Pointwise almost periodicity in in a generalized shift dynamical system

Fatemah Ayatollah Zadeh Shirazi 1 and Meysam Miralaei 2
1 Faculty of Math., Stat. and Computer Science, College of Science, University of Tehran, Tehran, Iran (fatemah@khayam.ut.ac.ir)
2 Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran (m.miralaei@math.iut.ac.ir)

Abstract. In the following text we prove that in a generalized shift dynamical system \((X^\Gamma, \sigma_\varphi)\) for discrete \(X\) with at least two elements, arbitrary nonempty \(\Gamma\) and bijection \(\varphi : \Gamma \to \Gamma\), the following statements are equivalent:
- \((X^\Gamma, \sigma_\varphi)\) is pointwise recurrent;
- \((X^\Gamma, \sigma_\varphi)\) is pointwise almost periodic;
- \((X^\Gamma, \sigma_\varphi)\) is pointwise regularly almost periodic;
- \((X^\Gamma, \sigma_\varphi)\) is compactly almost periodic;
- \(\text{Per}(\varphi) = \Gamma\) (\(\varphi : \Gamma \to \Gamma\) is pointwise periodic).

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1. Preliminaries

Let \(Y\) be an arbitrary set. We call the collection \(\mathcal{F}\) of subsets of \(Y \times Y\) a uniformity on \(Y\) if:
- for all \(\alpha \in \mathcal{F}\) we have \(\Delta_Y \subseteq \alpha\);
- for all \(\alpha, \beta \in \mathcal{F}\) we have \(\alpha \cap \beta \in \mathcal{F}\);

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• for all $\alpha \in \mathcal{F}$ and $\beta \subseteq Y \times Y$ with $\alpha \subseteq \beta$ we have $\beta \in \mathcal{F}$;
• for all $\alpha \in \mathcal{F}$ there exists $\beta \in \mathcal{F}$ with $\beta \circ \beta^{-1} \subseteq \alpha$;

where $\Delta_Y = \{(y, y) : y \in Y\}$, $\alpha \circ \beta = \{(x, y) : \text{there exists } z \text{ such that } (x, z) \in \alpha \text{ and } (z, y) \in \beta\}$, and $\alpha^{-1} = \{(y, x) : (x, y) \in \alpha\}$ for $\alpha, \beta \subseteq Y \times Y$. We call an element of uniformity $\mathcal{F}$ an index. Moreover, for $\alpha \in \mathcal{F}$ and $x \in Y$ let $\alpha[x] = \{y : (x, y) \in \alpha\}$.

If $\mathcal{F}$ is a uniformity on $Y$, we call $(Y, \mathcal{F})$ a uniform space and equip it with topology $T = \{U \subseteq Y : \text{for all } x \in U \text{ there exists } \alpha \in \mathcal{F} \text{ with } \alpha[x] \subseteq U\}$ (topology generated by $\mathcal{F}$). For nonempty set $\Lambda$ if $\{(Y_\theta, \mathcal{F}_\theta) : \theta \in \Lambda\}$ is a collection of uniform spaces, then $\prod_{\theta \in \Lambda} Y_\theta$ under product topology is a uniform space too, and we may consider the following uniformity over it:

$$\gamma \subseteq \prod_{\theta \in \Lambda} Y_\theta \times \prod_{\theta \in \Lambda} Y_\theta : \text{there exist } \theta_1, \ldots, \theta_n \in \Lambda \text{ and } \alpha_1 \in \mathcal{F}_{\theta_1}, \ldots, \alpha_n \in \mathcal{F}_{\theta_n} \text{ with } \gamma \subseteq \kappa(\alpha_1, \ldots, \alpha_n),$$

where $\kappa(\alpha_1, \ldots, \alpha_n)$ is the following set:

$$\{(x_\theta)_{\theta \in \Lambda}, (y_\theta)_{\theta \in \Lambda}) \in \prod_{\theta \in \Lambda} Y_\theta \times \prod_{\theta \in \Lambda} Y_\theta : \forall i \in \{1, \ldots, n\} ((x_{\theta_i}, y_{\theta_i}) \in \alpha_i)\}.$$

The topological space $W$ is uniformizable if there exists a uniformity $\mathcal{G}$ on $W$ such that the topology generated by $\mathcal{G}$ coincides with original topology of $W$ and in this case we call $\mathcal{G}$ an admissible uniformity on $W$.

