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A general class of one-parametric with memory method for solving nonlinear equations

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Abstract. In this work, we have created the four families of memory methods by convergence rates of three, six, twelve, and twenty-four. Every member of the proposed class has a self-accelerator parameter. And, it has approximated by using Newton's interpolating polynomials. The new iterative with memory methods have a 50% improvement in the order of convergence.

Keywords: Nonlinear equations, Self-accelerator, Order of convergence, With memory method.

2000 Mathematics subject classification: 65H04, 65H05.

1. Introduction

Solving nonlinear equations is very important in many physical problems and mathematical sciences. One of the main problems in computational mathematics is solving nonlinear equations. It is not always possible to get the analytical solution for nonlinear problems. Hence numerical iterative methods are suited for the purpose. Newton, Ostrowski [50] and Traub [66] are some of the oldest people who used iterative methods to solve nonlinear equations. Traub [66] divided the numerical iterative

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methods into two classes: One-Point iterative methods with and without memory and multipoint iterative methods with and without memory. After Traub discovered the efficiency index in 1964. This criterion is that compares iterative methods. The definition is as follows:

Definition 1.1. Let θ be the number of new pieces of information required by a method. A "piece of information" typically is any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [54] and is defined by

$$E = p^{\frac{1}{\theta}} \tag{1.1}$$

where p is the order of the method.

Another criterion for comparing iterative methods of solving nonlinear equations is their optimality and non-optimality. In this case, we also need the following conjecture and corollary:

Definition 1.2. Kung-Traub's conjecture: Multipoint iterative methods without memory, requiring $n+1$ evaluation function per iteration, have the order of convergence at most 2^n.

Corollary 1.3. *Each optimal single-step without memory, with n evaluation function, reaches the convergence order n.*

One can see references [2, 3, 19, 57, 58] for more information on repetitive methods that solving nonlinear equations numerically.
In this work, we have presented four families of iterative methods derived by using Geum-Kim's method [32] and have applied the class of methods to construct families of iterative methods with memory for nonlinear equations. Memory-based methods are a range of repetitive methods with a higher efficiency index. The methods of Secant [54], Traub [66], and Neta [48] can be called the first with-memory methods. In this manuscript, much of our work is on building and developing with-memory methods. Therefore, we have suggested a new class of second-, fourth, eighth, and sixteenth-order derivative-free methods in Section 2. Then, in Section 3, we have approximated the free parameter and have constructed the with-memory methods by convergence rates of $3, 6, 12,$ and 24. Numerical examples will be employed in Section 4 to show the effectiveness of our new members from the suggested with-memory classes. Finally, we have obtained numerical results that our proposed methods perform far better than the existing methods.

2. Without Memory Methods

In year 2011 Geum and Kim [32] exhibitted Steffesen-type without memory optimal methods with orders 16:

$$\begin{cases} y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k)-f(y_k)}, z_k = y_k + g(x_k),\ k = 0,1,2,\dots, \\ K(x_k) = g(x_k)\frac{f(x_k)f(z_k)}{(f(x_k)-f(z_k))(f(y_k)-f(z_k))},\ s_k = z_k + H(x_k) + K(x_k), \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k)\frac{f(q_k)}{(f(x_k)-f(q_k))(f(y_k)-f(q_k))(f(z_k)-f(q_k))}, \\ H(x_k) = T(x_k)((f(x_k)f(y_k) + f(z_k)^2 - f(z_k)f(q_k)), \\ h1_k = (f(z_k)(f(z_k) - f(s_k)) + f(x_k)f(y_k))f(z_k), \\ h2_k = f(q_k)(f(q_k) - f(s_k))(-f(x_k) - f(y_k) + f(q_k) + f(s_k)), \\ t1_k = f(x_k)f(y_k)(h1_k - h2_k) + f(z_k)f(q_k) \\ .(f(z_k) - f(q_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ t2_k = (f(x_k) - f(s_k))(f(y_k) - f(s_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ W(x_k) = T(x_k)f(s_k)\frac{t1_k}{t2_k},\ x_{k+1} = s_k + W(x_k). \end{cases} \tag{2.1}$$

These methods can be rewritten as four memory-free methods with different convergence rates of two, four, eight, and sixteen:

(1) A one parameter family of second order one-step iterative methods (TM2):

$$\begin{cases} y_k = x_k + \theta f(x_k),\ \theta \in \mathbb{R},\ k = 0,1,2,\dots, \\ g(x_k) = \frac{\theta f(x_k)f(y_k)}{f(x_k)-f(y_k)},\ x_{k+1} = y_k + g(x_k). \end{cases} \tag{2.2}$$

(2) We have a new method of fourth-order convergence (TM4):

$$\begin{cases} y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k)f(y_k)}{f(x_k)-f(y_k)},\ \theta \in \mathbb{R},\ k = 0,1,2,\dots, \\ z_k = y_k + g(x_k),\ K(x_k) = g(x_k)\frac{f(x_k)f(z_k)}{(f(x_k)-f(z_k))(f(y_k)-f(z_k))},\ x_{k+1} = z_k + K(x_k). \end{cases} \tag{2.3}$$

(3) We can construct a new family three-point of methods by (2.1) as follows (TM8) :

$$\begin{cases} y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k)f(y_k)}{f(x_k)-f(y_k)},\ \theta \in \mathbb{R},\ k = 0,1,2,\dots, \\ z_k = y_k + g(x_k),\ K(x_k) = g(x_k)\frac{f(x_k)f(z_k)}{(f(x_k)-f(z_k))(f(y_k)-f(z_k))}, \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k)\frac{f(q_k)}{(f(x_k)-f(q_k))(f(y_k)-f(q_k))(f(z_k)-f(q_k))}, \\ H(x_k) = T(x_k)((f(x_k)f(y_k) + f(z_k)^2 - f(z_k)f(q_k)), \\ x_{k+1} = z_k + H(x_k) + K(x_k). \end{cases} \tag{2.4}$$

(4) A sixteenth-order four-step iterative method suggested by Geum and Kim (2011) is given by (GKM):

$$\begin{cases} y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k) - f(y_k)},\ \theta \in \mathbb{R},\ k = 0, 1, 2, \ldots, \\ z_k = y_k + g(x_k),\ K(x_k) = g(x_k) \frac{f(x_k) f(z_k)}{(f(x_k) - f(z_k))(f(y_k) - f(z_k))}, \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k) \frac{f(q_k)}{(f(x_k) - f(q_k))(f(y_k) - f(q_k))(f(z_k) - f(q_k))}, \\ H(x_k) = T(x_k)(f(x_k) f(y_k) + f(z_k)^2 - f(z_k) f(q_k)),\ s_k = z_k + H(x_k) + K(x_k), \\ h1_k = (f(z_k)(f(z_k) - f(s_k)) + f(x_k) f(y_k)) f(z_k), \\ h2_k = f(q_k)(f(q_k) - f(s_k))(-f(x_k) - f(y_k) + f(q_k) + f(s_k)), \\ t1_k = f(x_k) f(y_k)(h1_k - h2_k) + f(z_k) f(q_k) \\ .(f(z_k) - f(q_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ t2_k = (f(x_k) - f(s_k))(f(y_k) - f(s_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ W(x_k) = T(x_k) f(s_k) \frac{t1_k}{t2_k},\ x_{k+1} = s_k + W(x_k). \end{cases} \tag{2.5}$$

We prove that the methods of the preceding family are second-, fourth- ,eighth- and sixteenth-order convergent through the theorem.

