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Evolution of Space Curves Using Type-3 Bishop Frame

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Abstract. In this paper, Firstly, we introduce a new version of Bishop frame using a common vector field as normal vector field of a regular curve and call this frame as type-3 Bishop frame. Geometric visualization for both Frenet frame and type-3 Bishop frame are plotted. Secondly, time evolution equation for type-3 Bishop frame and type three bishop curvatures are obtained and we plotted the evolving curve via the numerical integration of type-3 Bishop Serret Frenet equations. Finally, surfaces generated by evolving a regular space curve in R^3 as it evolves over time with arbitrary velocities attached to type-3 Bishop frame are obtained. We plotted these surfaces via numerical integration of Gauss-Weingarten equations.

Keywords: Type-3 Bishop frame, Serret-Frenet equations, Time evolution equation, Gauss-Weingarten equations.

2000 Mathematics subject classification: 14H50; Secondary 53A04.

1. Introduction

Hasimoto established a connection between integrable equation and a moving curve in the context of a vortex filament motion [1] which got mapped to the nonlinear Schrödinger equation (NLSE). Soon afterward, Lamba [2] presented the evolution of space curves in three dimensional space by considering tow sets of Frenet-Serret equations that describe the

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evolution for the tangent, normal, and binormal vectors to the curve with respect to arc-lenght s and time t. He obtained coupled nonlinear partial differential equations for the curvature κ and torsion τ of the curve by applying compatibility conditions on these vectors. Under certain conditions showed that these turn out to be nonlinear Schrödinger equation, sine-Gordon equation and Hirota equation. Subsequently, Doliwa and Santini [3] established a connection between the motion of inextensible curves and solitonic systems via an embedding in a space of constant curvature.

In the same period, Langer and Perline [4] showed that the dynamics of non-stretching vortex filament in R^3 leads to the NLS hierarchy. Lakshmanan et al. [5], derived the Heisenberg spin chain equation via the spatial motion of a curve. Doliwa and Santini [6] studied geometric characterization of the integrable motions of a curve on s^2 and s^3.

Recently, Nakayama et al.[7, 8] investigated motion of plane and space curves and obtained time evolution equation of the moving frame and the curvatures of the evolving curve.

All above authors are used Frenet frame that associated to a space curve to mobilize the curve in three dimensional space while T. Körpinar and E. Turhan [9] used type-2 Bishop Frame in E^3 to study inextensible flows of a space curve according to type-2 Bishop Frame. They obtained time evolution equation for type-2 Bishop Frame and type-2 Bishop curvatures which governing the evolution of the curve.

In this paper, Firstly, we introduce a new version of Bishop frame using a common vector field as normal vector field of a regular curve and call this frame as type-3 Bishop frame. Geometric visualization for both Frenet frame and type-3 Bishop frame are plotted. Secondly, time evolution equation for type-3 Bishop frame and type three bishop curvatures are obtained and we plotted the evolving curve via the numerical integration of type-3 Bishop Serret Frenet equations. Finally, surfaces generated by evolving a regular space curve in R^3 as it evolves over time with arbitrary velocities attached to type-3 Bishop frame are obtained. We plotted these surfaces via numerical integration of Gauss-Weingarten equations.

The article is organized as follows. In section 2 we introduce differential geometry of curves focusing on Serret-Frenet frame and type-3 Bishop frame along a space curve. In section 3, the evolution of curves is represented by to sets of type-3 Bishop frame equations. By applying compatibility condition on these vectors, partial differential equations for type-3 Bishop frame and the curvatures are derived. In section 4, we introduce differential geometry of surfaces focusing on the first and second fundamental forms, Gaussian and mean curvatures. In section 5, surfaces generated by evolving a regular space curve in R^3 as it evolves over time with arbitrary velocities attached to type-3 Bishop frame are constructed and plotted.

2. Bishop frame along a space curve

The theory of space curves are mainly extended by the Frenet-Serret theorem which expresses the derivative of a geometrically chosen basis of E^3 by the aid of itself is proved. Recently, L.R. Bishop [10] in 1975 established a frame that associated to a space curve using a common vector field as tangent vector field of a regular space curve.

There are a catalogue of frames that associated to a space curve such as Frenet frame [12], Bishop frame [10, 13, 14], Kepler frame[15] and quasi frame [16] . In this section we, we introduce a new version of Bishop frame using a common vector field as normal vector field of a regular curve and call this frame as type-3 Bishop frame.

Let $\vec{\mathbf{r}} = \vec{\mathbf{r}}(s)$ be a space curve embedded in three-dimensional space, generated by the vector $\vec{\mathbf{r}}(s) = (\mathbf{x_1}(s), \mathbf{x_2}(s), \mathbf{x_3}(s))$, where s is the arc-length. Let $\vec{\mathbf{t}}, \vec{\mathbf{n}}$ and $\vec{\mathbf{b}}$ denotes, respectively, the unit tangent, the unit normal, the unit binormal vectors on the curve. The orthogonal right-handed triad $(\vec{\mathbf{t}}, \vec{\mathbf{n}}, \vec{\mathbf{b}})$ evolving on the curve according to Serret-Frenet equations [11]

$$\frac{d}{ds}\begin{pmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{n}} \\ \vec{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{n}} \\ \vec{\mathbf{b}} \end{pmatrix}. \tag{2.1}$$

The curvature κ and the torsion τ are function of s and given by

$$\begin{aligned} \kappa &= \vec{\mathbf{n}}.\frac{d\vec{\mathbf{t}}}{ds}, \\ \tau &= -\vec{\mathbf{n}}.\frac{d\vec{\mathbf{b}}}{ds}, \end{aligned} \tag{2.2}$$

its well known that a curve can be uniquely represented by specifying κ and τ for a stated orientation.

