)} |u|^{q_1^+}\, dx \\ &\le \int_\Omega |u|^{q_1^-}\, dx + \int_\Omega |u|^{q_1^+}\, dx \\ &\le c_1\Big(||u||^{q_1^-} + \|u\|^{q_1^+}\Big), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} |u|^{q_2(x)} dx &\le \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^-}\, dx + \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^+}\, dx \\ &\le \int_\Omega |u|^{q_2^-}\, dx + \int_\Omega |u|^{q_2^+}\, dx \\ &\le c_2\Big(\|u\|^{q_2^-} + \|u\|^{q_2^+}\Big). \end{aligned} \tag{3.2}$$

In view of (**M1**) and relations (3.1) and (3.2), for $\|u\|$ sufficiently small, noting Proposition 2.3, we have

$$\begin{aligned} I(u) &\ge \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)} dx - \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)} dx - \int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)} dx \\ &\ge \frac{m_0}{p^+}\|u\|^{p^+} - \frac{|\lambda|_{L^\infty(\Omega)}}{q^-} C(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}) \\ &\ge \Big[c_3\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_1^-} + \|u\|^{q_1^+})\Big] \\ &\quad + \Big[c_3\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_2^-} + \|u\|^{q_2^+})\Big]. \end{aligned}$$

Since the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = c_3 - c_4 t^{q_2^- - p^+} - c_4 t^{q_2^+ - p^+}$$

is positive in a neighborhood of the origin, it follows that there exists $0 < \rho < 1$ such that $g(\rho) > 0$. On the other hand, defining

$$\lambda^* = \min\Big\{1, \frac{c_3}{2c_4} \min\{\rho^{p^+ - q_1^-}, \rho^{p^+ - q_1^+}\}\Big\}, \tag{3.3}$$

we deduce that there exists $\delta > 0$ such that for any $u \in X$ with $\|u\| = \rho$ we have $I(u) \geq \delta > 0$ provided $|\lambda|_{L^\infty(\Omega)} < \lambda^*$. $\square$

Lemma 3.6. *Let $q(x)$, $\lambda(x)$, and $M(t)$ be as in Theorem 2.7, then there exists $\psi \in X$, $\psi \neq 0$ such that $\lim_{t\to\infty} I(t\psi) \to -\infty$.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and there exist $x_1 \in \Omega \backslash B_R(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_1) \subset (\Omega \backslash B_R(x_0))$ we have $\psi(x) > 0$. When $t > t_0$, from (**M2**) we can easily obtain that

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{(1-\mu)}}} := Ct^{\frac{1}{(1-\mu)}},$$

where t_0 is an arbitrary positive constant. Thus, for $t > 1$ we have

$$\begin{aligned}
I(t\psi) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla t\psi|^{p(x)} dx\Big) - \int_\Omega \frac{\lambda(x)}{q(x)} |t\psi|^{q(x)} dx \\
&\leq c_6 \Big(\int_\Omega |t\nabla\psi|^{p(x)} dx\Big)^{\frac{1}{(1-\mu)}} - \int_{\Omega\backslash B_R(x_0)} \frac{\lambda(x)}{q(x)} |t\psi|^{q(x)} dx \\
&\leq c_6 t^{\frac{p^+}{(1-\mu)}} \Big(\int_\Omega |\nabla\psi|^{p(x)} dx\Big)^{\frac{1}{(1-\mu)}} - t^{q_2^-} \int_{\Omega\backslash B_R(x_0)} \frac{\lambda(x)}{q(x)} |\psi|^{q(x)} dx \\
&\to -\infty \quad \text{as} \quad t \to \infty,
\end{aligned}$$

due to $\frac{p^+}{1-\mu} < q_2^-$. $\square$

By Lemmas 3.5 and 3.6 and the mountain pass theorem of Ambrosetti and Rabinowitz [2], we deduce the existence of a sequence (u_n) such that

$$I(u_n) \to c_7 > 0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty. \tag{3.4}$$

We prove that (u_n) is bounded in X. Assume for the sake of contradiction, if necessary to a subsequence, still denote by (u_n), $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n.

By Proposition 2.3, we may infer that for n large enough

$$\begin{aligned}
1 + c_8 + \|u_n\| &\geq I(u_n) - \frac{1}{q_2^-}\langle I'(u_n), u_n\rangle \\
&= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right) - \int_\Omega \frac{\lambda(x)}{q(x)}|u_n|^{q(x)}\,dx \\
&\quad - \frac{1}{q_2^-}\Big[M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&\quad - \int_\Omega \lambda(x)|u_n|^{q(x)}\,dx\Big] \\
&\geq \frac{(1-\mu)}{p^+}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&\quad - \int_\Omega \frac{\lambda(x)}{q(x)}|u_n|^{q(x)}\,dx - \frac{1}{q_2^-}\Big[M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big) \\
&\quad \int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega \lambda(x)|u_n|^{q(x)}\,dx\Big] \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&\quad + \int_{B_r(x_0)}\left(\frac{1}{q_2^-} - \frac{1}{q_1(x)}\right)\lambda(x)|u_n|^{q_1(x)}\,dx \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\|u_n\|^{p^-} \\
&\quad - \lambda^*\left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right)\int_{B_r(x_0)}|u_n|^{q_1(x)}\,dx \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\|u_n\|^{p^-} \\
&\quad - c_1\lambda^*\left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right)\left(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\right) \\
&\geq m_0\left(\frac{1-\mu}{p^+} - \frac{1}{q_2^-}\right)\|u_n\|^{p^-} - c_8\Big(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\Big).
\end{aligned}$$