Theorem 2.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the Steffensen-type method by* (2.2) *has the order of convergence two and satisfies the error equation*

$$e_{k+1} = (1 + \theta f'(\alpha)) c_2 e_k^2 + O(e_k^3), \tag{2.6}$$

the convergence order of the method (2.3) *is 4 and the error equation for the method is given as*

$$e_{k+1} = (1 + \theta f'(\alpha))^2 c_2 (2c_2^2 - c_3) e_k^4 + O(e_k^5), \tag{2.7}$$

and the convergence order of the method (2.4) *is 8 and the error equation for the method is given as*

$$e_{k+1} = (1 + \theta f'(\alpha))^4 c_2^2 (2c_2^2 - c_3)(5c_2^3 - 5c_2 c_3 + c_4) e_k^8 + O(e_k^9), \tag{2.8}$$

the method (2.5) *is at least of sixteenth-order, and satisfies the error equation*

$$\begin{aligned} e_{k+1} =& (1 + \theta f'(\alpha))^8 (2c_2^2 - c_3)^2 (5c_2^3 - 5c_2 c_3 + c_4)(14c_2^4 - 21c_2^2 c_3 + 3c_3^2 \\ & + 6c_2 c_4 - c_5) c_2^4 e_k^{16} + O(e_k^{17}). \end{aligned} \tag{2.9}$$

Proof. We have earned the convergence order of these classes employing Taylor's expansions for the different expressions of the iterative method. Therefore, we show the Mathematica code for obtaining the mentioned Taylor's series. First, we define the Taylor series of $f(x)$ as follows:

$$In[1] : f[e\_] = fla(e + c_2 e^2 + \cdots + c_{16} e^{16}),$$

where $e = x - \alpha,\ fla = f'(\alpha)$. Note that since α is a simple zero of $f(x)$, the $f'(\alpha) \neq 0, f(\alpha) = 0$. We define

$$In[2] : ey = Series[e + \theta f[e], \{e, 0, 16\}]//FullSimplify$$

$$Out[2] : ey = (1 + \theta fla)e \tag{2.10}$$

$$In[3] : g[e] = Series[\frac{\theta f[e] f[ey]}{f[e] - f[ey]}, \{e, 0, 16\}];$$

$$In[4] : ez = ey + Series[g[e],\ \{e, 0, 16\}]//FullSimplify$$

$$Out[4] : ey = (1 + \theta fla)c_2 e^2 + O(e^3) \tag{2.11}$$

$$In[5] : K[e] = Series[(\frac{g[e]f[e]f[ez]}{(f[e] - f[ez])(f[ey] - f[ez])}, \{e, 0, 16\}];$$

$$In[6] : eq = ez + Series[K[e],\ \{e, 0, 16\}]//FullSimplify$$

$$Out[6] : eq = (1 + \theta f'(\alpha))^2 c_2(2c_2^2 - c_3)e^4 + O(e^5) \tag{2.12}$$

$$In[7] : T[e] = Series[(\frac{K[e]f[eq]}{(f[e] - f[eq])(f[ey] - f[eq])(f[ez] - f[eq])}, \{e, 0, 16\}];$$

$$In[8] : H[e] = Series[T[e](f[e]f[ey] + (f[ez])^2 - f[ez]f[eq]), \{e, 0, 16\}];$$

$$In[9] : es = ez + Series[K[e] + H[e],\ \{e, 0, 16\}]//FullSimplify$$

$$Out[9] : es = (1 + \theta f'(\alpha))^4 c_2^2(2c_2^2 - c_3)(5c_2^3 - 5c_2c_3 + c_4)e^8 + O(e^9) \tag{2.13}$$

$$In[10] : h1 = Series[(f[ez](f[ez] - f[es]) + f[e]f[ey])f[ez]),\ \{e, 0, 16\}];$$

$$In[11] : h2 = Series[f[eq](f[eq] - f[es])(-f[e] - f[ey] + f[eq] + f[es]),\ \{e, 0, 16\}];$$

$$In[12]: t1 = Series[f[e]f[ey](h1-h2)+f[ez]f[eq](f[ez]-f[eq])(f[ez]-f[es])(f[eq]-f[es]),\{e,0,16\}];$$

$$In[13]: t2 = Series[(f[e]-f[es])(f[ey]-f[es])(f[ez]-f[es])(f[eq]-f[es]),\{e,0,16\}];$$

$$In[14]: W[e] = Series[T[e]f[es]\frac{t1}{t2},\{e,0,16\}];$$

$$In[15]: e_{k+1} = es + Series[W[e],\ \{e,0,16\}]//FullSimplify$$

$$Out[15]: e_{k+1} = (1+\theta f'(\alpha))^8(2c_2^2-c_3)^2(5c_2^3-5c_2c_3+c_4)(14c_2^4-21c_2^2c_3+3c_3^2+6c_2c_4-c_5)c_2^4e^{16}+O(e^{17}) \tag{2.14}$$

The outputs of (2.11), (2.12), (2.13) and (2.14) prove the relationships of (2.2), (2.3), (2.4) and (2.5), respectively.

□

All of the error equations (2.6), (2.7), (2.8), and (2.9) have a free parameter. This self-accelerating parameter has an important role in the further acceleration convergence for the new families of methods, e. g. (2.2), (2.3), (2.4) and (2.5). We have used Traub's idea in the next section to improve the convergence degree of these methods.

3. New Class of Iterative Methods with Memory

It is clear from error equations (2.6), (2.7), (2.8), and (2.9) that the order of convergence of the family (2.2), (2.3), (2.4) and (2.5) is two, four, eight and sixteen respectively, when $(1+\theta f'(\alpha)) \neq 0$ Therefore, it is possible to increase the convergence speed of the proposed class (2.2), (2.3), (2.4), and (2.5), if $(1+\theta f'(\alpha)) = 0$ of $f'(\alpha)$ is not available in practice and such acceleration is not possible. Instead of that, we could use approximations $f'(\alpha) \approx \tilde{f}'(\alpha)$ and calculated by already available information. Therefore, by setting $\theta = \frac{1}{\tilde{f}'(\alpha)}$ convergence order increase without using any new functional evaluation. Hence, the main idea in constructing with memory methods consists of the calculation of the parameter $\theta = \theta_k$ as the iteration proceeds by the formula $\theta_k = \frac{1}{\tilde{f}'(\alpha)}$ for $k = 2, 3, \cdots$. We consider the following formula for θ_k:

Method $TM3$:

$$\theta_k = -\frac{1}{\tilde{f}'(\alpha)} = -\frac{1}{N_2'(x_k)}, \tag{3.1}$$

where $N_2'(x_k)$ is Newton's interpolation polynomial go through the nodes $\{x_k, x_{k-1}, y_{k-1}\}$.

Method $TM6$:

$$\theta_k = -\frac{1}{\tilde{f}'(\alpha)} = -\frac{1}{N_3'(x_k)}, \tag{3.2}$$

where $N_3'(x_k)$ is Newton's interpolation polynomial go through the nodes $\{x_k, x_{k-1}, y_{k-1}, z_{k-1}\}$.

Method $TM12$:

$$\theta_k = -\frac{1}{\tilde{f}'(\alpha)} = -\frac{1}{N_4'(x_k)}, \tag{3.3}$$

where $N_4'(x_k)$ is Newton's interpolation polynomial go through the nodes $\{x_k, x_{k-1}, y_{k-1}, z_{k-1}, q_{k-1}\}$.

Method $TM24$:

$$\theta_k = -\frac{1}{\tilde{f}'(\alpha)} = -\frac{1}{N_5'(x_k)}, \tag{3.4}$$

where $N_5'(x_k)$ is Newton's interpolation polynomial go through the nodes $\{x_k, x_{k-1}, y_{k-1}, z_{k-1}, q_{k-1}, s_{k-1}\}$.