If we rotate Frent frame $(\vec{\mathbf{t}}, \vec{\mathbf{n}}, \vec{\mathbf{b}})$ by angel ρ about the normal we get a new frame denoted by $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$, then the relation between two frames is given by

$$\begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix} = \begin{pmatrix} \cos\rho(s) & 0 & \sin\rho(s) \\ 0 & 1 & 0 \\ -\sin\rho(s) & 0 & \cos\rho(s) \end{pmatrix} \begin{pmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{n}} \\ \vec{\mathbf{b}} \end{pmatrix}. \tag{2.3}$$

Thus,

$$\begin{pmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{n}} \\ \vec{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \cos\rho(s) & 0 & -\sin\rho(s) \\ 0 & 1 & 0 \\ \sin\rho(s) & 0 & \cos\rho(s) \end{pmatrix} \begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix}. \tag{2.4}$$

Let,

$$F = \begin{pmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{n}} \\ \vec{\mathbf{b}} \end{pmatrix}, A_1 = \begin{pmatrix} \cos\rho(s) & 0 & \sin\rho(s) \\ 0 & 1 & 0 \\ -\sin\rho(s) & 0 & \cos\rho(s) \end{pmatrix}, A_2 = \begin{pmatrix} \cos\rho(s) & 0 & -\sin\rho(s) \\ 0 & 1 & 0 \\ \sin\rho(s) & 0 & \cos\rho(s) \end{pmatrix},$$

$$B = \begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix}, K = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \tag{2.5}$$

then 2.1, 2.3 and 2.4 can be written as

$$\begin{aligned} \frac{\partial F}{\partial s} &= KF, \\ B &= A_1 F, \\ F &= A_2 Q. \end{aligned} \tag{2.6}$$

By using the above equation the rate of change of type-3 Bishop Serret-Frenet frame, is given by

$$\frac{dB}{ds} = (\frac{dA_1}{ds}.B + A_1.K.A_2)B, \tag{2.7}$$

by using equation 2.5 we have

$$\frac{d}{ds}\begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix} = \begin{pmatrix} 0 & (\kappa\cos\rho(s) - \tau\sin\rho(s)) & \frac{d\rho}{ds} \\ -(\kappa\cos\rho(s) - \tau\sin\rho(s)) & 0 & (\kappa\sin\rho(s) + \tau\cos\rho(s)) \\ -\frac{d\rho}{ds} & -(\kappa\sin\rho(s) + \tau\cos\rho(s)) & 0 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix}. \tag{2.8}$$

Thus we can write the above equation in the form,

$$\frac{\mathbf{d}}{\mathbf{d}s}\begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{pmatrix} \begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix}, \tag{2.9}$$

where type-3 Bishop curvatures are

$$\begin{aligned} \kappa_1 &= \kappa\cos\rho(s) - \tau\sin\rho(s), \\ \kappa_2 &= \kappa\sin\rho(s) + \tau\cos\rho(s), \end{aligned} \tag{2.10}$$

where

$$\rho(s) = \rho_0 = c \tag{2.11}$$

The frame $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$ is properly oriented, We shall call the set $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}}, \kappa_1, \kappa_2)$ as type-3 Bishop invariants of the curve $\vec{\mathbf{r}} = \vec{\mathbf{r}}(s)$.

There is a more compact way of writing the type-3 Bishop Frenet equations which gives a better insight into the structure of these equations. The evolution of an orthonormal type-3 Bishop frame $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$ defined along the curve may be specified by its angular velocity $\vec{\omega}$ through the relations

$$\begin{aligned} \frac{d\vec{\mathbf{b_1}}}{ds} &= \vec{\omega} \times \vec{\mathbf{b_1}}, \\ \frac{d\vec{\mathbf{b_2}}}{ds} &= \vec{\omega} \times \vec{\mathbf{b_3}}, \\ \frac{d\vec{\mathbf{b_3}}}{ds} &= \vec{\omega} \times \vec{\mathbf{b_3}}. \end{aligned} \tag{2.12}$$

Since $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$ comprise a basis for R^3 we can write

$$\vec{\omega} = \omega_1 \vec{\mathbf{b_1}} + \omega_2 \vec{\mathbf{b_2}} + \omega_3 \vec{\mathbf{b_3}}, \tag{2.13}$$

thus equations 2.12 becomes

$$\begin{aligned} \frac{d\vec{\mathbf{b_1}}}{ds} &= -\omega_2 \vec{\mathbf{b_3}} + \omega_3 \vec{\mathbf{b_2}}, \\ \frac{d\vec{\mathbf{b_2}}}{ds} &= \omega_1 \vec{\mathbf{b_3}} - \omega_3 \vec{\mathbf{b_1}}, \\ \frac{d\vec{\mathbf{b_3}}}{ds} &= -\omega_1 \vec{\mathbf{b_2}} + \omega_2 \vec{\mathbf{b_1}}. \end{aligned} \tag{2.14}$$

