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Fixed Point Theorems in Midconvex Subgroups of a Hilbert group

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Abstract. In this paper, after introducing inner products on groups, first, we define a Hilbert group using the inner products and in the last section, we present some fixed points for closed and midconvex subgroups of such Hilbert groups.

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1. Introduction

Let $\mathcal{J}$ be a group and $\vartheta : \mathcal{J} \to \mathcal{J}$ be a mapping. An element $w \in \mathcal{J}$ is called a fixed point of ϑ if $\vartheta(w) = w$. Let $w_0 \in \mathcal{J}$ be an arbitrary element. Define the Picard iterative sequence $\{w_n\}$ in $\mathcal{J}$ as follows

$$w_{n+1} = \vartheta(w_n), \qquad (n = 0, 1, 2...).$$

We note that the convergence of this sequence plays a significant role in the existence of a fixed point for mapping ϑ. Define the n^{th} iterate of ϑ as $\vartheta^0 = I$ (the identity map) and $\vartheta^n = \vartheta^{n-1} o \vartheta$, for $n \geq 1$.

The origin of fixed point theory is known as the Banach contraction

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principle. Banach contraction principle states that any contraction on a complete metric space has a unique fixed point [1]. In 1968 Kannan [10] proved that a self-mapping υ on a complete metric space $(\mathcal{T}, d)$ satisfying

$$d(t,s) \leq \eta(d(t,\upsilon(t)) + d(s,\upsilon(s))),$$

for all $t, s \in \mathcal{T}$ where $0 < \eta < \frac{1}{2}$ has a unique fixed point. A similar conclusion was also obtained by Chaterjee in 1972 [5]. Koparde and Wghmode [11] proved a fixed point theorem for a self-mapping υ on a complete metric space $(\mathcal{T}, d)$ satisfying the Kannan type condition

$$d^2(\upsilon(t),\upsilon(s)) \leq \eta(d^2(t,\upsilon(t)) + d^2(s,\upsilon(s))),$$

for all $t, s \in \mathcal{T}$ where $0 < \eta < \frac{1}{2}$.

In this paper, we tend to investigate some fixed points on groups. Group-norms have played a role in the theory of topological groups [2, 8]. The Birkhoff-Kakutani's metrization theorem for groups states that each first-countable Hausdorf group has a right invariant metric [3]. The term group-norm probably first appeared in Pettis's paper in 1950 [12]. In our further considerations, we will generalize fixed point theorems for Hilbert groups. First, we need to define inner products and complete norms on groups.

The following definitions will be frequently used in the sequel. First, we need to know the definition of a normed group.

Definition 1.1. [8] Let $\mathcal{J}$ be a group. A norm on a group $\mathcal{J}$ is a function $\|.\| : \mathcal{J} \to \mathbb{R}$ with the following properties:

(1) $\|w\| \geq 0, \ for\ all\ w \in \mathcal{J};$
(2) $\|w\| = \|w^{-1}\|, \ for\ all\ w \in \mathcal{J};$
(3) $\|wq\| \leq \|w\| + \|q\|, \ for\ all\ w, q \in \mathcal{J};$
(4) $\|w\| = 0\ if\ and\ only\ if\ w = e.$

A normed group $(\mathcal{J}, \|.\|)$ is a group $\mathcal{J}$ equipped with a norm $\|.\|$. A group norm $\|.\|$ is an abelian norm if $\|wq\| = \|qw\|$ for all w, q in $\mathcal{J}$. By setting $d(w,q) := \|w^{-1}q\|$, it is easy to see that norms are in bijection with left invariant metrics on $\mathcal{J}$.

Note that the group norm generates a right and a left norm topology via the right-invariant and left invariant metrics $d_r(w,q) := \|wq^{-1}\|$ and $d_l(w,q) := \|w^{-1}q\| = d_r(w^{-1}, q^{-1})$.

Now, after defining a normed group, it is natural to define a complete normed group and a Banach group. Although these definitions are similar to those used in vector spaces, since they are defined for the first time, it seems necessary to mention them.

Definition 1.2. A normed group $(\mathcal{J}, \|.\|)$ is called complete if any Cauchy sequence in $\mathcal{J}$ converges to an element of $\mathcal{J}$, i.e. it has a limit in group $\mathcal{J}$.

Definition 1.3. [2] A Banach group is a normed group $(\mathcal{J}, \|.\|)$, which is complete with respect to the metric

$$d(w, q) = \|wq^{-1}\|, \quad (w, q \in \mathcal{J}).$$

For example, let $\mathcal{J}$ be a banach space. Then $\|w\| = d(w, e)$ defines a group norm on $\mathcal{J}$ and is also complete normed group.

Definition 1.4. [9] Let $\mathcal{J}$ be a group with identity element e. The order of an element $w \in \mathcal{J}$ is the smallest $n \in \mathbb{N}$ such that $w^n = e$. If no such n exists, w is said to have infinite order. An abelian group $\mathcal{J}$ is said to be torsion-free if no element other than the identity e is of finite order.