Using the pair of relationships (2.2) and (3.1), (2.3) and (3.2), (2.4) and (3.3), (2.5) and (3.4), we can define new memory methods as follows:

$$\begin{cases} \theta_k = -\frac{1}{N_2'(x_k)},\ k = 1, 2, 3, \cdots, \\ y_k = x_k + \theta f(x_k),\ k = 0, 1, 2, \ldots, \\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k) - f(y_k)},\ x_{k+1} = y_k + g(x_k), \end{cases} \tag{3.5}$$

$$\begin{cases} \theta_k = -\frac{1}{N_3'(x_k)},\ k = 1, 2, 3, \cdots, \\ y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k) - f(y_k)},\ k = 0, 1, 2, \ldots, \\ z_k = y_k + g(x_k),\ K(x_k) = g(x_k) \frac{f(x_k) f(z_k)}{(f(x_k) - f(z_k))(f(y_k) - f(z_k))},\ x_{k+1} = z_k + K(x_k). \end{cases} \tag{3.6}$$

$$\begin{cases} \theta_k = -\frac{1}{N_4'(x_k)},\ k = 1,2,3,\cdots, \\ y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k)-f(y_k)},\ k = 0,1,2,\ldots, \\ z_k = y_k + g(x_k),\ K(x_k) = g(x_k)\frac{f(x_k)f(z_k)}{(f(x_k)-f(z_k))(f(y_k)-f(z_k))}, \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k)\frac{f(q_k)}{(f(x_k)-f(q_k))(f(y_k)-f(q_k))(f(z_k)-f(q_k))}, \\ H(x_k) = T(x_k)(f(x_k)f(y_k) + f(z_k)^2 - f(z_k)f(q_k)), \\ x_{k+1} = z_k + H(x_k) + K(x_k). \end{cases} \tag{3.7}$$

$$\begin{cases} \theta_k = -\frac{1}{N_5'(x_k)},\ k = 1,2,3,\cdots, \\ y_k = x_k + \theta f(x_k),\ g(x_k) = \frac{\theta f(x_k) f(y_k)}{f(x_k)-f(y_k)},\ k = 0,1,2,\ldots, \\ q_k = z_k + K(x_k),\ T(x_k) = K(x_k)\frac{f(q_k)}{(f(x_k)-f(q_k))(f(y_k)-f(q_k))(f(z_k)-f(q_k))}, \\ H(x_k) = T(x_k)(f(x_k)f(y_k) + f(z_k)^2 - f(z_k)f(q_k)),\ s_k = z_k + H(x_k) + K(x_k), \\ h1_k = (f(z_k)(f(z_k) - f(s_k)) + f(x_k)f(y_k))f(z_k), \\ h2_k = f(q_k)(f(q_k) - f(s_k))(-f(x_k) - f(y_k) + f(q_k) + f(s_k)), \\ t1_k = f(x_k)f(y_k)(h1_k - h2_k) + f(z_k)f(q_k)(f(z_k) - f(q_k)) \\ .(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ t2_k = (f(x_k) - f(s_k))(f(y_k) - f(s_k))(f(z_k) - f(s_k))(f(q_k) - f(s_k)), \\ W(x_k) = T(x_k)f(s_k)\frac{t1_k}{t2_k},\ x_{k+1} = s_k + W(x_k). \end{cases} \tag{3.8}$$

Before expressing and confirming the main theorem of convergence of with memory methods, let us first consider the following lemma used to prove the theorem.

Lemma 3.1. *The following estimates are satisfied:*

$$\begin{cases} (1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}, & for \quad Eq.\ (3.5) \\ (1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}e_{k-1,z}, & for \quad Eq.\ (3.6) \\ (1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}, & for \quad Eq.\ (3.7), \\ (1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}e_{k-1,s}, & for \quad Eq.\ (3.8), \end{cases} \tag{3.9}$$

Proof. Suppose that there are $s+1$ nodes $t_0, t_1, \cdots, t_s$ from the interval $D = [a, b]$, where a is the minimum and b is the maximum of these nodes, respectively. Then the error of Newton's interpolation polynomial $N_s(t)$ of degree s is given by

$$f(t) - N_s(t) = \frac{f^{(s+1)}(\alpha)}{(s+1)!}\prod_{j=0}^{s}(t - t_j). \tag{3.10}$$

For $s = 2$ the above equation assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = y_{k-1}$, $t_2 = x_k$)

$$f(t) - N_2(t) = \frac{f^{(3)}(\alpha)}{3!}(t - x_k)(t - x_{k-1})(t - y_{k-1}). \tag{3.11}$$

Using above result and with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_2'(x_k) = \frac{f'''(\alpha)}{3!}(x_k - x_{k-1})(x_k - y_{k-1}). \tag{3.12}$$

Now

$$x_k - x_{k-1} = (x_k - \alpha) - (x_{k-1} - \alpha) = e_k - e_{k-1}.$$

Similarly

$$x_k - y_{k-1} = e_k - e_{k-1,y}.$$

Substituting these relations in Equation (3.12) and simplifying we get

$$\begin{aligned} N_2'(x_k) =& f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + ...) - \frac{f'''(\alpha)}{3!}(e_k - e_{k-1})(e_k - e_{k-1,y}) \\ &\sim f'(\alpha)(1 + 2c_2e_k - c_3e_{k-1}e_{k-1,y}). \end{aligned} \tag{3.13}$$

And thus

$$1 + f'(\alpha)\theta_k \sim 1 - \frac{1}{1 + 2c_2e_k - c_3e_{k-1}e_{k-1,y}} \sim e_{k-1}e_{k-1,y}. \tag{3.14}$$

Proof of relationship $(1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}$, for Equation (3.5).

For $s = 3$ the above equation(3.10) assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = y_{k-1}$, $t_2 = z_{k-1}$, $t_3 = x_k$)

$$f(t) - N_3(t) = \frac{f^{(4)}(\alpha)}{4!}(t - x_k)(t - z_{k-1})(t - y_{k-1})(t - x_{k-1}). \tag{3.15}$$

Using above result and with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_3'(x_k) = \frac{f^{(4)}(\alpha)}{4!}(x_k - x_{k-1})(x_k - y_{k-1})(x_k - z_{k-1}). \tag{3.16}$$

Now

$$x_k - x_{k-1} = (x_k - \alpha) - (x_{k-1} - \alpha) = e_k - e_{k-1}.$$

Similarly

$$x_k - y_{k-1} = e_k - e_{k-1,y},$$
$$x_k - z_{k-1} = e_k - e_{k-1,z}.$$

Substituting these relations in Equation (3.16) and simplifying we get

$$N_3'(x_k) = f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + ...) - \frac{f^{(4)}(\alpha)}{4!}(e_k - e_{k-1})(e_k - e_{k-1,y})$$
$$.(e_k - e_{k-1,z}) \sim f'(\alpha)(1 + 2c_2e_k + c_4e_{k-1}e_{k-1,y}e_{k-1,z}). \qquad (3.17)$$

And thus

$$1 + f'(\alpha)\theta_k \sim 1 - \frac{1}{1 + 2c_2e_k + c_4e_{k-1}e_{k-1,y}e_{k-1,z}} \sim e_{k-1}e_{k-1,y}e_{k-1,z}. \qquad (3.18)$$

Proof of relationship $(1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}$ Equation (3.6).

For $s = 4$ the above Equation (3.10) assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = y_{k-1}$, $t_2 = z_{k-1}$, $t_3 = q_{k-1}$ $t_4 = x_k$)

$$f(t) - N_4(t) = \frac{f^{(5)}(\alpha)}{5!}(t - x_k)(t - q_{k-1})(t - z_{k-1})(t - y_{k-1})(t - x_{k-1}). \qquad (3.19)$$

Using above result and with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_4'(x_k) = \frac{f^{(5)}(\alpha)}{5!}(x_k - x_{k-1})(x_k - y_{k-1})(x_k - z_{k-1})(x_k - q_{k-1}) \qquad (3.20)$$

Now

$$x_k - x_{k-1} = (x_k - \alpha) - (x_{k-1} - \alpha) = e_k - e_{k-1}.$$

Similarly

$$x_k - y_{k-1} = e_k - e_{k-1,y},$$
$$x_k - z_{k-1} = e_k - e_{k-1,z},$$
$$x_k - q_{k-1} = e_k - e_{k-1,q}.$$

Substituting these relations in Equation (3.20) and simplifying we get

$$N_4'(x_k) = f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + ...) - \frac{f^{(5)}(\alpha)}{5!}$$
$$(e_k - e_{k-1})(e_k - e_{k-1,y})(e_k - e_{k-1,z})(e_k - e_{k-1,q})$$
$$\sim f'(\alpha)(1 + 2c_2e_k - c_5e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}) \qquad (3.21)$$

And thus

$$1 + f'(\alpha)\theta_k \sim 1 - \frac{1}{1 + 2c_2e_k - c_5e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}} \sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}. \qquad (3.22)$$

Proof of relationship $(1+\theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}$ for Equation (3.7).