Comparing equations 2.14 with equations 2.9 we have

$$\begin{aligned} \omega_1 &= \kappa_2, \\ \omega_2 &= 0, \\ \omega_3 &= \kappa_1. \end{aligned} \tag{2.15}$$

thus the Darbuox vector for type-3 Bishop frame is given by

$$\vec{\omega} = \kappa_2 \vec{\mathbf{b_1}} + \kappa_1 \vec{\mathbf{b_3}}. \tag{2.16}$$

Thus when the triad evolves along the curve the $(\vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$-couple rotates around $(\vec{\mathbf{b_1}}$ with an angular velocity κ_2 and the $(\vec{\mathbf{b_2}}, \vec{\mathbf{b_1}})$-couple rotates around $(\vec{\mathbf{b_3}}$ with an angular velocity κ_1.

2.1. **Example 1.** In this subsection, first we show how to find type-3 Bishop trihedra for a space curve,

$$\vec{\mathbf{r}} = (s\cos(s), s\sin(s), s). \tag{2.17}$$

The Frenet frame and curvatures are calculated and given by

$$\begin{cases} \vec{\mathbf{t}} = (\frac{\cos s - s\sin s}{\sqrt{s^2+2}}, \frac{\cos s + s\sin s}{\sqrt{s^2+2}}, \frac{1}{\sqrt{s^2+2}}) \\ \vec{\mathbf{n}} = (\frac{-(s^2+4)\sin s - (s^3+3s)\cos s}{\sqrt{(s^2+2)(s^4+5s^2+8)}}, \frac{(s^2+4)\sin s - (s^3+3s)\cos s}{\sqrt{(s^2+2)(s^4+5s^2+8)}}, \frac{-s}{\sqrt{(s^2+2)(s^4+5s^2+8)}}) \\ \vec{\mathbf{b}} = (\frac{s\sin s - 2s\cos s}{(s^4+5s^2+8)}, \frac{-2\sin s + 2s\cos s}{(s^4+5s^2+8)}, \frac{s^2+2}{(s^4+5s^2+8)}) \\ \kappa = \sqrt{\frac{8+5s^2+s^4}{(2+s^2)}} \\ \tau = \frac{(6+s^2)^{3/2}}{8+5s^2+s^4} \end{cases} \tag{2.18}$$

Type-3 Bishop frame and curvatures are calculated and given by

$$\begin{cases} \vec{\mathbf{b_1}} = ((\frac{\cos s - s\sin s}{\sqrt{s^2+2}})\cos\rho + (\frac{s\sin s - 2s\cos s}{(s^4+5s^2+8)})\sin\rho, \\ (\frac{\cos s + s\sin s}{\sqrt{s^2+2}})\cos\rho + (\frac{-2\sin s + 2s\cos s}{(s^4+5s^2+8)})\sin\rho, \\ (\frac{1}{\sqrt{s^2+2}})\cos\rho + (\frac{s^2+2}{(s^4+5s^2+8)})\sin\rho), \\ \\ \vec{\mathbf{b_2}} = (\frac{\cos s + s\sin s}{\sqrt{s^2+2}}, \frac{-\cos s + s\sin s}{\sqrt{s^2+2}}, 0), \\ \\ \vec{\mathbf{b_3}} = (\frac{s\sin s - 2s\cos s}{(s^4+5s^2+8)})\cos\rho + (\frac{\cos s - s\sin s}{\sqrt{s^2+2}})\sin\rho, \\ (\frac{-2\sin s + 2s\cos s}{(s^4+5s^2+8)})\cos\rho + (\frac{\cos s + s\sin s}{\sqrt{s^2+2}})\sin\rho, \\ (\frac{s^2+2}{(s^4+5s^2+8)})\cos\rho + (\frac{1}{\sqrt{s^2+2}})\sin\rho), \\ \\ \kappa_1 = \sqrt{\frac{8+5s^2+s^4}{(2+s^2)}}\cos\rho + \frac{(6+s^2)^{3/2}}{8+5s^2+s^4}\sin\rho, \\ \\ \kappa_2 = \sqrt{\frac{8+5s^2+s^4}{(2+s^2)}}\sin\rho + \frac{(6+s^2)^{3/2}}{8+5s^2+s^4}\cos\rho, \end{cases} \tag{2.19}$$

The geometric visualization for both Frenet frame and type-3 Bishop frame are displayed in figure 1.

