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$\oplus$-supplemented modules with respect to images of a fully invariant submodule

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Abstract. Lifting modules and their various generalizations as some main concepts in module theory have been studied and investigated extensively in recent decades. Some authors tried to present some homological aspects of lifting modules and $\oplus$-supplemented modules. In this work, we shall present a homological approach to $\oplus$-supplemented modules via fully invariant submodules. Lifting modules and H-supplemented modules with respect to images of a fixed fully invariant submodule of a module where investigated in first author's last works. We intend here to introduce and study a module M such that $\varphi(F)$ has a supplement as a direct summand of M, for every endomorphism φ of M where F is a fixed fully invariant submodule of M.

Keywords: $\oplus$-supplemented module, E-$\oplus$-supplemented module, $\mathcal{I}_F$-lifting module, $\mathcal{I}_F$-$\oplus$-supplemented module, endomorphisms ring.

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1. Introduction

All rings considered in this paper are associative with an identity element and all modules are unitary right modules unless otherwise stated.

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Let R be a ring and M an R-module. Then $S = End_R(M)$ will denote the ring of all R-endomorphisms of M. We use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M$, we have $L + N \neq M$). A module M is called *hollow* if every proper submodule of M is small in M. The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\forall \phi \in End_R(M)$, $\phi(N) \subseteq N$). $Rad(M)$ and $Soc(M)$ denote the radical and the socle of a module M, respectively.

Let M be a module and L, K be submodules of M. We say that K *lies above* L in M, if $K/L \ll M/L$. A module M is called *lifting* , provided every submodule K of M lies above a direct summand L of M ([3]). A submodule K of M is said to be a *supplement* of L in M, if $M = K + L$ and $K \cap L \ll K$. Based on this definition, M is said to be *(⊕-)supplemented* provided every submodule of M has a supplement in M(which is a direct summand of M).

Recently, Lee, Rizvi and Roman introduced the notion of Rickart modules and dual Rickart modules in [5, 6]. Let M be a module. Then M is called *(dual) Rickart* in case for every endomorphism φ of M, $(Im\varphi)$ $Ker\varphi$ is a direct summand of M. (Dual) Rickart modules determine the importance of idempotents in the ring of all endomorphisms of a module. In particular as a famous result, a module M is Rickart and dual Rickart if and only if $End_R(M)$ is a von Neumann regular ring. A homological approach related to lifting modules was presented in [1]. Amouzegar introduced a generalization of dual Rickart modules as $\mathcal{I}$-lifting modules. A module is said to be *$\mathcal{I}$-lifting* in case Imf lies above a direct summand D of M in M, for every endomorphism f of M. The author in [1], investigated some properties of $\mathcal{I}$-lifting modules. After introducing a homological version of lifting modules, that was natural to define the similar case for H-supplemented modules. So, we can see a homological approach to H-supplemented modules in [8]. A module M is said to be *E-H-supplemented* provided, for every endomorphism φ of M there is a direct summand D of M such that $Im\varphi + X = M$ if and only if $D + X = M$, for every submodule X of M.

Recently, Amouzegar and the first author in [2] introduced a generalization of both lifting modules and dual Rickart modules via fully invariant submodules which they called them $\mathcal{I}_F$-lifting modules. Let M be a module and F a fixed fully invariant submodule of M. Then M is called *$\mathcal{I}_F$-lifting* if image of every endomorphism of M restricted to F, lies above a direct summand of M in M. It is shown that, if M is a projective $\mathcal{I}_{Rad(M)}$-lifting module, then $Rad(M) \ll M$. There is another work that will be helpful here. Moniri and Amouzegar in [7] introduced a new generalization of both H-supplemented modules and

$\mathcal{I}_F$-lifting modules entitled as $\mathcal{I}_F$-H-supplemented modules. A module M is called *$\mathcal{I}_F$-H-supplemented* provided for every endomorphism φ of M, there is a direct summand D of M such that $\varphi(F) + X = M$ if and only if $D + X = M$ for every submodule X of M. In [4], the authors introduced a homological aspect of $\oplus$-supplemented modules namely E-$\oplus$-supplemented modules. A module M is E-$\oplus$-supplemented provided the image of every endomorphism of M has a supplement which is a direct summand of M. It is not hard to check that every $\mathcal{I}$-lifting module is E-H-supplemented and any E-H-supplemented module is E-$\oplus$-supplemented, as well.

Motivating by mentioned works on lifting modules and $\oplus$-supplemented modules via a homological approach, we are interested to study on E-$\oplus$-supplemented modules via images of a fixed fully invariant submodule. In fact, in the definition of a E-$\oplus$-supplemented module M one can replace image of M by image of a fully invariant submodule of M. We say a module M is *$\mathcal{I}_F$-$\oplus$-supplemented* provided for every endomorphism φ of M, there is a direct summand D of M such that $\varphi(F) + D = M$ and $\varphi(F) \cap D \ll D$. In what follows by F, we mean a fully invariant submodule of M.

