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Coretractable modules relative to δ

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Abstract. Let R be a ring and M a right R-module. We call M, a $\delta(M)$-coretractable module if for every proper submodule N of M containing $\delta(M)$, there is a nonzero homomorphism from M/N to M. We investigate some conditions which under two concepts $\delta(M)$-coretractable and coretractable coincide. For a ring R, we prove that R is right Kasch if and only if R_R is $\delta(R_R)$-coretractable.

Keywords: coretractable modules, $\delta(M)$-coretractable module.

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1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. Let M be an R-module and N a submodule of M. We use $End_R(M)$, $ann_r(M)$, $ann_l(M)$ to denote the ring of endomorphisms of M, the right annihilator in R of M and the left annihilator in R of M, respectively. Let M be a module and K a submodule of M. Then K is essential in M denoted by $K \leq_e M$, if $L \cap K \neq 0$ for every nonzero submodule L of M. Dually, K is small in M ($K \ll M$), in case $M = K + L$ implies that $L = M$. We also recall that a module M is a small module in case there

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is a module L containing M such that $M \ll L$. It is well-known that a module M is small if and only if M is a small submodule of its injective hull.

Let M be a module and N a submodule of M. Following [13], N is δ-small in M (denoted by $N \ll_\delta M$), in case $M = N + K$ with M/K singular implies that $M = K$. Note that by definitions, every small submodule of M is δ-small in M. The sum of all δ-small submodules of M is denoted by $\delta(M)$. Also $\delta(M)$ is the reject of the class of all simple singular modules in M.

A submodule N of a module M is called a *δ-supplement* in M, if there is a submodule K of M such that $M = N + K$ and $N \cap K \ll_\delta N$. A module M is called *δ-supplemented* if every submodule of M has a δ-supplement in M. A module M is called *amply δ-supplemented*, in case $M = A+B$ implies A contains a δ-supplement A' of B in M. The reader can find more details about classes of all versions of δ-supplemented modules in [4].

Let R be a ring, M an right R-module. Recall that a module M is singular provided that $Z(M) = M$ where $Z(M) = \{x \in M \mid xI = 0, I \leq_e R_R\}$. Suppose that $\mathcal{S}$ denotes the class of all small right R-modules. In [9] the authors defined $\overline{Z}(M)$ as the reject of $\mathcal{S}$ in M, i.e. $\overline{Z}(M) = \cap\{Kerf \mid f : M \to U, U \in \mathcal{S}\}$. In this way, M is called *(non-)cosingular*, in case ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$. They investigated some general properties of $\overline{Z}(M)$.

Following [2], a module M is said to be retractable in case for every nonzero submodule N of M, there is a nonzero homomorphism $f : M \to N$, i.e $Hom_R(M, N) \neq 0$. Retractable modules and their various generalizations were widely studied and investigated (for example, see [3, 11, 12]). Amini, Ershad and Sharif in [1] defined dual notation of retractable modules namely coretractable modules. A module M is *coretractable* provided that, $Hom_R(M/N, M) \neq 0$ for every proper submodule N of M. Some general properties of coretractable modules were investigated in [1]. The class of Kasch rings is also characterized by means of coretractable modules. In [6], the author introduced coretractable modules relative to their $\overline{Z}$. According to [6], a module M is called $\overline{Z}(M)$-coretractable in case for every proper submodule K of M containing $\overline{Z}(M)$, there exists a nonzero homomorphism from M/K to M. Some conditions which under coretractable modules and $\overline{Z}$-coretractable modules coincide, were also presented. For a commutative semiperfect ring R, the author proved that R is Kasch if and only if every simple cosingular R-module can be embedded in R ([6, Corollary 2.14]). Inspired by last work, the same author and Talebi tried to generalize $\overline{Z}$-coretractable modules to a general submodule. It means that, in [7] a module M is called N-coretractable in case for every proper submodule K of M containing N there is a nonzero homomorphism $g : M/K \to M$. They proved that a right GV-ring R is a Kasch ring if and only if R is a semisimple ring. They also presented some statements that guaranteed that a module M is N-coretractable if and only if M is coretractable.

Inspired by mentioned works, we focus just on nonzero homomorphisms from M/K to M where K contains $\delta(M)$. We present some conditions to prove that when two concepts coretractable and $\delta(M)$-coretractable coincide. Among them, we show that if $\delta(M)$ is δ-small in M or it is a coretractable module, then M is coretractable if and only if M is $\delta(M)$-coretractable. We show that R_R is $\delta(R_R)$-coretractable if and only if every simple right R-module that annihilated by $\delta(R_R)$, can be embedded in R_R.