If $Y$ is compact Hausdorff, then it admits a unique admissible uniformity $\{\alpha \subseteq Y \times Y : \Delta_Y \subseteq \text{a subset of interior of } \alpha\}$. See [7] for more details.

By a (topological) dynamical system $((Z, \mu), h)$ or briefly $(Z, h)$ we mean a Hausdorff uniform topological space $Z$ (phase space) equipped with uniformity $\mu$ and a homeomorphism $h : Z \to Z$. In dynamical system $(Z, h)$, we call nonempty subset $W$ invariant if $h(W) = W$. We call the dynamical system $(Z, h)$ [8], [9]:

• **periodic**, if there exists $n \geq 1$ with $h^n = \text{id}_Z$, where $\text{id}_Z : Z \to Z$ is the identity map, $\text{id}_Z(x) = x, x \in Z$;
• **pointwise periodic**, if for all $x \in Z$ there exists $n \geq 1$ with $h^n(x) = x$;
• **pointwise recurrent**, if for all $z \in Z$ and all open neighborhood $U$ of $z$ there exists $n \geq 1$ such that $h^n(z) \in U$;
• **pointwise almost periodic**, if for all $z \in Z$ and all open neighborhood $U$ of $z$, there exists $N \geq 1$ such that for all $p \in Z$ there exists $n \in \{p, p + 1, \ldots, p + N - 1\}$ with $h^n(z) \in U$;
• **pointwise regularly almost periodic**, if for all $z \in Z$ and all open neighborhood $U$ of $z$, there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $h^{nm}(z) \in U$ for all $m \in \mathbb{N}$.
• **recurrent** (or **uniformly recurrent**) if for all $\alpha \in \mu$, there exists $n \geq 1$ with $\{(h^n(z), z) : z \in X\} \subseteq \alpha$;
• **almost periodic** (or **uniformly almost periodic**), if for all $\alpha \in \mu$, there exists $N \geq 1$ such that for all $p \in \mathbb{Z}$ there exists $n \in \{p, p+1, \ldots, p+N-1\}$ with $\{(h^n(z), z) : z \in Z\} \subseteq \alpha$;
• **regularly almost periodic** (or **uniformly regularly almost periodic**), if for all $\alpha \in \mu$, there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $\{(h^{nm}(z), z) : z \in Z, m \in \mathbb{N}\} \subseteq \alpha$;
• **compactly almost periodic** (or **uniformly compactly almost periodic**), if for all compact subset $B$ of $Z$, $\bigcup\{h^n(B) : n \in \mathbb{Z}\}$ is compact and for all compact invariant subset $W$ of $Z$, $(W, h|_W)$ is almost periodic;
• **compactly recurrent** (or **uniformly compactly recurrent**), if for all compact subset $B$ of $Z$, $\bigcup\{h^n(B) : n \in \mathbb{Z}\}$ is compact and for all compact invariant subset $W$ of $Z$, $(W, h|_W)$ is recurrent (the concept of compactly recurrence is introduced here, imitating the concept of compactly almost periodicity in [3]). For nonempty arbitrary sets $\Gamma$, $X$ and map $\varphi : \Gamma \to \Gamma$, we call $\sigma_{\varphi} : X^\Gamma \to X^\Gamma$ with $\sigma_{\varphi}, (x_\alpha)_{\alpha \in \Gamma} = (x_{\varphi(\alpha)})_{\alpha \in \Gamma}$ (for $(x_\alpha)_{\alpha \in \Gamma} \in X^{\Gamma}$), a generalized shift [3]. Whenever $\Gamma = \mathbb{N}$ and $\varphi(n) = n + 1$ ($n \in \mathbb{N}$), $\sigma_{\varphi} : X^\mathbb{N} \to X^\mathbb{N}$ is the familiar one sided shift, also whenever $\Gamma = \mathbb{Z}$ and $\varphi(n) = n + 1$ ($n \in \mathbb{Z}$), $\sigma_{\varphi} : X^\mathbb{Z} \to X^\mathbb{Z}$ is the well-known two sided shift. On the other hand, if $X$ is a topological space and $X^\Gamma$ is equipped with product topology, then $\sigma_{\varphi} : X^\Gamma \to X^\Gamma$ is continuous.

