Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(2)(2022), 368-380 (Research paper)

# Solving Defender-Attacker Game with Multiple Decision Makers Using Expected-Value Model

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Abstract. Defender-attacker game is a model for conflicting between a defender and an attacker. Defender tries to prevent attacking an opponent by assigning limited security resources. In real world the utility values of the defender-attacker game are assigned by experts which usually are uncertain. According to that the assigned values by several experts may be slightly different and conflicting, we consider a set of all their viewpoints. This approach is similar to hesitant fuzzy environment. Also, each of the experts may have the different weights; AHP method is used to determine the weights of each of the experts. A weighted sum method is applied to obtain a game with aggregated payoffs. An expected value of the fuzzy numbers is introduced to convert the problem into defender-attacker game with interval payoffs. According to this, we proposed a method to solve security game in fuzzy environment. It is shown that the optimal solution of the expected value model is the optimal solution of the original model. Finally, a practical example is illustrated to solve by the proposed method.

Keywords: Defender-attacker game, Fuzzy sets, Bilevel programming.

*2020 Mathematics subject classification:* 91A05

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Received: 10 February 2020
Revised: 31 March 2020
Accepted: 04 April 2020

## 1. Introduction

Due to limited security resources for allocating to different targets, game theory provides a mathematical approach. Research on these problems extensively made by Milind Tambe [23]. The utility values in these games were introduced by the experts of the relevant field. Fuzzy set theory is used to model decision making problems involving vagueness due to the lack of information and/or imprecision of the information on the problem situation. In fact, most of the real world problems that can be modeled as games have imprecise or vague information about its elements. Studies of fuzzy games have been made by incorporating fuzzy set theory. In the field of fuzzy games, considerable studies have been made (for example see [1, 3, 4, 5, 6, 7, 8, 9, 10, 18]). Also, the authors [6] proposed a method to solve security games with fuzzy payoffs. In these methods only single membership degree were used. In the real world, each of the experts may have different membership degrees due to incomplete information. One of the methods is to model in hesitane fuzzy environment. Torra [25] introduced the hesitant fuzzy sets (HFS) which permites the membership to have a set of possible values. Hesitant fuzzy set can reflect the human's hesitancy more objectively than the the other versions of fuzzy sets. Furtheremore, Torra introduced relationships and basic operations among HFS's using the concept of fuzzy set theory and its practical applications.

Recently, some researchers worked on the problems of game theory in hesitant fuzzy environment. For sample, Bhaumik and Roy [3] studied Intuitionistic interval-valued hesitant fuzzy matrix games with a new aggregation operator for solving management problem. Jana and Roy [17] considered Dual hesitant fuzzy matrix games based on new similarity measure.

In this paper, we first consider hesitant fuzzy sets and defender-attacker game. The model of defender-attacker game in hesitant fuzzy environment is proposed and a solution of games using an ordering method is introduced. The proposed method is illustrated by practical example. The remainder of the paper is organized as follows. In section 2, some preliminaries, necessary notations and definitions of hesitant fuzzy sets, interval arithmetic and the KKT conditions for linear programming problems with interval-valued objective functions are presented. In section 3, a method is proposed to solve defender-attacker games with multiple decision makers in fuzzy environment using expected value model. In section 4, a numerical example is presented to illustrate the mentioned approach. Conclusion is made in section 5.

## 2. Preliminaries

2.1. **Hesitant fuzzy sets.** In this subsection, we recall some notations and preliminaries of hesitant fuzzy sets according to [25, 26].
Let $X$ denote a universal set. A fuzzy subset $\tilde{a}$ of $X$ is defined by its membership function $\mu_{\tilde{a}} : X \to [0,1]$, which assigns to each element $x \in X$ a real number $\mu_{\tilde{a}}(x)$ in the interval $[0,1]$. The value of $\mu_{\tilde{a}}(x)$ represents the grade of membership of $x$ in $\tilde{a}$. The fuzzy subset $\tilde{a}$ can be characterized as the set of ordered pairs of elements $x$ and grades $\mu_{\tilde{a}}(x)$, and is often written as $\tilde{a} = \{(x, \mu_{\tilde{a}}(x)) | x \in X\}$.
A fuzzy number is a convex normalized fuzzy set of the real line $\mathbb{R}$ whose membership function is piecewise continuous. From the definition of a fuzzy number $\tilde{a}$, it is significant to note that each $\alpha$-cut $\tilde{a}_\alpha$ of a fuzzy number $\tilde{a}$ is a closed interval $[a^L_\alpha, a^R_\alpha]$.
A triangular fuzzy number $\tilde{a} = (a^l, a^m, a^r)$ is a special fuzzy number, whose membership function is given by