In Section 2, we present some examples of $\mathcal{I}_F$-$\oplus$-supplemented modules and study some properties of these modules. We show that a module M is $\mathcal{I}_F$-$\oplus$-supplemented if and only if F is E-$\oplus$-supplemented for a fully invariant direct summand F of M. Some characterizations of $\mathcal{I}_F$-$\oplus$-supplemented are provided. We also discuss about homomorphic images of $\mathcal{I}_F$-$\oplus$-supplemented modules.

2. $\mathcal{I}_F$-$\oplus$-supplemented modules

In this section we introduce and study a generalization of $\oplus$-supplemented modules via image of fully invariant submodules.

Definition 2.1. Let M be a module and F be a fully invariant submodule of M. We say M is *$\mathcal{I}_F$-$\oplus$-supplemented* if for every $\varphi \in End_R(M)$, the submodule $\varphi(F)$ has a supplement which is a direct summand of M.

It is clear that every $\oplus$-supplemented module is $\mathcal{I}_F$-$\oplus$-supplemented. Obviously, M is $\mathcal{I}_M$-$\oplus$-supplemented if and only if M is E-$\oplus$-supplemented. Note that every module M is clearly $\mathcal{I}_0$-$\oplus$-supplemented.

The following contains some examples of $\mathcal{I}_F$-$\oplus$-supplemented modules.

Example 2.2. (1) Let F be a fully invariant submodule of a module M. If F is small in M, then M is $\mathcal{I}_F$-$\oplus$-supplemented. In particular, every hollow module M is $\mathcal{I}_F$-$\oplus$-supplemented for every fully invariant submodule F of M. For example the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ is $\mathcal{I}_{(1/p+\mathbb{Z})}$-$\oplus$-supplemented. Note that $Soc(M) = (1/p + \mathbb{Z})$.

(2) Let p be a prime number. Then the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^2}$ is not a dual Rickart module. Now, $Rad(M) = (p) \neq 0$. Since M is a hollow module, M is $\mathcal{I}_{Rad(M)}$-$\oplus$-supplemented.

Example 2.2(1), provides a rich source of $\mathcal{I}_F$-$\oplus$-supplemented modules which are $\oplus$-supplemented.

Example 2.3. *Let M be a non-supplemented finitely generated module. Then by Example 2.2, M is $\mathcal{I}_{Rad(M)}$-$\oplus$-supplemented while M is not $\oplus$-supplemented (note that M is not supplemented). For instance, we may consider a non-semiperfect ring R which is $\mathcal{I}_{J(R)}$-$\oplus$-supplemented.*

Next, we shall present a characterization of $\mathcal{I}_F$-$\oplus$-supplemented modules when F is a direct summand of M.

Theorem 2.4. *Let M be a module and let F be a fully invariant direct summand of M. Then M is $\mathcal{I}_F$-$\oplus$-supplemented if and only if F is $\mathcal{I}$-$\oplus$-supplemented.*

Proof. ($\Rightarrow$) Let $g : F \to F$ be an endomorphism of F and $F \oplus F' = M$ for a submodule F' of M. Then $h = j \circ g \circ \pi_F : M \to M$ is an endomorphism of M where $j : F \to M$ is the inclusion and $\pi_K : M \to F$ is the projection map on F. It is straightforward that $h(F) = g(F)$. Since M is $\mathcal{I}_F$-$\oplus$-supplemented, there exists a decomposition $D \oplus D' = M$ of M such that $g(F) + D = M$ and $g(F) \cap D \ll D$. Now modularity implies $g(F) + (F \cap D) = F$. Note here that $F = (F \cap D \oplus (F \cap D')$ since F is fully invariant in M. It is easy to verify that $g(F) \cap D = g(F) \cap D \cap F \ll D \cap F$ as $F \cap D$ is a direct summand of M.

($\Leftarrow$) Let F be $\mathcal{I}$-$\oplus$-supplemented and f be an endomorphism of M. Consider $q = \pi_F \circ f \circ j : F \to F$, which is an endomorphism of F, where $j : F \to M$ is the inclusion and $\pi_F : M \to F$ is the projection on F. Being F a fully invariant submodule of M implies that $q(F) = f(F)$. As F is $\mathcal{I}$-$\oplus$-supplemented, there is a direct summand D of F (so that of M) such that $q(F) + D = F$ and $q(F) \cap D \ll D$. Set $D \oplus K = F$. So that $q(F) + D + F' = M$. Now it remains to prove that $q(F) \cap (D + F') \ll (D + F')$. As $q(F)$ and D are both contained in F, then $q(F) \cap (D + F') = q(F) \cap D$ which is a small submodule of $D + F'$ as well as D. Note also that $D + F'$ is a direct summand of M.