For $s = 5$ the above Equation(3.10) assumes the form (keeping in the mind $t_0 = x_{k-1}$, $t_1 = y_{k-1}$, $t_2 = z_{k-1}$, $t_3 = q_{k-1}$, $t_4 = s_{k-1}$, $t_5 = x_k$)

$$f(t) - N_5(t) = \frac{f^{(6)}(\alpha)}{6!}(t - x_k)(t - s_{k-1})(t - q_{k-1})(t - z_{k-1})(t - y_{k-1})(t - x_{k-1}). \tag{3.23}$$

Using above result and with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_5'(x_k) = \frac{f^{(6)}(\alpha)}{6!}(x_k - x_{k-1})(x_k - y_{k-1})(x_k - z_{k-1})(x_k - q_{k-1})(x_k - s_{k-1}) \tag{3.24}$$

Now

$$x_k - x_{k-1} = (x_k - \alpha) - (x_{k-1} - \alpha) = e_k - e_{k-1}.$$

Similarly

$$\begin{aligned} x_k - y_{k-1} &= e_k - e_{k-1,y}, \\ x_k - z_{k-1} &= e_k - e_{k-1,z}, \\ x_k - q_{k-1} &= e_k - e_{k-1,q}, \\ x_k - s_{k-1} &= e_k - e_{k-1,s}. \end{aligned}$$

Substituting these relations in Equation (3.20) and simplifying we get

$$\begin{aligned} N_5'(x_k) &= f'(\alpha)(1 + 2c_2e_k + 3c_3e_k^2 + ...) \\ &- \frac{f^{(6)}(\alpha)}{6!}(e_k - e_{k-1})(e_k - e_{k-1,y})(e_k - e_{k-1,z})(e_k - e_{k-1,q})(e_k - e_{k-1,s}) \\ &\sim f'(\alpha)(1 + 2c_2e_k + c_6e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}e_{k-1,s}) \end{aligned} \tag{3.25}$$

And thus

$$\begin{aligned} 1 + f'(\alpha)\theta_k &\sim 1 - \frac{1}{1 + 2c_2e_k + c_6e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}e_{k-1,s}} \\ &\sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}e_{k-1,s}. \end{aligned} \tag{3.26}$$

Proof of relationship $(1 + \theta_k f'(\alpha)) \sim e_{k-1}e_{k-1,y}e_{k-1,z}e_{k-1,q}e_{k-1,s}$ for Equation (3.8) finishes. □

Now we can state the following convergence theorem with memory methods (3.5), (3.6), (3.7), and (3.8).

Theorem 3.2. *If an initial guess x_0 is sufficiently close to the zero α of $f(x)$ and the parameters θ_k in the iterative scheme* (3.5), (3.6), (3.7) *and* (3.8) *are recursively calculated then, the R-order of convergence of with memory methods are at least 3, 6, 12 and 24 respectively.*

Proof. First, we assume that the R-order of convergence of sequence x_k, y_k, z_k, q_k and s_k are at least r, r_1, r_2, r_3, and, r_4, respectively. Hence

$$e_{k+1} \sim e_k^r \sim (e_{k-1}^r)^r \sim e_{k-1}^{r^2}, \tag{3.27}$$

and

$$e_{k,y} \sim e_k^{r_1} \sim (e_{k-1}^r)^{r_1} \sim e_{k-1}^{rr_1}, \tag{3.28}$$

similarly

$$e_{k,z} \sim e_k^{r_2} \sim (e_{k-1}^r)^{r_2} \sim e_{k-1}^{rr_2}, \tag{3.29}$$

$$e_{k,q} \sim e_k^{r_3} \sim (e_{k-1}^r)^{r_3} \sim e_{k-1}^{rr_3}, \tag{3.30}$$

$$e_{k,s} \sim e_k^{r_4} \sim (e_{k-1}^r)^{r_4} \sim e_{k-1}^{rr_4}, \tag{3.31}$$

We will present the proof of each part of the case separately:

Method TM3

Combining (2.10) and , (2.11) we find

$$e_{k,y} \sim (1 + f'(\alpha)\theta_k)e_k, \tag{3.32}$$

$$e_{k+1} \sim (1 + f'(\alpha)\theta_k)e_k^2. \tag{3.33}$$

Using the results of lemma (3.1) in the Equations (3.32) and (3.33), we obtain

$$e_{k,y} \sim e_{k-1}^{r_1+r+1}, \tag{3.34}$$

$$e_{k+1} \sim e_{k-1}^{(r_1+1)+2r}. \tag{3.35}$$

By comparing exponents of e_{k-1} appearing in two pairs of relations (3.28, 3.34) and (3.27, 3.35), we get the following system of equations

$$\begin{cases} rr_1 - r_1 - r - 1 = 0, \\ r^2 - (r_1 + 1) - 2r = 0. \end{cases} \tag{3.36}$$

Positive solution of the system (3.36) is given by $r_1 = 2$ and, $r = 3$, so that convergence order of the family (3.5) of one-point methods with memory is three.

Method TM6.

Considering (2.10), (2.11) and , (2.12), we have

$$e_{k,y} \sim (1 + f'(\alpha)\theta_k)e_k, \tag{3.37}$$

$$e_{k,z} \sim (1 + f'(\alpha)\theta_k)e_k^2, \tag{3.38}$$

$$e_{k+1} \sim (1 + f'(\alpha)\theta_k)^2 e_k^4. \tag{3.39}$$

Using the results of lemma (3.1) and the relationships (3.37)-(3.39), we have

$$e_{k,y} \sim e_{k-1}^{r_2+r_1+1+r}, \tag{3.40}$$

$$e_{k,z} \sim e_{k-1}^{(r_2+r_1+1)+2r}, \tag{3.41}$$

$$e_{k+1} \sim e_{k-1}^{2(r_2+r_1+1)+4r}. \tag{3.42}$$

Comparing the exponents of e_{k-1} on the right hand sides of (3.28, 3.40), (3.29, 3.41),and (3.27, 3.42), we form the system of three equations in $r_1,\ r_2$, and r

$$\begin{cases} rr_1 - r_2 - r_1 - 1 - r = 0, \\ rr_2 - (r_2 + r_1 + 1) - 2r = 0, \\ r^2 - 2(r_2 + r_1 + 1) - 4r = 0. \end{cases}$$

This system has the solutions $r_1 = 2,\ r_2 = 3,\ r = 6.$ Thus, we can conclude that the lower bound of the R-order of the with memory methods (3.6) is six.

Methed TM12.

From (2.10), (2.11), (2.12) and (2.13), we obtain

$$e_{k,y} \sim (1 + f'(\alpha)\theta_k)e_k, \tag{3.43}$$

$$e_{k,z} \sim (1 + f'(\alpha)\theta_k)e_k^2, \tag{3.44}$$

$$e_{k,q} \sim (1 + f'(\alpha)\theta_k)^2 e_k^4, \tag{3.45}$$

$$e_{k+1} \sim (1 + f'(\alpha)\theta_k)^4 e_k^8, \tag{3.46}$$

Using the results of lemma (3.1) in the Equations (3.43)-(3.46), we conclude

$$e_{k,y} \sim e_{k-1}^{r_3+r_2+r_1+1+r}, \tag{3.47}$$

$$e_{k,z} \sim e_{k-1}^{(r_3+r_2+r_1+1)+2r}, \tag{3.48}$$

$$e_{k,q} \sim e_{k-1}^{2(r_3+r_2+r_1+1)+4r}, \tag{3.49}$$

$$e_{k+1} \sim e_{k-1}^{4(r_3+r_2+r_1+1)+8r}. \tag{3.50}$$

Comparing the exponents of e_{k-1} on the right hand sides of (3.28, 3.47), (3.29 3.48), (3.30, 3.49), and (3.27, 3.50), we form the system of three equations in $r_1,\ r_2,\ r_3$, and r

$$\begin{cases} rr_1 - r_3 - r_2 - r_1 - 1 - r = 0, \\ rr_2 - (r_3 + r_2 + r_1 + 1) - 2r = 0, \\ rr_3 - 2(r_3 + r_2 + r_1 + 1) - 4r = 0, \\ r^2 - 4(r_3 + r_2 + r_1 + 1) - 8r = 0. \end{cases} \tag{3.51}$$

The positive real solution of (3.51) is $r_1 = 2$, $r_2 = 3$, $r_3 = 6$, $r = 12$. Therefore, the convergence R-order for Equation (3.7) is 12.