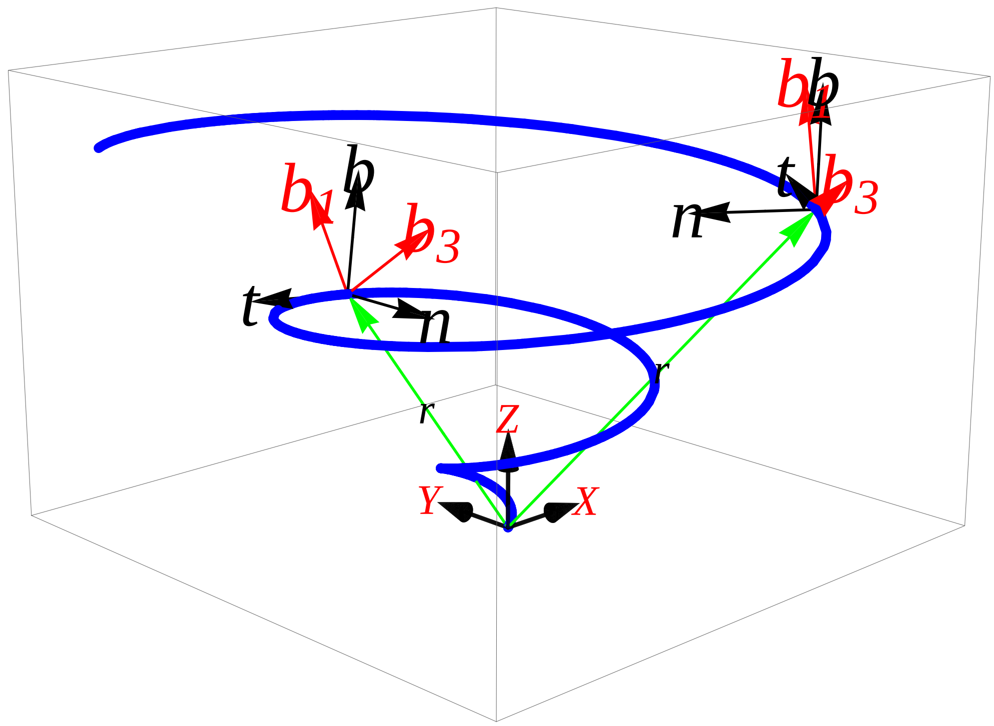

FIGURE 1. Frenet frame and type-3 Bishop frame at two point along the curve defended by the position vector $\vec{\mathbf{r}}(s) = (s\cos(s), s\sin(s), s)$.

3. Evolution of a space curve with time by Bishop frame

In this section we study the evolution of a regular space curve using Bishop frame. We derive time evolution equation for Bishop frame and Bishop curvatures.

Let, the curve evolving in the space according to

$$\frac{\partial \vec{\mathbf{r}}}{\partial t} = \mu_1 \vec{\mathbf{b_1}} + \mu_2 \vec{\mathbf{b_2}} + \mu_3 \vec{\mathbf{b_3}}, \tag{3.1}$$

where μ_1, μ_2, and μ_3 are functions of s and t, correspond to the normal, binormal and tangent projections of the velocity. Below we restrict our attention to a purely local form for these velocities as in the form

$$\begin{aligned}\mu_1 &= \mu_1(\kappa, \kappa_s, ..., \tau, \tau_s, ...),\\ \mu_2 &= \mu_2(\kappa, \kappa_s, ..., \tau, \tau_s, ...),\\ \mu_3 &= \mu_3(\kappa, \kappa_s, ..., \tau, \tau_s, ...).\end{aligned}$$

If the curve evolves in time t, then the general temporal evolution in which the triad $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$ remains orthonormal adopts the form [18]

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{\mathbf{b_1}}\\ \vec{\mathbf{b_2}}\\ \vec{\mathbf{b_3}}\end{pmatrix} = \begin{pmatrix}0 & \gamma_1 & 0\\ -\gamma_1 & 0 & \gamma_2\\ 0 & -\gamma_2 & 0\end{pmatrix}\begin{pmatrix}\vec{\mathbf{b_1}}\\ \vec{\mathbf{b_2}}\\ \vec{\mathbf{b_3}}\end{pmatrix}, \tag{3.2}$$

where γ_1, γ_2 and γ_3 are geometric parameters which are in general functions of s and t. These describe the evolution in t of the type-3 Bishop Serret-Frenet frame $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$ on the curve.

From section 2 the rate of change of type-3 Bishop Serret-Frenet frame, is given by

$$\frac{\partial}{\partial s}\begin{pmatrix}\vec{\mathbf{b_1}}\\ \vec{\mathbf{b_2}}\\ \vec{\mathbf{b_3}}\end{pmatrix} = \begin{pmatrix}0 & \kappa_1 & 0\\ -\kappa_1 & 0 & \kappa_2\\ 0 & -\kappa_2 & 0\end{pmatrix}\begin{pmatrix}\vec{\mathbf{b_1}}\\ \vec{\mathbf{b_2}}\\ \vec{\mathbf{b_3}}\end{pmatrix}. \tag{3.3}$$

Applying the compatibility condition

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{pmatrix}\vec{\mathbf{b_1}}\\ \vec{\mathbf{b_2}}\\ \vec{\mathbf{b_3}}\end{pmatrix} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{pmatrix}\vec{\mathbf{b_1}}\\ \vec{\mathbf{b_2}}\\ \vec{\mathbf{b_3}}\end{pmatrix}, \tag{3.4}$$

a short calculation using Eqs. 3.2, 3.3 and 3.4 leads to

$$\begin{aligned}\frac{\partial\kappa_1}{\partial t} &= \frac{\partial\gamma_1}{\partial s},\\ \frac{\partial\kappa_2}{\partial t} &= \frac{\partial\gamma_2}{\partial s},\\ 0 &= \gamma_2\kappa_1 - \gamma_1\kappa_2.\end{aligned} \tag{3.5}$$