$$\mu_{\tilde{a}}(x) = \begin{cases} (x-a^l)/(a^m-a^l) & a^l \le x \le a^m \\ (a^r-x)/(a^r-a^m) & a^m \le x \le a^r \\ 0 & otherwise, \end{cases} \tag{2.1}$$

where $a^m$ is the core of $\tilde{a}$, and $a^l$ and $a^r$ are the left and right extreme points of $supp(\tilde{a})$, respectively. $\tilde{a} = (a^l, a^m, a^r)$ is called a non-negative triangular fuzzy number if $a^l \ge 0$ and $a^r > 0$. Let $\tilde{a} = (a^l, a^m, a^r)$ and $\tilde{b} = (b^l, b^m, b^r)$ be two triangular fuzzy numbers. By the extension principle of Zadeh [29], the addition of $\tilde{a}$ and $\tilde{b}$ is given by

$$\tilde{a} + \tilde{b} = (a^l + b^l, a^m + b^m, a^r + b^r),$$

and the scalar multiplication of $\tilde{a}$ by the scalar $\lambda \in \mathbb{R}$ is given by

$$\lambda\tilde{a} = \begin{cases} (\lambda a^l, \lambda a^m, \lambda a^r) & \lambda \ge 0 \\ (\lambda a^r, \lambda a^m, \lambda a^l) & \lambda < 0. \end{cases}$$

When people make a decision, they are usually hesitant and irresolute for one thing or another, which makes it difficult to reach a final aggreement. Hesitant fuzzy sets, introduced by Torra [25, 26], is a useful tool to handle this hesitance.

**Definition 2.1.** [25, 26] Let $X$ be a fixed set. A hesitant fuzzy set (HFS) on $X$ is in terms of a function that when applied to $X$ returns a subset of $[0,1]$. In mathematical term, $\tilde{A} = \{< x, h_A(x) > | x \in X\}$ where $h_A$ is set of some values in $[0,1]$, is called the possible membership values of the element $x \in X$.

In defender-attacker game, each of the experts presents a payoff for any output which is a fuzzy number. In other words, there is a set of the fuzzy numbers for every output. Symbolically, it is expressed as,

$\tilde{A} = \{< x, \tilde{h}_A(x) > | x \in X\}$, where $\tilde{h}_A(x)$ is a set of the fuzzy numbers, which takes value from $[0, 1]$. Each $\tilde{h}_A(x)$ is called the hesitant fuzzy element.
Let $\tilde{h}_A$ be the triangular hesitant fuzzy element and $\tilde{H}$ be the set of the triangular hesitant fuzzy elements. If $\tilde{h}_A \in \tilde{H}$ then $\tilde{h}_A(x) = \{\tilde{u} | \tilde{u} = (u^l, u^m, u^r)\}$.

Now, we recall requaired interval arithmetic in this papar.

**Definition 2.2.** Let $a = \left[a^L, a^R\right]$, $b = \left[b^L, b^R\right]$ be two intervals. The order relations $\preceq_{LR}$ and $\prec_{LR}$ between $a$ and $b$ are defined as
(i) $a = b$ if and only if $a^L = b^L$ and $a^R = b^R$.
(ii) $a \preceq_{LR} b$ if and only if $a^L \leq b^L$ and $a^R \leq b^R$.
(iii) $a \prec_{LR} b$ if and only if $a \preceq_{LR} b$ and $a \neq b$.

We say $b$ is better than $a$ if $a \prec_{LR} b$ .

2.2. **The interval-valued optimization problem.** In this subsection we consider the following interval-valued optimization problem:

$$\begin{array}{ll} \min & f(x) = [f^L(x), f^R(x)] \\ s.t. & x \in X = \{x \in \mathbb{R}^n | g_i(x) \leq 0,\ i = 1, ..., m\}. \end{array} \tag{2.2}$$

where $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, ..., m$, are convex real-valued functions.
The feasible set $X$ obviously is a convex set.

**Definition 2.3.** [28] The feasible point $x^*$ is said to be an LR optimal solution of the problem (2.2) if there exists no $x \in X$ such that $f(x) \prec_{LR} f(x^*)$.

We say that the real-valued constraint functions $g_i, i = 1, ..., m$, satisfy the KKT assumptions at x* if $g_i$'s are convex on $\mathbb{R}^n$ and continuously differentiable at $x^*$ for $i = 1, ..., m$. KKT optimality conditions are stated as follows (for the definitions of convexity and differentiability for interval-valued functions refer to [28]).