**Remark 1.1.** For nonempty arbitrary sets $\Gamma$, $X$ and map $\varphi : \Gamma \to \Gamma$, with $|X| \geq 2$, the map $\sigma_{\varphi} : X^\Gamma \to X^\Gamma$ is bijective if and only if $\varphi : \Gamma \to \Gamma$ is bijective. Hence if $X$ is a topological space and $X^\Gamma$ is equipped with product topology, then $\sigma_{\varphi} : X^\Gamma \to X^\Gamma$ is a homeomorphism if and only if $\varphi : \Gamma \to \Gamma$ is bijective.

For mapping $f : A \to A$, we call $a \in A$ a **periodic** point of $f : A \to A$ if there exists $n \geq 1$ with $f^n(a) = a$. Let $\text{Per}(f) = \{a \in A : a$ is a periodic point of $f : A \to A\}$.

**In the following text suppose $X$ is a discrete topological space with at least two elements, $\Gamma$ is an infinite set, $\varphi : \Gamma \to \Gamma$ is bijective, and consider $X^\Gamma$ under product (pointwise convergence) topology.**

2. **Pointwise periodicity in generalized shift dynamical systems**

In this section we prove that the generalized shift $(X^\Gamma, \sigma_{\varphi})$ is pointwise periodic (resp. periodic) if and only if $\varphi : \Gamma \to \Gamma$ is periodic.
Remark 2.1. For maps \( \lambda, \theta : \Gamma \to \Gamma \), we have \( \sigma_\lambda = \sigma_\theta \) if and only if \( \lambda = \theta \).

Proof. If \( \theta \neq \lambda \), there exists \( \beta \in \Gamma \) with \( \theta(\beta) \neq \lambda(\beta) \). Choose distinct \( p, q \in X \), and let \( x_\alpha = p \) for \( \alpha \neq \beta \) and \( x_\beta = q \). Then for \( (y_\alpha)_{\alpha \in \Gamma} := \sigma_\theta((x_\alpha)_{\alpha \in \Gamma}) \) and \( (z_\alpha)_{\alpha \in \Gamma} := \sigma_\lambda((x_\alpha)_{\alpha \in \Gamma}) \) we have \( z_\beta = x_\theta(\beta) = q \) and \( y_\beta = x_\lambda(\beta) = p \) (since \( \lambda(\beta) \neq \theta(\beta) \)). So \( z_\beta \neq y_\beta \) and \( \sigma_\theta((x_\alpha)_{\alpha \in \Gamma}) \neq \sigma_\lambda((x_\alpha)_{\alpha \in \Gamma}) \), which leads to \( \sigma_\theta \neq \sigma_\lambda \) and completes the proof. \( \square \)

Remark 2.2. For maps \( \lambda, \theta : \Gamma \to \Gamma \), we have \( \sigma_\lambda \circ \sigma_\theta = \sigma_{\theta \circ \lambda} \).

Theorem 2.3. In the generalized shift dynamical system \((X^\Gamma, \sigma_\varphi)\), the following statements are equivalent:

1. \((X^\Gamma, \sigma_\varphi)\) is periodic (i.e., there exists \( n \geq 1 \) such that \( \sigma_\varphi^n = \text{id}_{X^\Gamma} \));
2. \((X^\Gamma, \sigma_\varphi)\) is pointwise periodic (i.e., \( \text{Per}(\sigma_\varphi) = X^\Gamma \));
3. \( \varphi : \Gamma \to \Gamma \) is periodic (i.e., there exists \( m \geq 1 \) with \( \varphi^m = \text{id}_\Gamma \)).