Applying the compatibility condition,

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\vec{\mathbf{r}} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\vec{\mathbf{r}} \tag{3.6}$$

yields,

$$\begin{aligned}
\frac{\partial \mu_1}{\partial s} &= \mu_2\kappa_1, \\
\frac{\partial \mu_2}{\partial s} &= \gamma_1 - \mu_1\kappa_1 + \mu_3\kappa_2, \qquad (3.7)\\
\frac{\partial \mu_3}{\partial s} &= \kappa_2\mu_2.
\end{aligned}$$

For a given velocity vector (μ_1, μ_2, μ_3), equations 3.5 and 3.7 form a set of 6 nonlinear first order partial differential equations which governing the evolution of the type-3 Bishop curvatures of the evolving curve in the space.

Equation 3.5 together gives

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial t} &= \frac{\partial \gamma_1}{\partial s}, \\
\frac{\partial \kappa_2}{\partial t} &= \gamma_1 \frac{\partial}{\partial s}\left(\frac{\kappa_2}{\kappa_1}\right) + \frac{\kappa_2}{\kappa_1}\frac{\partial \gamma_1}{\partial s}.
\end{aligned} \qquad (3.8)$$

From equation 3.7 γ_1 is given by

$$\gamma_1 = \frac{\partial \mu_2}{\partial s} + \mu_1\kappa_1 - \mu_3\kappa_2, \qquad (3.9)$$

$$\frac{\partial \gamma_1}{\partial s} = \frac{\partial^2 \mu_2}{\partial s^2} + \frac{\partial \mu_1}{\partial s}\kappa_1 + \mu_1\frac{\partial \kappa_1}{\partial s} - \frac{\partial \mu_3}{\partial s}\kappa_2 - \mu_3\frac{\partial \kappa_2}{\partial s}, \qquad (3.10)$$

using first and third equation in the system 3.7 and substituting in 3.10, we have

$$\frac{\partial \gamma_1}{\partial s} = \frac{\partial^2 \mu_2}{\partial s^2} + \mu_2\kappa_1^2 + \mu_1\frac{\partial \kappa_1}{\partial s} - \mu_2\kappa_2^2 - \mu_3\frac{\partial \kappa_2}{\partial s}. \qquad (3.11)$$

By substitution from 3.9 and 3.11 into 3.8 , the time evolution equation of type-3 Bishop curvatures κ_1 and κ_2 are given by,

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial t} &= \frac{\partial^2 \mu_2}{\partial s^2} + \mu_2\kappa_1^2 + \mu_1\frac{\partial \kappa_1}{\partial s} - \mu_2\kappa_2^2 - \mu_3\frac{\partial \kappa_2}{\partial s}, \\
\frac{\partial \kappa_2}{\partial t} &= \left(\frac{\partial \mu_2}{\partial s} + \mu_1\kappa_1 - \mu_3\kappa_2\right)\frac{\partial}{\partial s}\left(\frac{\kappa_2}{\kappa_1}\right) + \frac{\kappa_2}{\kappa_1}\frac{\partial}{\partial s}\left(\frac{\partial^2 \mu_2}{\partial s^2} + \mu_2\kappa_1^2 + \mu_1\frac{\partial \kappa_1}{\partial s} - \mu_2\kappa_2^2 - \mu_3\frac{\partial \kappa_2}{\partial s}\right),
\end{aligned} \qquad (3.12)$$

while the time evolution of type-3 Bishop frame $(\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, \vec{\mathbf{b_3}})$ on the curve is given by,

$$\begin{aligned}
\frac{\partial \vec{\mathbf{b_1}}}{\partial t} &= (\frac{\partial \mu_2}{\partial s} + \mu_1\kappa_1 - \mu_3\kappa_2)\vec{\mathbf{b_2}}, \\
\frac{\partial \vec{\mathbf{b_2}}}{\partial t} &= -(\frac{\partial \mu_2}{\partial s} + \mu_1\kappa_1 - \mu_3\kappa_2)\vec{\mathbf{b_1}} + \frac{\kappa_2}{\kappa_1}(\frac{\partial \mu_2}{\partial s} + \mu_1\kappa_1 - \mu_3\kappa_2)\vec{\mathbf{b_2}}, \\
\frac{\partial \vec{\mathbf{b_3}}}{\partial t} &= -\frac{\kappa_2}{\kappa_1}(\frac{\partial \mu_2}{\partial s} + \mu_1\kappa_1 - \mu_3\kappa_2)\vec{\mathbf{b_2}}.
\end{aligned} \tag{3.13}$$

3.1. **model 1.** If, the velocity vector $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ is given by

$$\begin{aligned}
\mu_1 &= 0, \\
\mu_2 &= 0, \\
\mu_3 &= 1,
\end{aligned} \tag{3.14}$$

, then type-3 Bishop curvatures evolves according to

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial t} &= -\frac{\partial \kappa_2}{\partial s}, \\
\frac{\partial \kappa_2}{\partial t} &= -\kappa_2 \frac{\partial}{\partial s}(\frac{\kappa_2}{\kappa_1}) - \frac{\kappa_2}{\kappa_1} - \frac{\partial \kappa_2}{\partial s}
\end{aligned} \tag{3.15}$$