**Theorem 2.4.** [28] *Assume that the real constraint functions $g_i, i = 1, ..., m$, of the problem (2.2) satisfy the KKT assumptions at $x^*$ and the interval-valued objective function $f : \mathbb{R}^n \to \Omega$ is LR-convex and weakly continuosly differentiable at $x^* \in X$. If there exist (Lagrange) multipliers $0 < \lambda^L, \lambda^R$ and $0 \leq \mu_i \in \mathbb{R}, i = 1, ..., m$, such that*

$$\begin{array}{l} (i)\ \lambda^L \nabla f^L(x^*) + \lambda^R \nabla f^R(x^*) + \sum\limits_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0 \\ (ii)\ \mu_i g_i(x^*) = 0, \forall i = 1, ..., m \end{array} \tag{2.3}$$

*then $x^*$ is an LR optimal solution of the problem (2.2).*

2.3. **Defender-attacker game.** A defender-attacker game models conflict between a defender and an attacker. In this game, first defender makes strategy and then attacker observing defender's strategy makes own strategy. The defender attempts to prevent attacks using the security resources. We show the security resources as a set of covering of targets which can be distributed in a continuos fashion among the targets. Consider the set of targets as $T = \{1, 2, ..., p\}$ together $m$ identical resources. The defender's strategy is a coverage vector $c = (c_1, c_2, ..., c_p)$ where $c_k$, (k=1,2,...,p) is the amount of coverage placed on target $k$ and represents the probability that the target $k$ is covered. However, attacker can observe this mixed strategy but does not know whether a target will be covered or not. The defender's and attacker's strategies are as follows, respectively:

$$C = \{c = (c_1, c_2, ..., c_p)|0 \leq c_k \leq 1, k = 1, ..., p, \quad \sum_{k=1}^{p} c_i \leq m, \},$$

and

$$A = \left\{ a = (a_1, a_2, \ldots, a_p)\, | a_k \geq 0,\ k = 1, ..., p,\ \sum_{k=1}^{p} a_k = 1 \right\}.$$

where $a_k$ is the probability of attacking to the target $k$ by attacker.

## 3. Defender-attacker game with multiple decision makers

In this game the defender first chooses a strategy for covering targets. Since the security resources are limited, the defender uses a random approach. In other words, the defender allocates a probability amount to each target which indicates how to assign them. When the defender and attacker choose their strategies, each of them receive payoffs. These payoffs are obtained based on a set of answers of some experts to a set of key questions. In fact, decision analysit should prepare a list of key questions to be answered by field experts. Since the obtained output includes several utilities, the utilities are in hesitant fuzzy environment. We consider the utilities as hesitant fuzzy sets. Now, assume that $\tilde{U}^{c,d}(k)$ is the defender's utility if $k$ is chosen by attacker and is fully covered by defender. If $k$ is uncovered, the defender's penalty is $\tilde{U}^{u,d}(k)$. The attacker's utility is denoted similarly by $\tilde{U}^{c,a}(k)$ and $\tilde{U}^{u,a}(k)$.

Let $c$ and $a$ be the defender's and attacker's strategies, respectively. The expected payoffs for both players are given by:

$$\tilde{U}^d(c, a) = \sum_{k=1}^{p} a_k \tilde{U}^d(c_k, k), \tag{3.1}$$

$$\tilde{U}^a(c,a) = \sum_{k=1}^{p} a_k \tilde{U}^a(c_k,k). \tag{3.2}$$

In Equations (3.1) and (3.2)

$$\tilde{U}^d(c_k,k) = c_k\tilde{U}^{c,d}(k) + (1-c_k)\tilde{U}^{u,d}(k),$$
$$\tilde{U}^a(c_k,k) = c_k\tilde{U}^{c,a}(k) + (1-c_k)\tilde{U}^{u,a}(k),$$

are the payoff received by the defender and attacker, respectively, if target $k$ is attacked and is covered with $c_k$ resources.

Note that the utilities are hesitant fuzzy sets. We represent the defender's utilities as follows:

$$\tilde{U}^{c,d}(k) = \{\tilde{U}_i^{c,d}(k),\ i=1,...,q\,,\ where\ \tilde{U}_i^{c,d}(k)\ is\ ordinary\, fuzzy\, set$$
$$introduced\, by\, the\, i-th\, expert\,\}$$

$$\tilde{U}^{u,d}(k) = \{\tilde{U}_i^{u,d}(k),\ i=1,...,q\,,\ where\ \tilde{U}_i^{u,d}(k)\ is\ ordinary\, fuzzy\, set$$
$$introduced\, by\, the\, i-th\, expert\,\}$$

The attacker's utilities are defined similarly.