Method TM24

With the use of (2.10), (2.11), (2.12), (2.13), and (2.14), we get

$$e_{k,y} \sim (1 + f'(\alpha)\theta_k)e_k, \tag{3.52}$$

$$e_{k,z} \sim (1 + f'(\alpha)\theta_k)e_k^2, \tag{3.53}$$

$$e_{k,q} \sim (1 + f'(\alpha)\theta_k)^2 e_k^4, \tag{3.54}$$

$$e_{k,s} \sim (1 + f'(\alpha)\theta_k)^4 e_k^8, \tag{3.55}$$

$$e_{k+1} \sim (1 + f'(\alpha)\theta_k)^8 e_k^{16}. \tag{3.56}$$

Using the results of lemma (3.1) in the Equations (3.52)-(3.56), we conclude

$$e_{k,y} \sim e_{k-1}^{r_4+r_3+r_2+r_1+1+r}, \tag{3.57}$$

$$e_{k,z} \sim e_{k-1}^{(r_4+r_3+r_2+r_1+1)+2r}, \tag{3.58}$$

$$e_{k,q} \sim e_{k-1}^{2(r_4+r_3+r_2+r_1+1)+4r}, \tag{3.59}$$

$$e_{k,s} \sim e_{k-1}^{4(r_4+r_3+r_2+r_1+1)+8r}, \tag{3.60}$$

$$e_{k+1} \sim e_{k-1}^{8(r_4+r_3+r_2+r_1+1)+16r}. \tag{3.61}$$

Comparing the exponents of e_{k-1} on the right hand sides of (3.28, 3.57), (3.29 3.58), (3.30, 3.59), (3.31, 3.60), and (3.27, 3.61), we form the system of three equations in r_1, r_2, r_3, r_4, and r

$$\begin{cases} rr_1 - r_4 - r_3 - r_2 - r_1 - 1 - r = 0, \\ rr_2 - (r_4 + r_3 + r_2 + r_1 + 1) - 2r = 0, \\ rr_3 - 2(r_4 + r_3 + r_2 + r_1 + 1) - 4r = 0, \\ rr_4 - 4(r_4 + r_3 + r_2 + r_1 + 1) - 8r = 0, \\ r^2 - 8(r_4 + r_3 + r_2 + r_1 + 1) - 16r = 0. \end{cases} \tag{3.62}$$

After solving these equations, we get $r_1 = 2$, $r_2 = 3$, $r_3 = 6$, $r_4 = 12$, $r = 24$. This indicates that the presented method (3.8) is of twenty fourth-order convergence. □

Remark 3.3. We found that the without memory method (2.2), (2.3), (2.4) and (2.5) have the optimal order. Their efficiency index are $EI = 2^{1/2} \simeq 1.41421$, $EI = 4^{1/3} \simeq 1.58740$, $EI = 8^{1/4} \simeq 1.68179$, $EI = 16^{1/5} \simeq 1.74110$, respectively. Therefore, the families of methods agrees with the conjecture of Kung-Traub.

Remark 3.4. Efficiency index with memory methods (3.5), (3.6), (3.7) and (3.8) equal to $EI = 3^{1/2} \simeq 1.73205, EI = 6^{1/3} \simeq 1.81712, EI = 12^{1/4} \simeq 1.86121$, $EI = 24^{1/5} \simeq 1.88818$, respectively. Therefore, which is much better than the optimal one until four-point optimal methods without memory.

4. Numerical results

Now, we further want to check the efficiency of the proposed scheme and validate the theoretical results. For this purpose, we use the some test functions [68] in Table 1. For better comparisons of our proposed methods with other existing ones, we have given two types of comparison tables in each test function:
(a) Absolute error between the two consecutive iterations $|x_{n+1} - x_n|$
(b)Absolute residual error in the corresponding function $(|f(x_n)|)$.
The parameter $\theta = 0.1$ is used in the first iteration. The errors of approximations to the corresponding zeros of test functions are displayed in Tables 2, 3, where $A(\pm h)$ denotes $A \times 10^{\pm h}$. These tables include the values of the computational order of convergence COC calculated by the formula [68]

$$COC = \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|} \tag{4.1}$$

The following abbreviations are used in tables two to seven:
(i) EF : Evaluation Fuction.
(ii) EI: Efficiency Index.
(iii) ρ: The order of convergence.
(iv) Iter: Total number of iteration.
(v) div: The corresponding iterative method is divergent for the initial guess.

5. conclusion

By theoretical analysis and numerical experiments, we demonstrate that the classes of Steffensen-like methods to achieve the optimal two-, fourth- eight-, and sixteen-order convergence for solving the simple root of nonlinear equations. The proposed families have a higher efficiency index than the methods obtained using Hermit and Newton interpolation. Four special cases,TM3 (3.5), TM6 (3.6), TM12 (3.7) and TM24 (3.8), of the proposed with-memory families, are also obtained. The new methods possess a very high computational efficiency index $EI = 3^{1/2} \simeq 1.73205, EI = 6^{1/3} \simeq 1.81712, EI = 12^{1/4} \simeq 1.86121$, $EI = 24^{1/5} \simeq 1.90365$, respectively, which are even higher than the efficiency of many

TABLE 1. Test functions

Nonlinear function	Zero	Initial guess
$f_1(x) = x\log(1+x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x)$	$\alpha = 0$	$x_0 = 0.6$
$f_2(x) = \sin(5x)e^x - 2$	$\alpha \approx 1.36$	$x_0 = 1.2$
$f_3(x) = 1 + \frac{1}{x^4} - \frac{1}{x} - x^2$	$\alpha = 1$	$x_0 = 1.4$
$f_4(x) = (x-2)(x^{10}+x+2)e^{-5x}$	$\alpha = 2$	$x_0 = 2.2$
$f_5(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1$	$\alpha = -1$	$x_0 = -1.65$
$f_6(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2}$	$\alpha = 1$	$x_0 = 1.5$
$f_7(x) = \log(1+x^2) + e^{-3x+x^2}\sin(x)$	$\alpha = 0$	$x_0 = 0.5$
$f_8(x) = x^3 + 4x^2 - 10$	$\alpha \approx 1.3652$	$x_0 = 1$
$f_9(x) = x\log(1-\pi+x^2) - \frac{1+x^2}{1+x^3}\sin(x^2) + \tan(x^2)$	$\alpha = \sqrt{\pi}$	$x_0 = 1.7$

of the developed with-memory methods. One can observe that the proposed methods competitive to methods [1, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 47, 48, 49, 50, 51, 52, 53, 55].