Proof. It is clear that if \((X^\Gamma, \sigma_\varphi)\) is periodic, then it is pointwise periodic. Now suppose \((X^\Gamma, \sigma_\varphi)\) is pointwise periodic and choose distinct \( p, q \in X \). Suppose \( \beta \in \Gamma \). Let \( x_\alpha = p \) for \( \alpha \neq \beta \) and \( x_\beta = q \). Since \( \sigma_\varphi \) is pointwise periodic, there exists \( n \geq 1 \) with \( \sigma_\varphi^n((x_\alpha)_{\alpha \in \Gamma}) = (x_\alpha)_{\alpha \in \Gamma} \). So \( (x_\alpha)_{\alpha \in \Gamma} = (x_{\varphi^n(\alpha)})_{\alpha \in \Gamma} \) and \( q = x_\beta = x_{\varphi^n(\beta)} \) which leads to \( \varphi^n(\beta) = \beta \). Hence \( \varphi : \Gamma \to \Gamma \) is periodic. For \( \alpha \in \Gamma \) let \( n_\alpha = \min\{n \geq 1 : \varphi^n(\alpha) = \alpha \} \), it’s evident that for all \( \alpha \in \Gamma \) we have \( n_\alpha = n_{\varphi(\alpha)} \) (note to the fact that \( \varphi : \Gamma \to \Gamma \) is bijective). In the following Claim we prove that \( \sup\{n_\alpha : \alpha \in \Gamma \} \) is finite.

Claim. \( \sup\{n_\alpha : \alpha \in \Gamma \} < \infty \).

Proof of Claim. If \( \sup\{n_\alpha : \alpha \in \Gamma \} = +\infty \), then there exists a strictly increasing sequence \( \{n_{\alpha_k}\}_{k \in \mathbb{N}} \). Let:

\[
x_\alpha = \begin{cases} 
q & \alpha \in \{\alpha_k : k \in \mathbb{N}\}, \\
p & \text{otherwise}.
\end{cases}
\]

Since \((X^\Gamma, \sigma_\varphi)\) is pointwise periodic, there exists \( m \geq 1 \) with \( \sigma_\varphi^m((x_\alpha)_{\alpha \in \Gamma}) = (x_\alpha)_{\alpha \in \Gamma} \). So for all \( k \geq 1 \) we have \( q = x_{\alpha_k} = x_{\varphi^m(\alpha_k)} \), which leads to \( \varphi^m(\alpha_k) \in \{\alpha_l : l \in \mathbb{N}\} \), using \( n_{\varphi^m(\alpha)} = n_\alpha \) and the fact that \( \{n_\alpha\}_{\alpha \in \Gamma} \) is one to one, we conclude \( \varphi^m(\alpha) = \alpha \). By \( \varphi^m(\alpha) = \alpha \) we have \( m \geq n_\alpha \), so \( \sup\{n_\alpha : \alpha \in \Gamma \} \leq m \), which is a contradiction, hence \( \sup\{n_\alpha : \alpha \in \Gamma \} < \infty \).

If \( N = \sup\{n_\alpha : \alpha \in \Gamma \} \), then for all \( \alpha \in \Gamma \) we have \( \varphi^N(\alpha) = \alpha \). Therefore, \( \varphi^N = \text{id}_\Gamma \), and \( \varphi : \Gamma \to \Gamma \) is periodic.

In order to complete the proof of theorem, note to the fact that if \( \varphi^n = \text{id}_\Gamma \), using Remarks 2.1 and 2.2 we have \( \sigma_{\varphi^n} = \sigma_{\varphi^n} = \sigma_{\text{id}_\Gamma} = \text{id}_{X^\Gamma} \), and \((X^\Gamma, \sigma_\varphi)\) is periodic. \( \square \)
3. Pointwise almost periodicity in generalized shift dynamical systems

In this section we prove that \((X^\Gamma, \sigma_\varphi)\) is any of pointwise recurrent, pointwise almost periodic, pointwise regularly almost periodic, compactly recurrent, compactly almost periodic if and only if \(\text{Per}(\varphi) = \Gamma\).