2. $\delta(M)$-coretractable modules

In this section we introduce a new generalization of coretractable modules namely, $\delta(M)$-coretractable modules.

Recall that a module M is *coretractable*, in case for every proper submodule N of M, there exists a nonzero homomorphism $f : M/N \to M$.

Definition 2.1. Let M be a module. We say M is $\delta(M)$-coretractable in case for every proper submodule N of M containing $\delta(M)$, there is a non-zero homomorphism from M/N to M.

Note that if for a module M we have $\delta(M) = 0$, then M is $\delta(M)$-coretractable if and only if M is coretractable.

Example 2.2. (1) Every coretractable module M is $\delta(M)$-coretractable. In particular every semisimple module M is $\delta(M)$-coretractable.

(2) Let M be a module with $\delta(M) = M$. Then clearly M is $\delta(M)$-coretractable. In other words, there is a module M with $\delta(M) = M$ such that M is not coretractable. Since $Hom_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$, then as an $\mathbb{Z}$-module $\mathbb{Q}$ is not coretractable. Note that $\delta(\mathbb{Q}) = \mathbb{Q}$.

Recall from [8] that a ring R is right GV (*generalized V-ring*), in case every simple singular right R-module is injective. In [10, Theorem 3.1] the authors proved that a ring R right GV if and only if every simple cosingular right R-module is projective.

Proposition 2.3. *Let R be a right GV-ring. If M is an indecomposable module with $0 \neq \frac{M}{\delta(M)}$ having a maximal submodule, then M is $\delta(M)$-coretractable if and only if M is simple projective.*

Proof. Let M be coretractable relative to its δ. By assumption there is a maximal submodule K of M containing $\delta(M)$. By assumption, there is a monomorphism $g : M/K \to M$. It follows that Img is a simple submodule of M. Then Img is either cosingular or injective. If Img is cosingular, then by [10, Theorem 3.1] Img is projective. It follows that K is a direct summand of M and hence $K = 0$ or $K = M$. So that $K = 0$. If Img is injective, then Img is a summand of M and since $g \neq 0$ we conclude that $Img = M$, a contradiction. The converse is obvious. □

We shall present some conditions that ensure us the two concepts $\delta(M)$-coretractable and coretractable, coincide.

Lemma 2.4. *Let M be a module. In each of the following cases M is $\delta(M)$-coretractable if and only if M is coretractable.*

(1) *$\delta(M) \ll_\delta M$ ($\delta(M) \ll M$).*

(2) *$\delta(M)$ is a coretractable module.*

Proof. (1) We shall prove the δ case. The other follows immediately. Let M be $\delta(M)$-coretractable and K a proper submodule of M. Suppose that $M \neq \delta(M) + K$. Since M is $\delta(M)$-coretractable, there is a homomorphism $f : M/(\delta(M) + K) \to M$. So that $f \circ \pi : M/K \to M$ is the required homomorphism where $\pi : M/K \to M/(\delta(M) + K)$ is natural epimorphism. Otherwise, $M = \delta(M) + K$. It follows from [13, Lemma 1.2], there is a decomposition $M = Y \oplus K$ where Y is a semisimple projective submodule of $\delta(M)$. Therefore, there is a monomorphism from M/K to M since K is a direct summand of M. Therefore, M is coretractable. The converse is clear.

(2) Let K be a proper submodule of M. Then $K + \delta(M) \neq M$ or $K + \delta(M) = M$. If first one happens, then similar to (1), we can construct a nonzero homomorphism. Now suppose that $K + \delta(M) = M$. Then $h : M/K \to \delta(M)/(\delta(M) \cap K)$ is an isomorphism induced from $M = \delta(M) + K$. Since $\delta(M)$ is coretractable, there is a nonzero homomorphism $g : \delta(M)/(\delta(M) \cap K) \to \delta(M)$. Therefore, $g \circ h \circ j : M/K \to M$ is a nonzero homomorphism where $j : N \to M$ is the inclusion. □

Proposition 2.5. *Let M be a module such that $M/\delta(M)$ is coretractable and can be embedded in M (for example, $M/\delta(M)$ is semisimple and $\delta(M)$ is a direct summand of M). Then M is $\delta(M)$-coretractable.*

Proof. Let K be a proper submodule of M containing $\delta(M)$. Then $K/\delta(M)$ is a proper submodule of $M/\delta(M)$. Since $M/\delta(M)$ is coretractable, there is a nonzero homomorphism $g : M/K \to M/\delta(M)$. Because, $M/\delta(M)$ can be embedded in M, we conclude that there will be a nonzero homomorphism from M/K to M. □

Let M be a module and $K \leq M$. We say M is $\delta(K)$-*coretractable* if for every proper submodule T of M containing $\delta(K)$, there is a non-zero homomorphism $g : M/T \to M$.