On solution of the above system is given by

$$\begin{aligned}
\kappa_1(s,t) &= \frac{-c_1}{c_2}(c_4 + c_5 \tanh(c_1 s + c_2 t + c_3)), \\
\kappa_2(s,t) &= c_4 + c_5 \tanh(c_1 s + c_2 t + c_3).
\end{aligned} \tag{3.16}$$

If we have the curvature and the torsion of a space curve as a functions of arc-length parameter, then by integrating Serret-Frenet we can reconstruct the curve up to its position in the space and this is an immediate consequence of the of the fundamental existence theorem for space curves[17]. Similarly, if we have the Bishob curvatures κ_1 and κ_2, then we can reconstruct the curve in the space via the integration of type-3 Bishop frame.

The figure 2 represent curve evolving in the space determined by $\kappa_1 = -\tanh(s+t)$ and $\kappa_2 = +\tanh(s+t)$ obtained by solving the type-3 Bishop Serret Frenet equations 2.9 for a specified κ_1 and κ_1 using Mathematica [22]. At every fixed t, we clearly have a representation of the corresponding (static) space curve at that instant. The program [22] generates static space curves corresponding to Serret-Frenet equations. It was extended slightly to generate space curves corresponding to type-3 Bishop Serret Frenet equations.

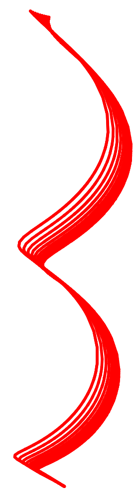

Figure 2. Curve evolving in the space determined by $\kappa_1 = -\tanh(s+t), \kappa_2 = +\tanh(s+t)$

4. Differential geometry of surfaces

To meet the requirements in the next sections, here, the basic elements of the theory of surfaces in the space E^3 are briefly presented a more complete elementary treatment can be found in [21].

We consider a surface imbedded in 3–dimensional Euclidean space $\mathbf{R}^3$, we denote local coordinates of the surface by (u^1, u^2). The surface is specified by the position vector $\mathbf{r}(u^1, u^2)$ where $u^1 = s$, $u^2 = t$. The metric on the surface is given by,

$$g_{\mu\nu} = \vec{\mathbf{r}}_\mu \cdot \vec{\mathbf{r}}_\nu \quad \mu, \nu = 1, 2, \tag{4.1}$$

we use the Einstein's convention for summation. Here $\vec{\mathbf{r}}_\nu$ is the tangent vector to the surface,

$$\vec{\mathbf{r}}_\mu = \frac{\partial \vec{\mathbf{x}}}{\partial u^\mu}, \quad \mu = 1, 2. \tag{4.2}$$

We denote the inverse of $g_{\mu\nu}$ by $g^{\mu\nu}$. At regular points where the tangent vectors $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent, we can define the unit normal vector $\mathbf{N}$ to the surface,

$$\mathbf{N} = \frac{\vec{\mathbf{x}}_1 \wedge \vec{\mathbf{x}}_2}{|\vec{\mathbf{x}}_1 \wedge \vec{\mathbf{x}}_2|}. \tag{4.3}$$

These vectors are related by the Gauss–Weingerten equations [17],

$$\begin{aligned}\frac{\partial}{\partial u^{\nu}}\vec{\mathbf{x}}_{\mu} &= \vec{\mathbf{x}}_{\lambda}\Gamma^{\lambda}_{\mu\nu} + \vec{\mathbf{N}}L_{\mu\nu},\\ \frac{\partial \vec{\mathbf{N}}}{\partial u^{\nu}} &= -\vec{\mathbf{x}}_{\lambda}g^{\lambda\mu}L_{\mu\nu}.\end{aligned} \tag{4.4}$$

In the above, the Christoffel's symbols $\Gamma^{\lambda}_{\mu\nu}$ and the second fundamental form are defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\frac{\partial}{\partial u^{\mu}}g_{\rho\nu} + \frac{\partial}{\partial u^{\nu}}g_{\mu\rho} - \frac{\partial}{\partial u^{\rho}}g_{\mu\nu}), \tag{4.5}$$

$$L_{\mu\nu} = \frac{\partial \vec{\mathbf{x}}_{\mu}}{\partial u^{\nu}} \cdot \vec{\mathbf{N}}. \tag{4.6}$$

The Gaussian curvature κ_g and the mean curvature κ_m are given by

$$\kappa_g = det(g^{\mu\nu}L_{\nu\lambda}) = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{4.7}$$

$$\kappa_m = \frac{1}{2}tr(g^{\mu\nu}L_{\nu\lambda}) = \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)}. \tag{4.8}$$

We recall that a curve in E^3 is uniquely determined by two quantities, curvature and torsion, as functions of arc length. Similarly, a surface in E^3 is uniquely determined by certain local invariant quantities called the first and second fundamental forms 4.1 and 4.1.