As we said, in this problem first defender chooses strategy. In fact he wants to allocate resources for covering targets. The defender commits to an optimal strategy, first, based on the assumption that the attacker will be able to observe this strategy and then choose an optimal response. Therefore, considering hesitant fuzzy utilities, the problem is formulated as follows:

$$\begin{array}{l} max_{c\in C}\tilde{U}^d(c,a) \\ \quad s.t. \sum\limits_{k=1}^{p} c_k \le m \\ \qquad 0 \le c_k \le 1 \quad k=1,...,p \\ \left.\begin{array}{l} where\, a\, solves \\ \quad max_a \tilde{U}^a(c,a) \\ \qquad \sum\limits_{k=1}^{p} a_k = 1 \\ \qquad a_k \ge 0 \end{array}\right\} \end{array} \tag{3.3}$$

According to our proposed method, the fuzzy utilities can be any kind of the fuzzy numbers. Here for simplicity in explanation, we suppose that the utilities are triangular hesitant fuzzy numbers. Then,

$$\tilde{U}^{c,d}(k) = \left\{\tilde{U}_i^{c,d}(k),\, i=1,...,q \,\middle|\, \tilde{U}_i^{c,d}(k) = ((u_i^{c,d})^l, (u_i^{c,d})^m, (u_i^{c,d})^r)\right\},$$

$$\tilde{U}^{u,d}(k) = \left\{\tilde{U}_i^{u,d}(k),\, i=1,...,q \,\middle|\, \tilde{U}_i^{u,d}(k) = ((u_i^{u,d})^l, (u_i^{u,d})^m, (u_i^{u,d})^r)\right\}.$$

The attacker's utilities are as above similarly. We consider $\alpha$-cut of the above numbers as follows(for attacker is similarly):

$$\left[\tilde{U}^{c,d}(k)\right]_\alpha = \left\{x \left|\tilde{U}_i^{c,d}(k)(x) \geq \alpha,\ i = 1, ..., q\right.\right\}$$
$$\left[\tilde{U}^{u,d}(k)\right]_\alpha = \left\{x \left|\tilde{U}_i^{u,d}(k)(x) \geq \alpha,\ i = 1, ..., q\right.\right\}$$

We use abbreviated notations such as $\tilde{U}_i^{c,d}(k) = \tilde{U}_i^{c,d}$ , $\tilde{U}^{c,d}(k) = \tilde{U}^{c,d}$, and so on.

According to the fact that the $\alpha$-cuts of triangular fuzzy numbers are closed intervals, then

$$\left[\tilde{U}^{c,d}\right]_\alpha = \left\{\left[(U_i^{c,d})_\alpha^L, (U_i^{c,d})_\alpha^R\right] |i = 1, ..., q\right\},$$
$$\left[\tilde{U}^{u,d}\right]_\alpha = \left\{\left[(U_i^{u,d})_\alpha^L, (U_i^{u,d})_\alpha^R\right] |i = 1, ..., q\right\}.$$

On the other hand, each of the experts may have different weights. There are different methods to determine these weights such as AHP [27]. We obtain the weights of the experts by AHP method. The decision analyst is as the consultant for commander of decision makers (or commander himself/herself) who makes paired comparisons. Assume that we obtain the weights of the experts by this method. Therefore, we aggregate the above hesitant fuzzy numbers as

$$\left[\tilde{U}^{c,d}\right]_\alpha^w = \left[\sum_{i=1}^q w_i(U_i^{c,d})_\alpha^L, \sum_{i=1}^q w_i(U_i^{c,d})_\alpha^R\right],$$
$$\left[\tilde{U}^{u,d}\right]_\alpha^w = \left[\sum_{i=1}^q w_i(U_i^{u,d})_\alpha^L, \sum_{i=1}^q w_i(U_i^{u,d})_\alpha^R\right].$$

where $w_i$ is the allocated weight to the i'th expert and $\sum_{i=1}^q w_i = 1$. Since for each $\alpha \in (0, 1]$ we obtain a utility, we should obtain appropriate $\alpha$ or $\alpha$-expected utilities for defender and attacker. Here, we propose a method to calculus expected utility.

$$E(\left[\tilde{U}^{c,d}\right]_\alpha^w) = \int_0^1 \alpha \left[\sum_{i=1}^q w_i(U_i^{c,d})_\alpha^L, \sum_{i=1}^q w_i(U_i^{c,d})_\alpha^R\right] d\alpha \tag{3.4}$$

$$E(\left[\tilde{U}^{u,d}\right]_\alpha^w) = \int_0^1 \alpha \left[\sum_{i=1}^q w_i(U_i^{u,d})_\alpha^L, \sum_{i=1}^q w_i(U_i^{u,d})_\alpha^R\right] d\alpha \tag{3.5}$$

Similarly, we calculate $\alpha$-expected utilities of attackers.