Lemma 3.1. If \((X^\Gamma, \sigma_\varphi)\) is pointwise recurrent, then \(\text{Per}(\varphi) = \Gamma\).

Proof. Suppose \(\theta \in \Gamma\) and \((X^\Gamma, \sigma_\varphi)\) is pointwise recurrent. Choose distinct \(p, q \in X\) and let:

\[
U_\alpha = \begin{cases} \{p\} & \alpha = \theta, \\ X & \alpha \neq \theta, \end{cases} \quad \text{and} \quad x_\alpha = \begin{cases} p & \alpha = \theta, \\ q & \alpha \neq \theta. \end{cases}
\]

Since \((X^\Gamma, \sigma_\varphi)\) is pointwise recurrent and \(\prod_{\alpha \in \Gamma} U_\alpha\) is an open neighborhood of \((x_\alpha)_{\alpha \in \Gamma}\) there exists \(n \geq 1\) with \((x_{\varphi^n(\alpha)})_{\alpha \in \Gamma} = \sigma_\varphi^n(x_\alpha)_{\alpha \in \Gamma} \subseteq \prod_{\alpha \in \Gamma} U_\alpha\). In particular \(x_{\varphi^n(\theta)} \in U_\theta = \{p\}\) and \(x_{\varphi^n(\theta)} = p\) which leads to \(\varphi^n(\theta) = \theta\) by (*), and \(\theta\) is periodic under \(\varphi\). □

Lemma 3.2. If \(\text{Per}(\varphi) = \Gamma\), then \((X^\Gamma, \sigma_\varphi)\) is pointwise regularly almost periodic.

Proof. Suppose \(\text{Per}(\varphi) = \Gamma\). Let \(w = (w_\alpha)_{\alpha \in \Gamma} \in X^\Gamma\) if \(U\) is an open neighborhood of \(w\), then there exist \(\theta_1, \ldots, \theta_k \in \Gamma\) such that \(\prod_{\alpha \in \Gamma} U_\alpha \subseteq U\), where:

\[
U_\alpha = \begin{cases} \{w_\alpha\} & \alpha = \theta_1, \ldots, \theta_k, \\ X & \text{otherwise}. \end{cases}
\]

For all \(i \in \{1, \ldots, k\}\) there exists \(r_i \geq 1\) such that \(\varphi^{r_i}(\theta_i) = \theta_i\). For all \(i \in \{1, \ldots, k\}\) and \(t \in \mathbb{Z}\) we have \(\varphi^{r_i \cdot t}(\theta_i) = \theta_i\), moreover if \((y_\alpha)_{\alpha \in \Gamma} = \sigma_{\varphi^{r_i \cdot t}}(w_\alpha)_{\alpha \in \Gamma} = (w_{\varphi^{r_i \cdot t}(\alpha)})_{\alpha \in \Gamma}\), then for \(i = 1, \ldots, k\) we have \(y_{\theta_i} = w_{\varphi^{r_i \cdot t}(\theta_i)} = w_{\theta_i}\), which leads to \((y_\alpha)_{\alpha \in \Gamma} \in \prod_{\alpha \in \Gamma} U_\alpha(\subseteq U)\) and completes the proof. □

Lemma 3.3. If \(\text{Per}(\varphi) = \Gamma\), and \(B\) is a compact subset of \(X^\Gamma\), then for each \(\alpha \in \Gamma\) there exists finite subset \(D_\alpha\) of \(X\) such that \(B \subseteq \bigcap_{\alpha \in \Gamma} D_\alpha =: D\) and \(D_\beta = D_{\varphi(\beta)}\) for all \(\beta \in \Gamma\), hence \(\sigma_\varphi(D) = D\).