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TABLE 2. Comparison of the absolute error of proposed methods for functions Table 1

$f_1(x) = x\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x), \alpha = 0, x_0 = 0.6$					
Methods	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	COC	EI
TM3, $\theta_0 = 0.1$	0.47811(0)	0.69702(−1)	0.30072(−3)	3.00000	1.73205
TM6, $\theta_0 = 0.1$	0.25323(0)	0.55497(−4)	0.38415(−26)	6.00000	1.81712
TM12, $\theta_0 = 0.1$	0.23230(−1)	0.86135(−19)	0.14421(−225)	12.00000	1.86121
TM24, $\theta_0 = 0.1$	0.10110(−2)	0.46606(−72)	0.54641(−1735)	24.00000	1.88818
$f_2(x) = \sin(5x)e^x - 2, \alpha \approx 1.36, x_0 = 1.2$					
TM3, $\theta_0 = 0.1$	0.40712(0)	0.51556(0)	0.45948(0)	3.00000	1.73205
TM6, $\theta_0 = 0.1$	0.27457(0)	0.17396(−1)	0.39732(−2)	6.00000	1.81712
TM12, $\theta_0 = 0.1$	0.33552(0)	0.45898(0)	0.45888(0)	12.00000	1.86121
TM24, $\theta_0 = 0.1$	0.34911(0)	0.45888(0)	0.45888(0)	24.00000	1.88818
$f_3(x) = 1 + \frac{1}{x^4} - \frac{1}{x} - x^2, \alpha = 1, x_0 = 1.4$					
TM3, $\theta_0 = 0.1$	0.60801(−1)	0.88768(−3)	0.78586(−9)	3.00000	1.73205
TM6, $\theta_0 = 0.1$	0.53415(−2)	0.24470(−12)	0.21083(−73)	6.00000	1.81712
TM12, $\theta_0 = 0.1$	0.13484(−3)	0.28949(−43)	0.25257(−518)	12.00000	1.86121
TM24, $\theta_0 = 0.1$	0.16059(−7)	0.38697(−185)	0.22015(−4449)	24.00000	1.88818
$f_4(x) = (x - 2)(x^{10} + x + 2)e^{-5x}, \alpha = 2, x_0 = 2.2$					
TM3, $\theta_0 = 0.1$	0.20469(−1)	0.14492(−6)	0.91831(−20)	3.00000	1.73205
TM6, $\theta_0 = 0.1$	0.11395(−2)	0.60291(−20)	0.77065(−124)	6.00000	1.81712
TM12, $\theta_0 = 0.1$	0.10770(−5)	0.73347(−77)	0.19249(−927)	12.00000	1.86121
TM24, $\theta_0 = 0.1$	0.41335(−11)	0.17804(−286)	0.86843(−6895)	24.00000	1.88818
$f_5(x) = e^{x^3-x} - \cos(x^2 - 1) + x^3 + 1, \alpha = -1, x_0 = -1.65$					
TM3, $\theta_0 = 0.1$	0.21585(0)	0.68092(−2)	0.44848(−6)	3.00000	1.73205
TM6, $\theta_0 = 0.1$	0.65889(−1)	0.40122(−6)	0.79121(−40)	6.00000	1.81712
TM12, $\theta_0 = 0.1$	0.75045(−2)	0.14249(−28)	0.15689(−349)	12.00000	1.86121
TM24, $\theta_0 = 0.1$	0.16552(−4)	0.14956(−112)	0.26039(−2710)	24.00000	1.88818

Table 3. Comparison of the absolute error of proposed methods for functions Table 1

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	COC	EI
$f_6(x)=\dfrac{-5x^2}{2}+x^4+x^5+\dfrac{1}{1+x^2}, \alpha=1, x_0=1.5$					
$\mathrm{TM3}, \theta_0=0.1$	0.41154(0)	0.10843(0)	0.72142(−2)	3.00000	1.73205
$\mathrm{TM6}, \theta_0=0.1$	0.21684(0)	0.66686(−2)	0.13601(−9)	6.00000	1.81712
$\mathrm{TM12}, \theta_0=0.1$	0.95464(−1)	0.40672(−6)	0.45673(−69)	12.00000	1.86121
$\mathrm{TM24}, \theta_0=0.1$	0.29138(−1)	0.70176(−22)	0.95469(−516)	24.00000	1.88818
$f_7(x)=\log(1+x^2)+e^{-3x+x^2}\sin(x), \alpha=0, x_0=0.5$					
$\mathrm{TM3}, \theta_0=0.1$	0.13943(−1)	0.13391(−3)	0.30563(−11)	3.00000	1.73205
$\mathrm{TM6}, \theta_0=0.1$	0.47232(−2)	0.23582(−16)	0.21634(−97)	6.00000	1.81712
$\mathrm{TM12}, \theta_0=0.1$	0.56889(−4)	0.36134(−47)	0.35711(−565)	12.00000	1.86121
$\mathrm{TM24}, \theta_0=0.1$	0.42915(−8)	0.70066(−193)	0.11629(−4628)	24.00000	1.88818
$f_8(x)=10-4x^2+x^3, \alpha\approx 1.3652, x_0=1$					
$\mathrm{TM3}, \theta_0=0.1$	0.27996(0)	0.28623(−2)	0.30018(−4)	3.00000	1.73205
$\mathrm{TM6}, \theta_0=0.1$	0.72203(−1)	0.30019(−4)	0.30013(−4)	6.00000	1.81712
$\mathrm{TM12}, \theta_0=0.1$	0.60804(−2)	0.30013(−4)	0.30013(−4)	12.00000	1.86121
$\mathrm{TM24}, \theta_0=0.1$	0.77617(−4)	0.30013(−4)	0.30013(−4)	24.00000	1.88818
$f_9(x)=x\log(1-\pi+x^2)-\dfrac{1+x^2}{1+x^3}\sin(x^2)+\tan(x^2), \alpha=\sqrt{\pi}, x_0=1.7$					
$\mathrm{TM3}, \theta_0=0.1$	0.11486(−1)	0.18439(−5)	0.11282(−16)	3.00000	1.73205
$\mathrm{TM6}, \theta_0=0.1$	0.19928(−3)	0.11380(−20)	0.11367(−125)	6.00000	1.81712
$\mathrm{TM12}, \theta_0=0.1$	0.53113(−7)	0.11781(−82)	0.16313(−991)	12.00000	1.86121
$\mathrm{TM24}, \theta_0=0.1$	0.19555(−14)	0.49671(−340)	0.10296(−8163)	24.00000	1.88818

TABLE 4. Comparison EF, EI and COC of proposed method (3.5) by one-step methods

$f_1(x) = x\log(1+x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x)$,					$\alpha = 0, \quad x_0 = 0.6$				
without memory methods	EF	Iter	COC	EI	with memory methods	EF	Iter	COC	EI
AM [1]	4	100	0.9260	0.9623	Traub,$\gamma_0 = 0.1$ [66]	2	3	2.4061	1.5512
Newton[49]	2	7	2.0000	1.4142	CPM [10]	2	3	3.0000	1.7321
Halley [54]	3	4	3.0000	1.4142	DPM[25]	2	5	3.0000	1.7321
Chebyshev [54]	3	5	3.0000	1.4422	Secant ,$x_1 = 0.4$	2	15	1.6180	1.6180
Homeier [34]	3	4	3.0000	1.4422	TM3 (3.5)	2	3	3.0000	1.7321
$f_2(x) = \sin(5x)e^x - 2$,					$\alpha \approx 1.36, \quad x_0 = 1.2$				
AM [1]	4	4	3.0000	1.4422	Traub,$\gamma_0 = -0.1$ [66]	2	3	2.3328	1.5274
Newton[49]	2	5	2.0000	1.4142	CPM [10]	2	3	3.0000	1.7321
Halley [54]	3	5	3.0000	1.4142	DPM[25]	2	5	3.0000	1.7320
Chebyshev [54]	3	5	3.0000	1.4422	Secant,$x_1 = 1$	2	17	1.6180	1.6180
Homeier [34]	3	5	3.0000	1.4422	TM3 (3.5)	2	3	3.0000	1.7321
$f_6(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2}$,					$\alpha = 1, \quad x_0 = 1.5$				
AM [1]	4	6	3.0000	1.4422	Traub,$\gamma_0 = 0.1$ [66]	2	4	2.0189	1.4209
Newton[49]	2	9	2.0000	1.4142	CPM [10]	2	3	3.0000	1.7321
Halley [54]	3	6	3.0000	1.4142	DPM[25]	2	4	3.0000	1.7320
Chebyshev [54]	3	6	3.0000	1.4422	Secant,$x_1 = 0.9$	2	16	1.6180	1.6180
Homeier [34]	3	5	3.0000	1.4422	TM3 (3.5)	2	3	3.0000	1.7321
$f_9(x) = x\log(1-\pi+x^2) - \frac{1+x^2}{1+x^3}\sin(x^2) + \tan(x^2)$,					$\alpha = \sqrt{\pi}, \quad x_0 = 1.7$				
methods	EF	Iter	COC	EI	methods	EF	Iter	COC	EI
AM [1]	2	5	1.0000	1.0000	Traub,$\gamma_0 = 0.1$ [66]	2	3	2.4452	1.5637
Newton[49]	2	5	2.0000	1.4142	CPM [10]	2	3	3.0000	1.7321
Halley [54]	3	4	3.0000	1.4142	DPM[25]	2	4	3.0000	1.7320
Chebyshev [54]	3	4	3.0000	1.4422	Secant,$x_1 = 1$	2	14	1.6180	1.6180
Homeier [34]	3	5	3.0000	1.4422	TM3 (3.5)	2	3	3.0000	1.7321