We only calculate lower bound integral of equation (3.4). The other cases are calculated similarly.

$$\begin{aligned}
&E^L([\tilde{U}^{c,d}]^w_\alpha) = \int_0^1 \alpha \sum_{i=1}^{q} w_i \left( (U_i^{c,d})^l + ((U_i^{c,d})^m - (U_i^{c,d})^l)\alpha \right) d\alpha \\
&= \sum_{i=1}^{q} w_i \left( (U_i^{c,d})^l \tfrac{\alpha^2}{2} + ((U_i^{c,d})^m - (U_i^{c,d})^l)\tfrac{\alpha^3}{3} \right) \Big|_0^1 \\
&= \sum_{i=1}^{q} w_i \tfrac{(U_i^{c,d})^l + 2(U_i^{c,d})^m}{6}
\end{aligned} \tag{3.6}$$

By calculating the other cases, we have

$$E([\tilde{U}^{c,d}]^w_\alpha) = \left[ \sum_{i=1}^{q} w_i \frac{(U_i^{c,d})^l + 2(U_i^{c,d})^m}{6}, \sum_{i=1}^{q} w_i \frac{(U_i^{c,d})^m + 2(U_i^{c,d})^r}{6} \right]$$

$$E([\tilde{U}^{u,d}]^w_\alpha) = \left[ \sum_{i=1}^{q} w_i \frac{(U_i^{u,d})^l + 2(U_i^{u,d})^m}{6}, \sum_{i=1}^{q} w_i \frac{(U_i^{u,d})^m + 2(U_i^{u,d})^r}{6} \right]$$

also

$$E([\tilde{U}^{c,a}]^w_\alpha) = \left[ \sum_{i=1}^{q} w_i \frac{(U_i^{c,u})^l + 2(U_i^{c,a})^m}{6}, \sum_{i=1}^{q} w_i \frac{(U_i^{c,a})^m + 2(U_i^{c,u})^r}{6} \right]$$

$$E([\tilde{U}^{u,a}]^w_\alpha) = \left[ \sum_{i=1}^{q} w_i \frac{(U_i^{u,a})^l + 2(U_i^{u,a})^m}{6}, \sum_{i=1}^{q} w_i \frac{(U_i^{u,a})^m + 2(U_i^{u,a})^r}{6} \right]$$

Using the above equations in ralations (3.1) and (3.2), we have expected utilities of the defender and attacker as follows:

$$\left[ E^L(\tilde{U}^d(c,a)), E^R(\tilde{U}^d(c,a)) \right] = \left[ \sum_{k=1}^{p} a_k E^L(\tilde{U}^d(c,a)), \sum_{k=1}^{p} a_k E^R(\tilde{U}^d(c,a)) \right]$$

and

$$\left[ E^L(\tilde{U}^a(c,a)), E^R(\tilde{U}^a(c,a)) \right] = \left[ \sum_{k=1}^{p} a_k E^L(\tilde{U}^a(c,a)), \sum_{k=1}^{p} a_k E^R(\tilde{U}^a(c,a)) \right]$$

Considering the expected-values of the utilities, the following expected value model is obtained.

$$\left.\begin{array}{l} max_{c\in C}\left[E^L(\tilde{U}^d(c,a)),E^R(\tilde{U}^d(c,a))\right] \\ \quad s.t. \sum_{k=1}^{p} c_k \leq m \\ \quad\quad 0\leq c_k\leq 1 \quad k=1,...,p \\ \quad where\, a\, solves \\ \quad\quad max_a \left[E^L(\tilde{U}^a(c,a)),E^R(\tilde{U}^a(c,a))\right] \\ \quad\quad\quad \sum_{k=1}^{p} a_k = 1 \\ \quad\quad\quad a_k\geq 0 \end{array}\right\} \tag{3.7}$$

According to Definition 2.3, a feasible solution $(c^*, a^*)$ is an $LR$ optimal solution to the model (3.3) if there exists no $(c, a)$ such that

$$E(\tilde{U}^d(c^*,a^*))\prec_{LR}E(\tilde{U}^d(c,a))$$

We use KKT conditions to solve the problem (3.7). According to Theorem 2.4, given the defender's strategy $c$, the attacker's optimal response strategies satisfy the optimality conditions

$$\begin{array}{l} \lambda^L \frac{\partial E^L(\tilde{U}^d(c,a))}{\partial a_k}+\lambda^R\frac{\partial E^R(\tilde{U}^d(c,a))}{\partial a_k}-\mu_0+\mu_k=0, \quad k=1,...,p \\ \mu_k a_k=0, \quad k=1,...,p \\ \mu_k\geq 0,\lambda^L,\lambda^R\geq 0,k=1,...,p, \end{array} \tag{3.8}$$

where

$$\begin{array}{l} \frac{\partial E^L(\tilde{U}^d(c,a))}{\partial a_k}=c_kE^L(\tilde{U}^{c,a}(k))+(1-c_k)E^L(\tilde{U}^{u,a}(k)), \\ \frac{\partial E^R(\tilde{U}^d(c,a))}{\partial a_k}=c_kE^R(\tilde{U}^{c,a}(k))+(1-c_k)E^R(\tilde{U}^{u,a}(k)). \end{array}$$