5. Surfaces generated by the evolution of a space curve using Bishob frame

There are a catalog of surfaces that can be described by integrable equations such as surfaces of constant negative Gaussian curvature, surfaces of constant mean curvature, minimal surfaces, affine spheres and Hasimoto surfaces which generated by evolving a regular space curve $\vec{\mathbf{r}} = \vec{\mathbf{r}}(s,t)$ by

$$\frac{\partial \vec{\mathbf{r}}}{\partial t} = \kappa\vec{\mathbf{b}}, \tag{5.1}$$

this is an evolution of the curve in its binormal direction with velocity equal to its curvature. Eq.5.1 Known as the vortex filament flow or localized induction equation (LIE). Here, $\vec{\mathbf{r}}(s,t)$ is a position vector for a point on the curve, t is the time, s is the arc-length parameter, κ is the curvature of $\vec{\mathbf{x}}(s,t)$, $\vec{\mathbf{b}}$ is the unit binormal. The authors in [19] constructed Hasimoto surfaces via integration for Frenet–Serret equations while the authors in [20] constructed Hasimoto surface from fundamental form coefficients via numerical integration of Gauss-Weingarten equations.

The present work continues the program by adding new surfaces to this catalog. These surfaces are obtained by evolving a regular space curve $\vec{\mathbf{r}}(s,t)$ in R^3 as it evolves over time according to the following evolution equation:

$$\frac{\partial \vec{\mathbf{r}}}{\partial t} = \mu_1 \vec{\mathbf{b_1}} + \mu_2 \vec{\mathbf{b_2}} + \mu_3 \vec{\mathbf{b_3}} \tag{5.2}$$

where the set $\{\lambda, \mu, \nu\}$ is the velocities. This our motivation in this paper.

The tangent space to S at an arbitrary point $P = \vec{\mathbf{r}}(s,t)$ of S is spanned by

$$\begin{aligned} \frac{\partial \vec{\mathbf{r}}}{\partial s} &= \vec{\mathbf{t}}, \\ \frac{\partial \vec{\mathbf{r}}}{\partial t} &= \lambda_1 \vec{\mathbf{b_1}} + \lambda_2 \vec{\mathbf{b_2}} + \lambda_3 \vec{\mathbf{b_3}}. \end{aligned} \tag{5.3}$$

Thus the moving frame along the surface is calculated and given by

$$\begin{pmatrix} \frac{\partial \vec{\mathbf{r}}}{\partial s} \\ \frac{\partial \vec{\mathbf{r}}}{\partial s} \\ \vec{\mathbf{N}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 \\ 0 & -\frac{\mu_3}{\sqrt{\mu_2+\mu_3^2}} & \frac{\mu_2}{\sqrt{\mu_2+\mu_3^2}} \end{pmatrix} \begin{pmatrix} \vec{\mathbf{b_1}} \\ \vec{\mathbf{b_2}} \\ \vec{\mathbf{b_3}} \end{pmatrix} \tag{5.4}$$

The coefficient of the first fundamental form and fundamental metric are calculated and given by

$$\begin{cases} g_{11} = 1, \\ g_{12} = \mu_1, \\ g_{22} = \mu_1^2 + \mu_2^2 + \mu_3^2, \\ g = \mu_2^2 + \mu_3^2. \end{cases} \tag{5.5}$$

A short calculation shows that the coefficients of the seconde fundamental form are given as

$$\begin{cases} L_{11} = \frac{-\mu_3\kappa_1+\mu_2\kappa_2}{\sqrt{\mu_2^2+\mu_3^2}}, \\ L_{12} = \frac{-\gamma_1\mu_3}{\sqrt{\mu_2^2+\mu_3^2}}, \\ L_{22} = \frac{\gamma_2\mu_2^2-\gamma_1\mu_1\mu_3+\gamma_2\mu_3^2-\frac{\partial\mu_2}{\partial t}\mu_3+\mu_2\frac{\partial\mu_3}{\partial t}}{\sqrt{\mu_2^2+\mu_3^2}} \\ L = \frac{1}{\mu_2^2+\mu 3^2}(-\gamma_1^2\mu 3^2 - \gamma_2^2\mu_2^2\mu_3\kappa_1 + \gamma_1\mu_1\mu_3^2\kappa_1 - \gamma_2\mu_3^3\kappa_1 - \gamma_2\mu_3\kappa_1 \\ +\mu_3^2\kappa_1\frac{\partial\mu_2}{\partial t} - \mu_2\mu_3\kappa_1\frac{\partial\mu_2}{\partial t}). \end{cases} \tag{5.6}$$

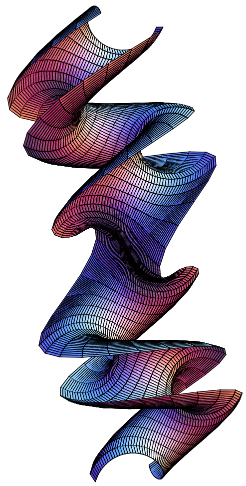

FIGURE 3. surface genrated by the curve evolving in the space with the curvatures, $\kappa_1 = -\tanh(s+t), \kappa_2 = \tanh(s+t)$ and fundamental forms 5.8

.