$\square$

Recall from [9] that an R-module M is *noncosingular (cosingular)* provided $\overline{Z}(M) = M$ ($\overline{Z}(M) = 0$) where $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U\}$ for all small R-modules U.

Corollary 2.5. (1) *Let M be a module such that $\overline{Z}(M)$ is a weak duo direct summand of M. Then M is $\mathcal{I}_{\overline{Z}(M)}$-$\oplus$-supplemented if and only if $\overline{Z}(M)$ is dual Rickart.*

(2) *Let M be a module such that $Soc(M)$ is a direct summand of M. Then M is $\mathcal{I}_{Soc(M)}$-$\oplus$-supplemented.*

Proof. (1) Let M be $\mathcal{I}_{\overline{Z}(M)}$-$\oplus$-supplemented. Then by Theorem 2.4, the submodule $\overline{Z}(M)$ is E-$\oplus$-supplemented. Note that since $\overline{Z}(M)$ is a direct summand of M, it is noncosinsgular. The result follows from [2, Theorem 2.18].

(2) It is clear as $Soc(M)$ is semisimple. □

We shall present a characterization of $\mathcal{I}_F$-$\oplus$-supplemented modules with no nonzero small submodules.

Theorem 2.6. *Let M be a module with $Rad(M) = 0$ and let $F \leq M$ be fully invariant. Then the following statements are equivalent:*

(1) *M is $\mathcal{I}_F$-$\oplus$-supplemented;*

(2) *F is a dual Rickart direct summand of M.*

Proof. (1) $\Rightarrow$ (2) Let φ be an arbitrary endomorphism of M. Then there exists a direct summand D of M such that $\varphi(F) + D = M$ and $\varphi(F) \cap D \ll D$. Since $Rad(M) = 0$ we conclude that $Rad(D) = 0$ as D is a direct summand of M. Therefore, $\varphi(F) \cap D = 0$ showing that $\varphi(F)$ is a direct summand of M. It follows that F is a direct summand of M. From Theorem 2.4, F is E-$\oplus$-supplemented. Since $Rad(M) = 0$, we conclude from [2, Proposition 2.11] that F is a dual Rickart module.

(2) $\Rightarrow$ (1) It follows directly from Theorem 2.4 and the fact that every dual Rickart module is E-$\oplus$-supplemented. □

From the previous theorem, we conclude that over a right V-ring R, a right R-module M is $\mathcal{I}_F$-$\oplus$-supplemented if and only if F is a dual Rickart direct summand of M.

Example 2.7. (1) Let F be a field and $R = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for each $i \in \mathbb{N}$. Then R is a von Neumann regular V-ring. Take $M = R$ and F be any finitely generated ideal of R. So that F is a direct summand of M. It is well-known that M is a dual Rickart module (see [6, Remark 2.2]) and hence F as a direct summand is also dual Rickart (see [6, Proposition 2.8]). Hence, M is $\mathcal{I}_F$-$\oplus$-supplemented module by Theorem 2.6.

(2) Let L be an V-ring and K be a field. Then $S = K \times L$ is an V-ring as well. Consider the central idempotent $e = (1, 0)$ of S. Then $Se = eS \cong K$ as both left S-module and right S-module. Let R be the ring $M_n(S)$ (the ring of all $n \times n$ matrices with entries from S). As R is

Morita-equivalent to S, it should be also an V-ring. Now, R has a central idempotent, $f = eI$ where I is the identity matrix of R. Then $fR = Rf$ is isomorphic to $M_n(Se)$ so that $fR = Rf \cong M_n(K)$. Note that $F = Rf$ is a two-sided ideal of R and also is a direct summand of R. Being K a field implies that $M_n(K)$ and hence F is semisimple and so is dual Rickart. It follows from Theorem 2.6 that R is a $\mathcal{I}_F$-$\oplus$-supplemented module.

The following contains some properties of fully invariant submodules of modules. There may be same results in related works. We shall present them here for the sake of completeness.

Lemma 2.8. *Let M be a module. Then the following assertions hold:*

(1) *Let $F \leq M$ be fully invariant and let K be a direct summand of M. Then $F \cap K$ is a fully invariant submodule of K.*

(2) *Let $F \leq M$ be fully invariant and K a direct summand of M contained in F. Then F/K is a fully invariant submodule of M/K.*

Proof. (1) Let F be a fully invariant submodule of M and $K \leq^{\oplus} M$. Consider a decomposition $M = K \oplus K'$ and an arbitrary endomorphism g of K. Now $h = j \circ g \circ \pi_K$ is an endomorphism of M where $j : K \to M$ is the inclusion and $\pi_K : M \to K$ is the canonical projection. Since F is fully invariant in M, we have $h(F) \subseteq F$. It is easy to verify that $h(F) = g(F \cap K)$ (note that $F = (F \cap K) \oplus (F \cap K')$ implies $\pi_K(F) = F \cap K$). Therefore, $g(F \cap K)$ is contained in $F \cap K$ and so $F \cap K$ is a fully invariant submodule of K.