Thus, the problem (3.7) is written as follows:

$$\begin{array}{ll} max_{c\in C} & \left(\left[E^L(\tilde{U}^d(c,a)),E^R(\tilde{U}^d(c,a))\right]\right) \\ s.t. & \sum_{k=1}^{p} c_k\leq m \\ & 0\leq c_k\leq 1, \quad k=1,...,p \\ & \lambda^L\frac{\partial E^L(\tilde{U}^a(c,a))}{\partial a_k}+\lambda^R\frac{\partial E^R(\tilde{U}^a(c,a))}{\partial a_k}-\mu_0+\mu_k=0,\ k=1,...,p, \\ & \mu_k a_k=0, \quad k=1,...,p, \\ & \sum_{k=1}^{p} a_k=1, \quad a_k\geq 0, \quad k=1,...,p, \\ & \mu_k\geq 0,\lambda^L,\lambda^R\geq 0, \quad k=1,...,p,. \end{array} \tag{3.9}$$

**Theorem 3.1.** *If $(c^*, a^*)$ be $LR$ optimal solution of the problem (3.9) then it is $LR$ optimal solution of the problem (3.3).*

*Proof.* The third up to sixth constraints of the problem (3.9) are KKT optimality conditions for the second level problem. Therefore, $(c^*, a^*)$ satisfies in the constraints of the problem (3.3). Thus, if $(c^*, a^*)$ be $LR$ optimal solution of the problem (3.9) then it is $LR$ optimal solution of the problem (3.3). □

We consider the problem (3.9) as a bi-objective mathematical programming problem by considering the left bound as the first objective and the right bound as the second objective. Because according to Definition (5), a better interval is an interval with bigger left and right bounds. We have the following problem:

$$
\begin{aligned}
& max\ E^L(\tilde{U}^d(c,a)) \\
& max\ E^R(\tilde{U}^d(c,a)) \\
& s.t.\ \ \sum_{k=1}^{p} c_k \le m \\
& \quad 0 \le c_k \le 1 \quad k = 1, ..., p \\
& \quad \lambda^L \tfrac{\partial U^{aL}(c,a)}{\partial a_k} + \lambda^R \tfrac{\partial U^{aR}(c,a_i)}{\partial a_k} - \mu_0 + \mu_k = 0,\ k = 1, ..., p, \\
& \quad \mu_k a_k = 0, \quad k = 1, ..., p, \\
& \quad \sum_{k=1}^{p} a_k = 1, \quad a_k \ge 0,\ \ k = 1, ..., p, \\
& \quad E^L(\tilde{U}^d(c,a)) \le E^R(\tilde{U}^d(c,a)), \\
& \quad \mu_k \ge 0, \lambda^L, \lambda^R \ge 0,\ \ k = 1, ..., p,
\end{aligned}
\tag{3.10}
$$

Therefore, calculating LR optimal solutions to the fuzzy bilevel programming problem turns out to solve the bi-objective optimization problem. Different methods of solving multi objective programming problems such as weighted sum, goal programming, and so on[20] can be used, to calculate LR optimal solutions of the fuzzy bilevel problem.

## 4. Numerical Example

Consider the security game among a defender and attacker. Suppose that there are three targets and two security resources (i.e. $p = 3, m = 2$). Suppose that exist three decision makers to compare strategies. The payoffs matrices of game are represented in Tables 1, 2, 3 for each of decision makers. Also, Table4 present the obtained payoff matrix by expected operator for the triangular fuzzy numbers.