For the curve evolving in space by the velocity vector $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$

$$\begin{aligned} \mu_1 &= 0, \\ \mu_2 &= 0, \\ \mu_3 &= 1, \end{aligned} \tag{5.7}$$

the coefficient of the first and seconde fundamental forms are given by,

$$\begin{cases} g_{11} = 1, \\ g_{12} = 0, \\ g_{22} = 1, \\ L_{11} = \tanh(s+t), \\ L_{12} = -\tanh(s+t), \\ L_{22} = -\tanh(s+t). \end{cases} \tag{5.8}$$

Thus, the Gaussian curvature κ_g and the mean curvature κ_m are

$$\begin{aligned} \kappa_g &= -2\tanh(s+t)^2, \\ \kappa_m &= 0, \end{aligned} \tag{5.9}$$

and the principle curvatures are

$$\begin{aligned}\kappa_{11} &= \sqrt{2}\tanh(s+t) = \sqrt{2}\kappa_2,\\ \kappa_{22} &= -\sqrt{2}\tanh(s+t) = \sqrt{2}\kappa_1.\end{aligned} \tag{5.10}$$

The surface in figure 3 is obtained via numerical integration of the system of Gauss-Weingarten equation 4.4 with the coefficient 5.8 using mathematica [23].

6. Main result

In this paper, Firstly, we introduce a new version of Bishop frame using a common vector field as normal vector field of a regular curve and call this frame as type-3 Bishop frame. Geometric visualization for both Frenet frame and type-3 Bishop frame are plotted. Secondly, time evolution equation for type-3 Bishop frame and type three bishop curvatures are obtained and we plotted the evolving curve via the numerical integration of type-3 Bishop Serret Frenet equations. Finally, surfaces generated by evolving a regular space curve in R^3 as it evolves over time with arbitrary velocities attached to type-3 Bishop frame are obtained. We plotted these surfaces via numerical integration of Gauss-Weingarten equations.

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References

[1] L.S. Da Rios, Sul moto d'un liquido indefinito con un filetto vorticoso, *Rend. Circ. Mat. Palermo.* **22** *(1906) 117-135.*

[2] G.L. Lamb Jr., Solitons on moving space curves, *J. Math. Phys.* **18***(1977), 1654-1661.*

[3] A. Doliwa and P. Santini, An elementary geometric characterisation of the integrable motions of a curve, *Phys. Lett. A* **185***(1994), 373-384.*

[4] J. Langer and R. Perline, Poisson geometry of the filament equation, *J. Nonlinear Sci.* **1***(1991), 71-93.*

[5] M. Lakshmanan, Th.W. Ruijgrok and C.J. Thompson, On the dynamics of a continuum spin system, *Physica A,* **84***(1976), 577-590.*

[6] A. Doliwa and P.M. Santini, An elementary geometric characterization of the integrable motions of a curve, *Phys. Lett. A* **185***(1994), 373-384.*

[7] Kazuaki Nakayama, Takeshi Iizuka, and Miki Wadati, Curve Lengthening Equation and Its Solutions, *J. Phys. Soc. Jpn.* **63***(1994), 1311-1321.*

[8] Kazuaki Nakayama and Miki Wadati, Motion of Curves in the Plane, *J. Phys. Soc. Jpn.* **62***(1993), 473-479.*

[9] Kiziltug, S., On Characterization of Inextensible Flows of Curves According to Type-2 Bishop Frame E^3. *Math. Comput. Appl.* **19***(2014), 69-77.*

[10] R. L. Bishop, There is more than one way to frame a curve,*Amer. Math. Monthly.* **82***(1975), 246-251.*

[11] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, *Dover Pub., Inc., New York, (1909).*

[12] M. do Carmo, Differential Geometry of Curves and Surfaces, *Prentice-Hall, Englewood Cliffs, 1976.*

[13] S. ülmaz, M. Turgut, A new version of Bishop frame and an application to spherical images, *J. Math. Anal. Appl.* **371***(2010), 764-776.*

[14] J. Bloomenthal, Calculation of Reference Frames Along a Space Curve, Graphics Gems, *Academic Press Professional, Inc., San Diego, CA. 1990.*

[15] Baur O. and E.W. Grafarend , Orbital rotations of a satellite case study, *GOCE, Artificial Satellites.* **40***(2005), 87-107.*

[16] M. Dede, C. Ekici and H. Tozak, Directional tubular surfaces, *International Journal of Algebra.* **9***(2015) 527-535.*

[17] M. M. Lipschutz, Theory and problems of Differential Geometry, *Schaum's Outline Series, 1969.*

[18] C. Rogers and W. K. Schief, Backlund and Darboux Transformations Geometry and Modern Application in Soliton Theory, *Cambridge University press, Cambridge, 2002.*

[19] Peter J. VassiliouIan and G. Lisle, Geometric approaches to differential equations, *Cambridge University Press 2000.*

[20] Nassar H. Abdel-All, R. A. Hussien and Taha Youssef, Hasimoto Surfaces, *Life Science Journal***3***(2012), 1170-1185.*

[21] J.J.Stoker, Differential Geometry, *Wiley-Interscience, New York, 1969.*

[22] A. Gray, Modern Differential Geometry of Curves and Surfaces with Mathematica, *CRC, New York, 1998.*

[23] Yoshihiko Tazawa, Introduction to the theory of curves and surfaces with Mathematica, *Pearson Education Japan, 1999.*