Lemma 2.6. (1) *Let* $M = \bigoplus_{i=1}^{n} M_i$ *be a* $\delta(M_i)$*-coretractable module for at least one* $i \in \{1, \ldots, n\}$. *Then* M *is* $\delta(M)$*-coretractable.*

(2) *Let* M *be* $\delta(M)$*-coretractable. If* $\delta(M)$ *contains no nonzero image of any endomorphism of* M*, then* $M/\delta(M)$ *is coretractable.*

(3) *If* $\frac{M}{\delta(M)}$ *has a maximal submodule, then* $Soc(M) \neq 0$. *In particular, if* M *is a finitely generated* $\delta(M)$*-coretractable, then* $Soc(M) \neq 0$.

Proof. (1) This is straightforward.

(2) Let $T/\delta(M)$ be a proper submodule of $M/\delta(M)$. Then $\delta(M) \subseteq T \subset M$. Since M is $\delta(M)$-coretractable, there exists a nonzero homomorphism $g : M/T \to M$. Now define $h : \frac{M/\delta(M)}{T/\delta(M)} \to M/\delta(M)$ by $h(x + \delta() + \frac{T}{\delta(M)}) = g(x + T)$ for every $x \in M$. If $Imh = \delta(M)$, then $Img \subseteq \delta(M)$, a contradiction. So that, $M/\delta(M)$ is coretractable.

(3) Let $\frac{K}{\delta(M)}$ be a maximal submodule of $\frac{M}{\delta(M)}$. Then K is a maximal submodule of M also containing $\delta(M)$. So there is a $h : \frac{M}{K} \to M$. It follows that Imh is a simple submodule of M.

□

Let M be a module and $N \leq M$. Then N is called *fully invariant, if for every* $f \in End_R(M)$, $f(N) \subseteq N$. *Some submodules of a module* M *are fully invariant such as* $Rad(M), Soc(M), \delta(M)$.

Proposition 2.7. (1) *Let* M *be a module and* $K, L \leq M$ *such that* K *is a fully invariant singular* δ*-supplement of* L *in* M. *If* M *is* $\delta(L)$*-coretractable, then* K *is coretractable.*

(2) *Let* M *be a module such that* $\delta(M)$ *has a fully invariant singular* δ*-supplement* K *in* M. *If* M *is* $\delta(M)$*-coretractable, then* K *is coretractable.*

(3) *If* $\delta(M)$ *is a direct summand of a coretractable module* M*, with* $M/\delta(M)$ *singular, then* $\delta(M)$ *is coretractable.*

Proof. (1) Let N be a proper submodule of K. Consider the submodule $N+\delta(L)$ of M. If $N+\delta(L) = M$, then by modularity $N+(K\cap\delta(L)) = K$ which implies that $N = K$, a contradiction (note that $K \cap \delta(L) \subseteq K \cap L \ll_\delta K$). It follows that $N + \delta(L)$ is a proper submodule of M. Being M, $\delta(L)$-coretractable, implies that there is non-zero homomorphism $g : M/(N + \delta(L)) \to M$. Now $(g \circ \pi)(K) \subseteq K$ as K is fully invariant where $\pi : M \to M/(N + (\delta(L))$ is the natural epimorphism. Define the homomorphism $h : K/N \to K$ by $h(x + N) = g(x + N + \delta(L))$. Since g is nonzero, there is a $x \in M \setminus (N + \delta(L))$ such that $g(x + N + \delta(L)) = h(x + N) \neq 0$. Hence K is coretractable.