(2) Suppose that $g : M/K \to M/K$ is an endomorphism of M/K. Then $f = j \circ h \circ g \circ \pi : M \to M$ is an endomorphism of M. Here we should note that $\pi : M \to M/K$ is the canonical projection, $h : M/K \to K'$ is the isomorphism induced by the decomposition $M = K \oplus K'$ and $j : K' \to M$ is the inclusion map. Let $g(F/K) = T/K$ for a submodule T of M containing K. Now, $f(F) = T \cap K' \subseteq F$. Suppose that $t + K \in T/K$. Since $K \oplus (T \cap K') = T$, we conclude that $t + K = x + K$ such that $x \in T \cap K'$. It follows that $x \in F$ and so $g(F/K) = T/K \subseteq F/K$. □

Proposition 2.9. *Let M be a weak duo module, F be a fully invariant submodule of M and K a direct summand of M contained in F. If M is $\mathcal{I}_F$-$\oplus$-supplemented, then M/K is $\mathcal{I}_{F/K}$-$\oplus$-supplemented.*

Proof. Let $g : M/K \to M/K$ be an endomorphism of M/K and $M = K \oplus K'$. Then $f = j \circ h \circ g \circ \pi : M \to M$ is an endomorphism of M. Note that $\pi : M \to M/K$ is the canonical projection, $h : M/K \to K'$ is the isomorphism induced by the decomposition $M = K \oplus K'$ and $j : K' \to M$ is the inclusion map. By assuming $g(F/K) = T/K$, one can see $f(F) = T \cap K'$. Since M is $\mathcal{I}_F$-$\oplus$-supplemented, there is a direct summand D of M such that $(T \cap K') + D = M$ and $(T \cap K') \cap D \ll D$. Set $M = D \oplus D'$. Then $M/K = (D + K)/K + (D' + K)/K$. As K is a fully invariant submodule of M, we have $K = (K \cap D) \oplus (K \cap D')$. Hence $(D+K) \cap (D'+K) = K$ which implies that $(D+K)/K$ is a direct summand of M/K. We shall prove that $T/K + (K + D)/K = M/K$ and $[T/K \cap (K + D)/K] = [K + (T \cap D)]/K \ll (K + D)/K$. To verify the last assertion, let $[K + (T \cap D)]/K + L/K = (D + K)/K$. Then $(T \cap D) + L = K + D$. As D is a fully invariant submodule of M and $M = K \oplus K'$, then $(K \cap D) \oplus (K' \cap D) = D$. By modular law we have $D \cap T = (K \cap D) \oplus (K' \cap D \cap T)$. The equality $(T \cap D) + (L \cap D) = D$ with last observation leads us to $(K \cap D) + (L \cap D) = D$. Then $L \cap D = D$ which implies D is contained in L. Therefore, $L = D + K$ proving that $[K + (T \cap D)]/K \ll (K + D)/K$. Obviously $T/K + (D + K)/K = M/K$. □

The following presents a characterization of a $\mathcal{I}_F$-⊕-supplemented module M in terms of finitely generated ideals of $End_R(M)$.

Proposition 2.10. *Let M be a module and let F be a fully invariant submodule of M. Then M is $\mathcal{I}_F$-⊕-supplemented if and only if for every finitely generated right ideal I of $End_R(M)$, the submodule $\sum_{\varphi \in I} \varphi(F)$ of M has a supplement which is a direct summand of M.*

Proof. (⇒) Let M be $\mathcal{I}_F$-⊕-supplemented and $I = \langle f_1, \ldots, f_k \rangle$ a finitely generated right ideal of S. It is easy to check that $\sum_{f \in I} f(F) = [f_1(F) + \ldots + f_k(F)] \subseteq F$. Set $f = f_1 + \ldots + f_k$. Then $f(F) = \sum_{\varphi \in I} \varphi(F)$. Since M is $\mathcal{I}_F$-⊕-supplemented, there exists a direct summand D of M such that $\sum_{\varphi \in I} \varphi(F) + D = M$ and $\sum_{\varphi \in I} \varphi(F) \cap D \ll D$.

(⇐) Let $f \in S$. Consider the cyclic right ideal $I = \langle f \rangle$ of S. By assumption there is a direct summand D of M such that $\sum_{\varphi \in I} \varphi(F) + D = M$ and $\sum_{\varphi \in I} \varphi(F) \cap D \ll D$. It is not hard to verify that $f(F) = \sum_{\varphi \in I} \varphi(F)$. Now, the proof is completed.

□

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