Assume that importance of decision makers are the same. Then we have $w_1 = w_2 = w_3 = \frac{1}{3}$. The mathematical programming problem

TABLE 1. The payoffs matrix of DM1 for defender-attacker game.

| | target1 | | target2 | | target3 | |
|---|---|---|---|---|---|---|
| | covered | uncovered | covered | uncovered | covered | uncovered |
| defender | (8,9,10) | (-3,-2,-1) | (7,8,9) | (-2,-1,0) | (7,8,9) | (-2,-1,0) |
| attacker | (-5,-3,-1) | (8,10,11) | (-3,-2,-1) | (7,8,9) | (-3,-2,0) | (6,7,8) |

TABLE 2. The payoffs matrix of DM2 for defender-attacker game.

| | target1 | | target2 | | target3 | |
|---|---|---|---|---|---|---|
| | covered | uncovered | covered | uncovered | covered | uncovered |
| defender | (7,8,10) | (-2,-1,0) | (7,8,10) | (-3,-2,-1) | (6,8,9) | (-3,-2,-1) |
| attacker | (-4,-3,-2) | (7,9,10) | (-4,-3,-2) | (8,9,11) | (-5,-4,-3) | (7,9,11) |

TABLE 3. The payoffs matrix of DM3 for defender-attacker game.

| | target1 | | target2 | | target3 | |
|---|---|---|---|---|---|---|
| | covered | uncovered | covered | uncovered | covered | uncovered |
| defender | (8,9,11) | (-4,-3,-2) | (9,10,11) | (-5,-4,-3) | (8,9,11) | (-4,-3,-1) |
| attacker | (-5,-3,-2) | (9,10,12) | (-5,-4,-3) | (10,11,12) | (-6,-4,-2) | (8,9,10) |

TABLE 4. The expected payoffs matrix of defender-attacker game.

| | target1 | | target2 | | target3 | |
|---|---|---|---|---|---|---|
| | covered | uncovered | covered | uncovered | covered | uncovered |
| defender | [75/18,88/18] | [-7/6,-2/3] | [75/18,86/18] | [-24/18,-15/18] | [71/18,82/18] | [-21/18,-5/9] |
| attacker | [-16/9,-19/18] | [92/18,95/18] | [-30/18,-21/18] | [81/18,92/18] | [-17/9,-10/9] | [71/18,83/18] |

(3.10) is as follows:

$$
\begin{aligned}
&\max\ \{96/18a_1c_1+99/18a_2c_2+92/18a_3c_3-21/18a_1-24/18a_2-21/18a_3\}\\
&\max\ \{100/18a_1c_1+101/18a_2c_2+92/18a_3c_3-12/18a_1-15/18a_2-10/18a_3\}\\
&s.t.\\
&\qquad \lambda^L(82/18-114/18c_1)+\lambda_1^R(95/18-114/18c_1)-\mu_0+\mu_1=0,\\
&\qquad \lambda^L(81/18-111/18c_2)+\lambda^R(81/18-102/18c_2)-\mu_0+\mu_2=0,\\
&\qquad \lambda^L(71/18-105/18c_3)+\lambda^R(83/18-103/18c_3)-\mu_0+\mu_3=0,\\
&\qquad \{96/18a_1c_1+99/18a_2c_2+92/18a_3c_3-21/18a_1-24/18a_2-21/18a_3\}\leq\\
&\qquad \{100/18a_1c_1+101/18a_2c_2+92/18a_3c_3-12/18a_1-15/18a_2-10/18a_3\},\\
&\qquad \mu_1a_1=0,\\
&\qquad \mu_2a_2=0,\\
&\qquad \mu_3a_3=0,\\
&\qquad 0\leq c_1\leq 1,\\
&\qquad 0\leq c_2\leq 1,\\
&\qquad 0\leq c_3\leq 1,\\
&\qquad c_1+c_2+c_3\leq 2,\\
&\qquad a_1+a_2+a_3=1,\\
&\qquad a_1,a_2,a_3\geq 0,\\
&\qquad \lambda^L,\lambda^R,\mu_1,\mu_2,\mu_3\geq 0.
\end{aligned}
\tag{4.1}
$$

We use weighted sum method [20] with equal weights 1/2 for both objectives to solve the problem (4.1). Using Lingo software yields

$$c_1 = 1, c_2 = 0.49, c_3 = 0.51.$$

This means that the defender should protect the target 1 completely, and the other security source had a 49% at second target and 51% at third tatget, randomly.

## 5. Conclusion

In this paper, defender-attacker game in the uncertain environment was considered where the payoffs of matrix game are hesitant fuzzy numbers. A method was proposed to solve mentioned game. First, this game is formulated as bilevel programming problem with fuzzy hesitant coefficients. By the alpha cuts of the fuzzy hesitant numbers and then using defined expected value, the bilevel problem was rewritten as bilevel programming problem with interval coefficients. The KKT optimality conditions in lower level of bilevel problem were applied. By this approach, the bilevel programming problem was transformed to a single level programming problem with interval coefficients in the objective functions. By solving this problem, optimal strategies of defender were obtained. It was shown that the defender's strategies in facing of attacker can be obtained by solving a programming problem. The main advantage of this method is computation efficiency because the proposed method provides a single-level objective optimization model which can be solved easily. Finally, Validity and applicability of the method are illustrated by a practical example. In this paper, we presented a mathematical analysis for modeling and solving Defender-Attacker game problem. According to that the bi-level programming is NP-hard, as future work, we propose to use Meta-heuristic approaches to solve Defender-Attacker game problems with large size.