Proof. For \(\alpha \in \Gamma\) let \(\text{orb}(\varphi, \alpha) := \{\varphi^n(\alpha) : n \in \mathbb{Z}\}\) suppose \(\pi_\alpha : X^\Gamma \to X\) is the projection map on the \(\alpha\)th coordinate. Since \(\alpha \in \text{Per}(\varphi)\), the set \(\text{orb}(\varphi, \alpha)\) is finite and \(\text{orb}(\varphi, \alpha) = \text{orb}(\varphi, \beta)\) for all \(\beta \in \text{orb}(\varphi, \alpha)\). If \(B\) is a compact nonempty subset of \(X^\Gamma\) and \(\alpha \in \Gamma\), then \(\pi_\alpha(B)\) is a compact
and hence finite subset of \( X \). Thus \( D_\alpha := \bigcup \{ \pi_\beta(B) : \beta \in \text{orb}(\varphi, \alpha) \} \) is finite and \( D_\alpha = D_\beta \) for all \( \beta \in \text{orb}(\varphi, \alpha) \) in particular, \( D_\alpha = D_{\varphi(\alpha)} = D_{\varphi^{-1}(\alpha)} \). Moreover, \( \sigma_\varphi(D) = \prod_{\alpha \in \Gamma} D_{\varphi(\alpha)} = \prod_{\alpha \in \Gamma} D_\alpha = D. \)

**Lemma 3.4.** If \( \text{Per}(\varphi) = \Gamma \), and \( B \) is a compact subset of \( X^{\Gamma} \), then \( \{\sigma^n_\varphi(B) : n \in \mathbb{Z}\} \) is compact.

**Proof.** Consider \( D = \prod_{\alpha \in \Gamma} D_\alpha \supseteq B \) as in Lemma 3.3. By the Tychonoff theorem \( D \) is a compact and hence closed subset of \( X^{\Gamma} \). Using \( \sigma_\varphi(D) = D \) and \( B \subseteq D \) we have \( \{\sigma^n_\varphi(B) : n \in \mathbb{Z}\} \subseteq D = D \), thus \( \{\sigma^n_\varphi(B) : n \in \mathbb{Z}\} \) is compact.

**Lemma 3.5.** If \( \text{Per}(\varphi) = \Gamma \), and \( B \) is a compact invariant subset of \( (X^{\Gamma}, \sigma_\varphi) \), then \( (B, \sigma_\varphi|_B) \) is regularly almost periodic. In particular \( (B, \sigma_\varphi|_B) \) is almost periodic and recurrent.

**Proof.** For \( H \subseteq \Gamma \) let
\[
\beta_H := \{(x_\alpha, y_\alpha)_{\alpha \in \Gamma} : (x_\alpha, y_\alpha)_{\alpha \in \Gamma} \in \prod_{\lambda \in \Gamma} \alpha_\lambda \}
\]
where \( \alpha_\lambda = \{(w, w) : w \in X\} \) for \( \lambda \in H \) and \( \alpha_\lambda = X \times X \) for \( \lambda \in \Gamma \setminus H \).

Consider \( D = \prod_{\alpha \in \Gamma} D_\alpha \supseteq B \) as in Lemma 3.3. We recall that since \( D_\lambda \)'s and \( \prod_{\lambda \in \Lambda} D_\lambda \) are compact Hausdorff, they admit a unique uniformity. If \( \alpha \) is an index of \( \prod_{\lambda \in \Lambda} D_\lambda \), then there exist \( \lambda_1, \ldots, \lambda_m \in \Gamma \) such that
\[
\beta_{\{\lambda_1, \ldots, \lambda_m\}} \cap (D \times D) \subseteq \alpha.
\]
Since \( \text{Per}(\varphi) = \Gamma \), there exists \( n \in \mathbb{N} \) such that \( \varphi^n(\lambda_i) = \lambda_i \) for all \( i \in \{1, \ldots, m\} \). For all \( x = (x_\lambda)_{\lambda \in \Gamma} \in D \), \( k \in \mathbb{Z} \), and for \( y = (y_\lambda)_{\lambda \in \Gamma} = \sigma_\varphi^{kn}(x) = (x_{\varphi^{kn}(\lambda)})_{\lambda \in \Gamma} \) we have:
\[
\forall i \in \{1, \ldots, m\} \quad y_{\lambda_i} = x_{\varphi^{kn}(\lambda_i)} = x_{\lambda_i} \in D_{\lambda_i},
\]
which leads to \( (x, \sigma_\varphi^{kn}(x)) = (x, y) \in \beta_{\{\lambda_1, \ldots, \lambda_m\}} \cap (D \times D) \). Hence \( (x, \sigma_\varphi^{kn}(x)) \in \alpha \) for all \( x \in D \), \( k \in \mathbb{Z} \) and \( (D, \sigma_\varphi|_B) \) is regularly almost periodic, therefore \( (B, \sigma_\varphi|_B) \) is regularly almost periodic.