(2) Similar to (1).
(3) Follows from (2). □

Proposition 2.8. *Let $M = M_1 \oplus \ldots \oplus M_n$. If each M_i is $\delta(M_i)$-coretractable, then M is $\delta(M)$-coretractable.*

Proof. The proof is exactly similar to proof of [1, Proposition 2.6]. Note that by [13, Lemma 1.5(3)], $\delta(M_1 \oplus \ldots \oplus M_n) = \delta(M_1) \oplus \ldots \oplus \delta(M_n)$. □

Let M be an R-module. A submodule K is said to be dense in M if, for any $y \in M$ and $0 \neq x \in M$, there exists $r \in R$ such that $xr \neq 0$ and $yr \in K$. Obviously, any dense submodule of M is essential. From [5, Proposition 8.6]*, K is dense in M if and only if $Hom_R(P/K, M) = 0$ for every submodule $P \supseteq K$.*

Theorem 2.9. *Let R be a ring. Then the following are equivalent:*
(1) *R_R is $\delta(R_R)$-coretractable;*
(2) *Every finitely generated free right R-module F is $\delta(F)$-coretractable;*
(3) *For every right ideal $I \supseteq \delta(R_R)$, $ann_l(I) \neq 0$;*
(4) *Every simple right R-module annihilated by $\delta(R_R)$ can be embedded in R_R.*

Proof. (1) $\Leftrightarrow$ (2) Follows from Proposition 2.8.

(1) $\Rightarrow$ (3) Let I be a right ideal containing $\delta(R_R)$. Since R_R is $\delta(R_R)$-coretractable, there is a nonzero homomorphism $f : R/I \to R$. Consider the endomorphism $g = f \circ \pi : R \to R$ where π is the natural epimorphism from R to R/I. Then there is an element $a \in R$ such that $g(x) = ax$. Let $y \in I$. Then $g(y) = ay = 0$ as $I \subseteq Kerg$.

(3) $\Rightarrow$ (1) Let I be a right ideal of R containing $\delta(R_R)$. Since $ann_l(I) \neq 0$, there exists an element of R such as a that $aI = 0$ and $a \neq 0$. Define $f : R/I \to R$ by $f(x + I) = ax$. It is easy to check that f is an R-homomorphism and in particular $f \neq 0$.

(1) $\Rightarrow$ (4) Let $M \cong R/K$ be a simple right R-module such that $M\delta(R_R) = 0$. It follows that $\delta(R_R) \subseteq K$. Since R is $\delta(R_R)$-coretractable, there is a nonzero homomorphism $f : R/K \to R$.

(4) $\Rightarrow$ (1) Let T be a proper right ideal of R containing $\delta(R_R)$. Now there exists a right maximal ideal K of R such that $\delta(R_R) \subseteq T \subseteq K$. Consider the simple right R-module $M = R/K$. Since $M\delta(R_R) = 0$, there is a nonzero homomorphism $g : R/K \to R$ by assumption. Being T a submodule of K, there exists $f : R/T \to R/K$ defined by $f(x+T) = x + K$. Hence gof is the desired homomorphism. □

Corollary 2.10. *The following statements are equivalent for a ring R;*
(1) *R is a right Kasch ring;*
(2) *Every finitely generated free right R-module F is $\delta(F)$-coretractable;*
(3) *For every right ideal $I \supseteq \delta(R_R)$, $ann_l(I) \neq 0$;*

(4) *Every simple right R-module annihilated by $\delta(R_R)$ can be embedded in R_R.*

Proof. The proof follows from Theorem 2.9 and Lemma 2.4 and the fact that R is a right Kasch ring if and only if R_R is a coretractable module. □

Corollary 2.11. *Let R be a right GV-ring. Then the following are equivalent:*
(1) *R_R is $\delta(R_R)$-coretractable;*
(2) *R is a right Kasch ring;*
(3) *R is a semisimple ring.*

Proof. Follows from [7, Proposition 2.26] and Theorem 2.9. □

Proposition 2.12. *Let R be a ring such that every free right R-module $R^{(A)}$ is $\delta(R)^{(A)}$-coretractable. Then for every right R-module M with $\delta(R_R) \subseteq ann_r(M)$, $Hom_R(M, R) \neq 0$.*

Proof. Let M be a right R-module such that $\delta(R_R) \subseteq ann_r(M)$. Then there is a free right R-module F and a submodule K of F such that $M \cong F/K$. Since $M\delta(R_R) = 0$, we have $\delta(R_R)^{(A)} \subseteq K$ where A is an indexed set. By assumption, there is a nonzero homomorphism $\lambda : F/K \to F$. Then the homomorphism $\pi \circ \lambda : M \to R$ is the required one where $\pi : F \to R$ is the natural epimorphism. □

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