TABLE 5. Comparison EF, EI and COC of proposed methods (3.6) by two-steps methods

without memory methods	EF	Iter	COC	EI	with memory methods	EF	Iter	COC	EI
$f_1(x) = x\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x)$,					$\alpha = 0, \quad x_0 = 0.6$				
CHMTM [20]	3	5	div	div	LMMRM, g_1, $\beta = 0, \gamma = 0.1$, [43]	3	4	6.0000	1.8171
CCGT [12], (4)	3	4	4.0000	1.5874	WM, $\lambda_0 = 0.1$, (24) [69]	3	4	div	div
KTM [40]	4	3	4.0000	1.5874	CPJM, $T_0 = 0.1$ [13]	3	4	5.0000	1.7100
DHM [24]	3	3	3.0000	1.4423	TM6, (3.6)	3	3	6.0000	1.8171
$f_2(x) = \sin(5x)e^x - 2$,					$\alpha \approx 1.36, \quad x_0 = 1.2$				
CHMTM [20]	3	5	div	div	LMMRM, g_1, $\beta = 0, \gamma = 0.1$, [43]	3	5	6.0000	1.8171
CCGT [12], (4)	3	4	4.0000	1.5874	WM, $\lambda_0 = 0.1$, (24) [69]	3	4	div	div
KTM [40]	4	3	4.0000	1.5874	CPJM, $T_0 = 0.1$ [13]	3	4	5.0000	1.7100
DHM [24]	3	3	div	div	TM6, (3.6)	3	3	6.0000	1.8171
$f_6(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2}$,					$\alpha = 1, \quad x_0 = 1.5$				
CHMTM [20]	3	6	4.0000	1.5874	LMMRM, g_1, $\beta = 0, \gamma = 0.1$, [43]	3	5	6.0000	1.8171
CCGT [12], (4)	3	4	4.0000	1.5874	WM, $\lambda_0 = 0.1$, (24) [69]	3	6	4.2361	1.6180
KTM [40]	4	4	4.0000	1.5874	CPJM, $T_0 = 0.1$ [13]	3	4	5.0000	1.7100
DHM [24]	3	31	3.0000	1.4423	TM6, (3.6)	3	3	6.0000	1.8171
$f_7(x) = \log(1 + x^2) + e^{x^2-3x}\sin(x)$,					$\alpha = 0, \quad x_0 = 0.5$				
CHMTM [20]	3	6	4.0000	1.5874	LMMRM, g_1, $\beta = 0, \gamma = 0.1$, [43]	3	4	6.0000	1.8171
CCGT [12], (4)	3	4	4.0000	1.5874	WM, $\lambda_0 = 0.1$, (24) [69]	3	8	4.2361	1.6180
KTM [40]	4	3	4.0000	1.5874	CPJM, $T_0 = 0.1$ [13]	3	4	5.0000	1.7100
DHM[24]	3	3	3.0000	1.4423	TM6, (3.6)	3	3	6.0000	1.8171
$f_9(x) = x\log(1 - \pi + x^2) - \frac{1+x^2}{1+x^3}\sin(x^2) + \tan(x^2)$,					$\alpha = \sqrt{\pi}, \quad x_0 = 1.7$				
CHMTM [20]	3	9	4.0000	1.5874	LMMRM, g_1, $\beta = 0, \gamma = 0.1$, [43]	3	4	6.0000	1.8171
CCGT [12], (4)	3	3	4.0000	1.5874	WM, $\lambda_0 = 0.1$, (24) [69]	3	6	4.2361	1.6180
KTM [40]	4	3	4.0000	1.5874	CPJM, $T_0 = 0.1$ [13]	3	4	5.0000	1.7100
DHM[24]	3	3	1.0000	1.0000	TM6, (3.6)	3	3	6.0000	1.8171

TABLE 6. Comparison EF, EI and COC of proposed methods (3.7) by three- and four- steps methods

$f_1(x) = x\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x)$,					$\alpha = 0, \quad x_0 = 0.6$				
without memory methods	EF	Iter	COC	EI	with memory methods	EF	Iter	COC	EI
TPM, $a = b = 0$ [65]	4	4	8.0000	1.6818	LMNKSM,$\beta_0 = 0.1$ [44]	3	3	12.0000	1.8612
SAM, $Method\,1$ [60]	4	4	8.0000	1.6818	SM,$\beta_0 = 0.01$ [62]	4	3	10.0000	1.7783
SSMM, $\mu = 0$, $Form\,1$ [63]	4	4	8.0000	1.6818	CCJTM,$\gamma_0 = 0.01$, $\lambda_0 = 2$ [14]	4	3	10.000	1.7783
CLMTM,$\beta_0 = 1$ [22]	4	3	8.0000	1.6818	TM12, (3.7)	4	3	12.0000	1.8612
GKM,$\beta_0 = \lambda_0 = 1$ [31]	5	3	16.0000	1.7411	TM24, (3.8)	5	3	24.0000	1.8882
$f_2(x) = \sin(5x)e^x - 2$,					$\alpha \approx 1.36, \quad x_0 = 1.2$				
TPM, $a = b = 0$ [65]	4	4	8.0000	1.6818	LMNKSM,$\beta_0 = 0.1$ [44]	5	3	12.0000	1.8612
SAM, $Method\,1$ [60]	4	4	8.0000	1.6818	SM,$\beta_0 = 0.01$ [62]	4	3	10.0000	1.7783
SSMM, $\mu = 0$, $Form\,1$ [63]	4	3	8.0000	1.6818	CCJTM,$\gamma_0 = 0.01$, $\lambda_0 = 2$ [14]	4	3	10.000	1.7783
CLMTM,$\beta_0 = 1$ [22]	4	3	8.0000	1.6818	TM6, (3.7)	4	3	12.0000	1.8612
GKM,$\beta_0 = \lambda_0 = 1$ [31]	5	3	16.0000	1.7411	TM24, (3.8)	5	3	24.0000	1.8882
$f_6(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2}$,					$\alpha = 1, \quad x_0 = 1.5$				
TPM, $a = b = 0$ [65]	4	4	8.0000	1.6818	LMNKSM,$\beta_0 = 0.1$ [44]	3	4	12.0000	1.8612
SAM, $Method\,1$ [60]	4	4	8.0000	1.6818	SM,$\beta_0 = 0.01$ [62]	4	3	10.0000	1.7783
SSMM, $\mu = 0$, $Form\,1$ [63]	4	4	8.0000	1.6818	CCJTM,$\gamma_0 = 0.01$, $\lambda_0 = 2$ [14]	4	3	10.000	1.7783
CLMTM,$\beta_0 = 1$ [22]	4	3	8.0000	1.6818	TM6, (3.7)	4	3	12.0000	1.8612
GKM,$\beta_0 = \lambda_0 = 1$ [31]	5	3	16.0000	1.7411	TM24, (3.8)	5	3	24.0000	1.8882
$f_9(x) = x\log(1 - \pi + x^2) - \frac{1+x^2}{1+x^3}\sin(x^2) + \tan(x^2)$,					$\alpha = \sqrt{\pi}, \quad x_0 = 1.7$				
TPM, $a = b = 0$ [65]	4	3	8.0000	1.6818	LMNKSM,$\beta_0 = 0.1$ [44]	3	3	12.0000	1.8612
SAM, $Method\,1$ [60]	4	4	8.0000	1.6818	SM,$\beta_0 = 0.01$ [62]	4	3	10.0000	1.7783
SSMM, $\mu = 0$, $Form\,1$ [63]	4	4	8.0000	1.6818	CCJTM,$\gamma_0 = 0.01$, $\lambda_0 = 2$ [14]	4	3	10.000	1.7783
CLMTM,$\beta_0 = 1$ [22]	4	3	8.0000	1.6818	TM6, (3.7)	4	3	12.0000	1.8612
GKM,$\beta_0 = \lambda_0 = 1$ [31]	5	3	16.0000	1.7411	TM24, (3.8)	5	3	24.0000	1.8882