## References

[1] C. R. Bector, S. Chandra, *Fuzzy mathematical programming and Fuzzy matrix games*, Springer Verlag, Berlin (2005).

[2] A. Bhaumik, S.K Roy, *Intuitionistic interval-valued hesitant fuzzy matrix games with a new aggregation operator for solving management problem*, Granul. Comput, (2019).

[3] H. Bigdeli, H. Hassanpour, *A satisfactory strategy of multi-objective two person matrix games with fuzzy payoffs*, Iranian Journal of Fuzzy Systems, 13, 17-33 (2016).

[4] H. Bigdeli, H. Hassanpour, *An approach to solve multiobjective linear production planning games with fuzzy parameters*, Yugoslav Journal of Operations Research, 28, 2(2018).

[5] H. Bigdeli, H. Hassanpour, and J. Tayyebi, *Constrained Bimatrix Games with Fuzzy Goals and its Application in Nuclear Negotiations*, Iranian journal of numerical analysis and optimization, 8, 1, 81–109(2018).

[6] H. Bigdeli, H. Hassanpour, J Tayyebi, *Multiobjective security game with fuzzy payoffs*, Iranian Journal of Fuzzy Systems, 16 (1), 89-101 (2019).

[7] M. Brown, B. An, C. Kiekintveld, F. Ordóñez, M. Tambe, *An extended study on multi-objective security games*, Auton Agent Multi-Agent Syst, 28, 31-71 (2014).

[8] D. Butnariu, *Fuzzy games; a description of the concept*, Fuzzy Sets and Systems, 1, 181-192 (1978).

[9] M. Clemente, F. R. Fernández, J. Puerto, *Pareto-optimal security strategies in matrix games with fuzzy payoffs*, Fuzzy Sets and Systems 176, 36-45 (2011).

[10] D. Dubois, H. Prade, *Fuzzy sets and systems: theory and applications*, Academic Press, New York (1980).

[11] Z. Fang, S. Liu, H. Shi, Y. Lin, *Grey game theory and its applications in economic decision making*, Auerbach publications, (2010).

[12] D. Fudenberg , J. Tirole, *Game theory*, The MIT Press (1991).

[13] N. Gatti, *Game theoretical insights in strategic patrolling: modeland algorithm in normal-form*, in ECAI-08, 403-407 (2008).

[14] A. Heifetz, *Game theory: interactive strategies in economic and managment* , Cambridge university press, (2012).

[15] J. Jana, S.K Roy *Dual hesitant fuzzy matrix games: based on new similarity measure*, Soft Comput 23, 8873–8886 (2019).

[16] K. Lye, J. M. Wing, *Game strategies in network security*, International Journal of Information Security, 4, 71-86 (2005).

[17] J. V. Neumann, O. Morgenstern, *Theory of games and economic behavior*, Wiley, New York (1944).

[18] I. Nishizaki, M. Sakawa, *Fuzzy and multiobjective games for conflict resolution*, springer-verlag Berlin Heidelberg (2001).

[19] G. Owen, *Game theory*, Academic Press, San Diego, Third Edition (1995).

[20] M. Sakawa, *Fuzzy sets and interactive multiobjective optimization*, Plenum press, New York and london, (1993).

[21] T. Sandler, D. G. A. M, *Terrorism and game theory*, Simulationand Gaming, 34, 319-337 (2003).

[22] R. Steuer, *Multiple criteria optimization: theory, computation, and application*, John Wiley & Sons, New York (1985).

[23] M. Tambe, *Security and game theory, algorithms, deployed systems, lessons learned*, Cambridge university press (2012).

[24] GS. Thakur, R. Thakur, R. Singh *New hesitant fuzzy operators*, Fuzzy Inf Eng, 6:379–392 (2014).

[25] V. Torra, *Hesitant fuzzy sets*, Int J Intell Syst, 25:529–539 (2010) .

[26] V. Torra, Y. Narukawa, *On hesitant fuzzy sets and decision in the 18th IEEE International Conference on Fuzzy Systems*, Jeju Island, Korea, pp 1378–1382 (2009).

[27] H. Wang, C. Xu, Z. Xu, *An approach to evaluate the methods of determining experts' objective weights based on evolutionary game theory*, Knowledge-Based Systems 182, 104862 (2019).

[28] H. Wu, *The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function*, Eur J Oper Res 176, 46-59 (2007).

[29] L. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Inform.Sci. 8, 199-249 (1978).