But, this cannot hold true since $p^- > 1$. Hence (u_n) is bounded in X. This information combined with the fact X is reflexive implies that there exists a subsequence, still denote by (u_n), and $u_1 \in X$ such that $u_n \rightharpoonup u_1$ in X. Since X is compactly embedded in $L^{q(x)}(\Omega)$, it follows that $u_n \to u_n$ in $L^{q(x)}(\Omega)$. Using Proposition 2.2 we deduce

$$\lim_{n\to\infty}\int_\Omega \lambda(x)|u_n|^{q(x)-2}u_n(u_n - u_1)dx = 0.$$

This fact and relation (3.4) yield

$$\lim_{n\to\infty} M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u_1)=0.$$

In view of (**M1**), we have

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u_1)=0.$$

Using Lemma 2.5, we find that $u_n\to u_1$ in X. Then by relation (3.4) we have

$$I(u_1)=c_7>0 \quad\text{and}\quad I'(u_1)=0,$$

that is u_1 is a non-trivial weak solution of (1.1).

We hope to apply Ekeland's variational principle [26] to get a non-trivial weak solution of problem (1.1).

Lemma 3.7. *Let all conditions in Theorem 2.7 hold. Then there exists $\varphi\in X$, $\varphi\neq 0$ such that $I(t\varphi)<0$ for $t>0$ small enough.*

Proof. Let $\varphi\in C_0^\infty(\Omega)$, $\varphi\geq 0$ and there exist $x_2\in B_r(x_0)$ and $\varepsilon>0$ such that for any $x\in B_\varepsilon(x_2)\subset B_r(x_0)$ we have $\varphi(x)>0$. For any $0<t<1$, we have

$$\begin{aligned} I(t\varphi) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}dx\Big)-\int_\Omega \frac{\lambda(x)}{q(x)}|t\varphi|^{q(x)}dx\\ &\leq c_{10}\Big(\int_\Omega |t\nabla\varphi|^{p(x)}dx\Big)^{\frac{1}{(1-\mu)}}-\int_{B_r(x_0)}\frac{\lambda(x)}{q(x)}|t\varphi|^{q(x)}dx\\ &\leq c_{10}t^{\frac{p^-}{(1-\mu)}}\Big(\int_\Omega |\nabla\varphi|^{p(x)}dx\Big)^{\frac{1}{(1-\mu)}}-t^{q_1^+}\int_{B_r(x_0)}\frac{\lambda(x)}{q_1(x)}|\varphi|^{q_1(x)}dx. \end{aligned}$$

So $I(t\varphi)<0$ for $t<\theta^{\frac{1}{(\frac{p^-}{1-\mu}-q_1^+)}}$, where

$$0<\theta<\min\Big\{1,\frac{\int_{B_r(x_0)}\frac{\lambda(x)}{q_1(x)}|\varphi|^{q_1(x)}dx}{\int_\Omega|\nabla\varphi|^{p(x)}dx}\Big\}.$$

□

Let $\lambda^*>0$ be defined as in (3.3) and assume $|\lambda|_{L^\infty(\Omega)}<\lambda^*$. By Lemma 3.5 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)=\{\omega\in X;\ ||\omega||<\rho\}$, we have

$$\inf_{\partial B_\rho(0)} I>0.$$

By Lemma 3.7, there exists $\varphi\in X$ such that

$$I(t\varphi)<0 \quad\text{for}\quad t>0 \quad\text{small enough}\ .$$

Moreover, for $u \in B_\rho(0)$,

$$I(u) \geq \Big[c_3\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_1^-} + \|u\|^{q_1^+})\Big] + \Big[c_3\|u\|^{p^+} - c_4|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_2^-} + \|u\|^{q_2^+})\Big].$$

It follows that

$$-\infty < c_{11} = \inf_{\overline{B_\rho(0)}} I < 0.$$

We let now $0 < \varepsilon < \inf_{\partial B_\rho(0)} I - \inf_{B_\rho(0)} I$. Applying Ekeland's variational principle [26] to the functional $I : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} I + \varepsilon$$
$$I(u_\varepsilon) < I(u) + \varepsilon||u - u_\varepsilon||, \;\; u \neq u_\varepsilon.$$

Since

$$I(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} I + \varepsilon \leq \inf_{B_\rho(0)} I + \varepsilon < \inf_{\partial B_\rho(0)} I,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $K : \overline{B_\rho(0)} \to \mathbb{R}$ by $K(u) = I(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of K and thus

$$\frac{K(u_\varepsilon + tv) - K(u_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and $v \in B_1(0)$. The above relation yields

$$\frac{I(u_\varepsilon + tv) - I(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle I'(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$ and we infer that $\|I'(u_\varepsilon)\| \leq \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_\rho(0)$ such that

$$I(v_n) \to c_{11} \quad \text{and} \quad I'(v_n) \to 0. \tag{3.5}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_2 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_2 in X. Actually, with similar arguments as those used in the proof that the sequence $u_n \to u_1$ in X we can show that $v_n \to u_2$ in X. Thus, by relation (3.5),

$$I(u_2) = c_{11} < 0 \quad \text{and} \quad I'(u_2) = 0,$$

i.e., u_2 is a non-trivial weak solution for problem (1.1).

Finally, since

$$I(u_1) = c_7 > 0 > c_{11} = I(u_2),$$

we see that $u_1 \neq u_2$. Thus, problem (1.1) has two non-trivial weak solutions.

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