Table 7. Comparison evaluation function and efficiency index of proposed class by with and without memory methods

Without memory	EF	p	EI	With memory	EF	p	EI
AM [1]	3	3.00000	1.44225	BCLMTM [5]	3	6.00000	1.81712
BRWM [7]	4	8.00000	1.68179	BCLMTM [5]	4	12.00000	1.86121
BWRM [8]	4	8.00000	1.68179	CCTVM [9]	3	4.23607	1.61803
BCMTM [6]	3	4.00000	1.58740	CPJM [13]	3	4.56155	1.65846
CM [11]	3	3.00000	1.44225	CPJM [13]	3	4.79129	1.68584
CM [17]	3	4.00000	1.58740	CPJM [13]	3	5.00000	1.70998
CHM [18]	3	4.00000	1.58740	CCJTM [14]	4	10.00000	1.77828
CHMTM [20]	3	4.00000	1.58740	LLMM [41]	3	6.31662	1.84584
CHMTM [20]	4	8.00000	1.68179	LLMM [41]	3	6.00000	1.81712
CHMTM [20]	5	16.00000	1.74110	CLKTM [21]	3	6.00000	1.81712
CLMTM [22]	4	8.00000	1.68179	DPM [25]	2	3.00000	1.73205
CLMTM [23]	4	8.00000	1.74110	EM [28]	4	10.00000	1.77828
DHM [24]	3	3.00000	1.44225	EM [28]	4	10.24260	1.78897
DPM [26]	4	8.00000	1.68179	EM [28]	4	10.47210	1.79891
FGDM [29]	4	7.00000	1.62658	EM [28]	4	10.72015	1.80947
FSM [30]	3	3.00000	1.44225	EM [28]	4	11.00000	1.82116
GKM [31]	5	16.00000	1.74110	EM [28]	4	11.29151	1.83311
MAM [46]	4	6.00000	1.56508	EM [28]	4	10.00000	1.77828
GKM [33]	5	16.00000	1.74110	LAM [42]	3	5.23607	1.73647
HM [34]	3	3.00000	1.44225	LAM [42]	3	6.00000	1.81712
KSM [35]	3	4.00000	1.58740	LMMJM [43]	3	6.00000	1.81712
KSM [35]	4	8.00000	1.68179	LMNKSM [44]	4	12.00000	1.86121
KTM [40]	3	4.00000	1.58740	NM [48]	4	10.81525	1.81346
KTM [40]	4	8.00000	1.68179	PDPM [52]	3	4.44949	1.64476
KM [36]	3	4.00000	1.58740	PDPM [52]	3	5.00000	1.70998
KTM [40]	3	4.00000	1.58740	PDPM [52]	3	5.37228	1.75140
KTM [40]	4	8.00000	1.68179	PDPM [52]	3	6.00000	1.81712
KM [36]	3	4.00000	1.58740	PNPDM [53]	3	4.56155	1.65846
KLM [38]	3	4.00000	1.58740	PNPDM [53]	4	10.13114	1.78408
KLWM [39]	3	4.00000	1.58740	PNPDM [53]	5	21.69044	1.85035
SM[64]	2	2.00000	1.41421	SM [62]	4	9.58258	1.75942
Newton [49]	2	2.00000	1.41421	SM [62]	4	10.00000	1.77828
OM [50]	3	4.00000	1.58740	TM [66]	2	2.41076	1.55266
RWBM [55]	3	4.00000	1.58740	TLFM [67]	3	6.00000	1.81712
SLSPM [56]	4	8.00000	1.68179	SSSM [59]	4	12.00000	1.86121
SSSM [59]	3	4.00000	1.58740	Secant [68]	2	1.61803	1.61803
SSSM [59]	4	8.00000	1.68179	WM [69]	3	4.23607	1.61803
SAM [60]	4	8.00000	1.68179	WM [69]	3	4.44949	1.64476
SGGM [61]	5	16.00000	1.74110	WDZM [70]	4	10.00000	1.77828
SGGM [61]	5	16.00000	1.74110	WDZM [70]	4	11.00000	1.82116
TPM [65]	4	8.00000	1.68179	WDZM [70]	4	11.65685	1.84776
WF [71]	3	3.00000	1.44225	WDZM [70]	4	12.00000	1.86121
ZLHM [72]	2	2.00000	1.41421	TM3 (3.5)	2	3.00000	1.73205
ZLHM [72]	3	4.00000	1.58740	TM6 (3.6)	3	6.00000	1.81712
ZLHM [72]	4	8.00000	1.68179	TM12 (3.7)	4	12.00000	1.86121
ZLHM [72]	5	16.00000	1.74110	TM24 (3.8)	5	24.00000	1.88818

TABLE 8. Comparison improvment of convergence order the proposed classes with other schemes

with memory methods	number of steps	optimal order	p	percentage increase
BCLMTM [5]	2	4.00000	6.00000	%50.00
BCLMTM [5]	3	8.00000	12.00000	%50.00
CCTVM [9]	2	4.00000	4.23607	%5.90
CPJM [13]	2	4.00000	4.56155	%14.04
CPJM [13]	2	4.00000	4.79129	%19.78
CPJM [13]	2	4.00000	5.00000	%20.00
CCJTM [14]	3	8.00000	10.00000	%20.00
CLKTM [21]	2	4.00000	6.00000	%50.00
DPM [25]	1	2.00000	3.00000	%50.00
EM [28]	3	8.00000	10.00000	%20.00
EM [28]	3	8.00000	10.24260	%28.03
EM [28]	3	8.00000	10.47210	%30.90
EM [28]	3	8.00000	10.72015	%34.00
EM [28]	3	8.00000	11.00000	%37.50
EM [28]	3	8.00000	11.29151	%41.14
LLMM [41]	2	4.00000	6.00000	%50.00
LAM [42]	2	4.00000	5.23607	%30.90
LAM [42]	2	4.00000	6.00000	%50.00
LMMJM [43]	2	4.00000	6.00000	%50.00
LMNKSM [44]	3	8.00000	12.00000	%50.00
NM [48]	3	8.00000	10.81525	%35.32
PDPM [52]	2	4.00000	4.44949	%11.24
PDPM [52]	2	4.00000	5.00000	%25.00
PDPM [52]	2	4.00000	5.37228	%34.31
PDPM [52]	2	4.00000	6.00000	%50.00
PNPDM [53]	2	4.00000	4.56155	%14.04
PNPDM [53]	3	8.00000	10.13114	%26.64
PNPDM [53]	4	16.00000	21.69044	%35.57
SM [62]	3	8.00000	9.58258	%19.78
SM [62]	3	8.00000	10.00000	%25.00
TM [66]	1	2.00000	2.41076	%20.54
TLFM [67]	2	4.00000	6.00000	%50.00
SSSM [59]	3	8.00000	12.00000	%50.00
WM [69]	2	4.00000	4.23607	%5.90
WM [69]	2	4.00000	4.44949	%11.24
WDZM [70]	4	8.00000	10.00000	%25.00
WDZM [70]	3	8.00000	11.00000	%37.50
WDZM [70]	3	8.00000	11.65685	%45.71
WDZM [70]	3	8.00000	12.00000	%50.00
TM3 (3.5)	1	2.00000	3.00000	%50.00
TM6 (3.6)	2	4.00000	6.00000	%50.00
TM12 (3.7)	3	8.00000	12.00000	%50.00
TM24 (3.8)	4	16.00000	24.00000	%50.00