**Corollary 3.6.** By Lemmas 3.4 and 3.5, if \( \text{Per}(\varphi) = \Gamma \), then \( (X^{\Gamma}, \sigma_\varphi) \) is compactly almost periodic and compactly recurrent.

**Theorem 3.7** (Main Theorem). The following statements are equivalent:

- \((X^{\Gamma}, \sigma_\varphi)\) is pointwise recurrent;
- \((X^{\Gamma}, \sigma_\varphi)\) is pointwise almost periodic;
• \((X^\Gamma, \sigma_\varphi)\) is pointwise regularly almost periodic;
• \((X^\Gamma, \sigma_\varphi)\) is compactly almost periodic;
• \((X^\Gamma, \sigma_\varphi)\) is compactly recurrent;
• \(\text{Per}(\varphi) = \Gamma\) (\(\varphi : \Gamma \to \Gamma\) is pointwise periodic).

Proof. Use Lemmas 3.1, 3.2, Corollary 3.6 and the fact that if \((X^\Gamma, \sigma_\varphi)\) is compactly almost periodic, then it is pointwise almost periodic and if \((X^\Gamma, \sigma_\varphi)\) is pointwise almost periodic or pointwise regularly almost periodic, then it is pointwise recurrent. \(\square\)

**Uniformly almost periodicity in generalized shift dynamical systems.** Let \(X\) is finite, then \(X^\Gamma\) is compact, hence by Lemma 3.5 and Theorem 3.7 the following statements are equivalent:
• \((X^\Gamma, \sigma_\varphi)\) is pointwise recurrent;
• \((X^\Gamma, \sigma_\varphi)\) is pointwise almost periodic;
• \((X^\Gamma, \sigma_\varphi)\) is pointwise regularly almost periodic;
• \((X^\Gamma, \sigma_\varphi)\) is recurrent;
• \((X^\Gamma, \sigma_\varphi)\) is almost periodic;
• \((X^\Gamma, \sigma_\varphi)\) is regularly almost periodic;
• \(\text{Per}(\varphi) = \Gamma\).

However even for infinite \(X\) if we equip \(X^\Gamma\) with uniformity generated by basis \(\{\beta_H : H\text{ is a finite subset of }\Gamma\}\) \((\beta_Hs\text{ are defined in (*) in Lemma 3.5})\), then the above statements are equivalent, using a similar method described in this section and Lemma 3.5.

**Example 3.8.** Define \(\varphi : \mathbb{N} \to \mathbb{N}\) with \(\varphi(1) = 1\), \(\varphi(k) = k + 1\) whenever \(2^n \leq k \leq 2^{n+1} - 1\) and \(\varphi(2^{n+1} - 1) = 2^n\) for \(n \in \mathbb{N}\). Then \(\varphi\) is pointwise periodic and it is not periodic, hence \((X^\mathbb{N}, \sigma_\varphi)\) is pointwise almost periodic and satisfies all equivalent conditions of Theorem 3.7 but \((X^\mathbb{N}, \sigma_\varphi)\) is not pointwise periodic and does not satisfy any of equivalent conditions of Theorem 2.3.


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References