Let $(\mathcal{J}, \|.\|)$ be a normed group. For $q \in \mathcal{J}$ the q-conjugate norm is defined by

$$\|w\|_q := \|qwq^{-1}\|.$$

Note that the group-norm is abelian iff the norm preserved under conjugacy [2]. It is obvious that each norm on an abelian group is an abelian norm.

Definition 1.5. [7] Let $\mathcal{J}$ be a group. An element $w \in \mathcal{J}$ is said to be divisible by $n \in \mathbb{Z}$ if $w = q^n$ has a solution q in $\mathcal{J}$. A group $\mathcal{J}$ is called infinitely divisible if each element in $\mathcal{J}$ is divisible by every positive integer.
A group-norm $\|.\|$ is $\mathbb{N}$-homogeneous if for each $n \in \mathbb{N}$,

$$\|w^n\| = n\|w\| \ (\forall w \in \mathcal{J}).$$

In a normed group $(\mathcal{J}, \|.\|)$ which the norm is $\mathbb{N}$-homogeneous and $\mathcal{J}$ is divisible, for $w, q \in \mathcal{J}$, let $w^n = q$ and $n \in \mathbb{N}$, then $\|w\| = \frac{1}{n}\|q\|$ and as $w^m = q^{\frac{m}{n}}$, we have $\|q^{\frac{m}{n}}\| = \|w^m\| = m\|w\| = \frac{m}{n}\|q\|$, i.e. for rational $p > 0$ we have $p\|w\| = \|w^p\|$. This shows that the $\mathbb{N}$-homogeneous norm-group is $\mathbb{Q}^+$-homogeneous.

Definition 1.6. [4] Let $(\mathcal{T}, d)$ be a metric space. We call a mapping $\upsilon : \mathcal{T} \to \mathcal{T}$ is sequentially convergent if for each sequence $\{t_n\}$ that $\{\upsilon(t_n)\}$ is convergent then $\{t_n\}$ is also convergent.

2. Inner product groups

In this section, we present the concept of inner product and semi-inner product on groups. To clarify these concepts, we provide examples and prove the results we need in the next section.

Definition 2.1. A semi-inner product on a group $\mathcal{J}$ with identity element e is a function that associates a real number $\langle w, q\rangle$ with each pair of elements w and q in $\mathcal{J}$ in such a way that the following axioms are satisfied for all elements w, q and z in $\mathcal{J}$:

(1) $\langle w, q\rangle = \langle q^{-1}, w^{-1}\rangle$ (Symmetry);
(2) $\langle wq, z\rangle = \langle w, z\rangle + \langle q, z\rangle$ (Distributivity);
(3) $\langle w, w\rangle \geq 0$ (Positivity).

A group with a semi-inner product will be called a semi-inner product group.

Note that combining (1) and (2) gives the equation

$$\langle w, qz\rangle = \langle w, q\rangle + \langle w, z\rangle \qquad (\forall w, q, z \in \mathcal{J}).$$

In the next example, we use the discrete Heisenberg group to show how a semi-inner product is defined on a non-abelian group.

Example 2.2. Let $\mathcal{J} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in Z \right\}$ denote the discrete Heisenberg group and let $\langle ., .\rangle : \mathcal{J} \to \mathbb{R}$ be defined by

$$\langle w, q\rangle = (a + c)(\acute{a} + \acute{c}) \qquad (\forall w, q \in \mathcal{J})$$

when $w = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & \acute{a} & \acute{b} \\ 0 & 1 & \acute{c} \\ 0 & 0 & 1 \end{pmatrix}$. Then $(\mathcal{J}, \langle ., .\rangle)$ is a semi-inner product group.

Usually, after defining a semi-inner product, the definition of an inner product is given. But the question is, can an inner product be defined on all groups?
Before defining an inner product on groups, it is necessary to determine on what types of groups one can define an inner product. It seems that there is an additional condition for groups with an inner product. In other words, we show that the definition of our inner product imposes an abelian structure on groups.

Definition 2.3. [9] Let $\mathcal{J}$ be a group. The subgroup of $\mathcal{J}$ generated by the set $\{wqw^{-1}q^{-1} | w, q \in \mathcal{J}\}$ is called commutator subgroup of $\mathcal{J}$ and denoted by $\acute{\mathcal{J}}$. The elements $wqw^{-1}q^{-1}(w, q \in \mathcal{J})$ are called commutators.

Theorem 2.4. [9] *Group $\mathcal{J}$ is abelian if and only if $\acute{\mathcal{J}} = \{e\}$.*

Now, let $(\mathcal{J}, \langle .,. \rangle)$ be an semi-inner product group and suppose that $\langle w, w \rangle = 0$ implies $w = e$. We show that $\acute{\mathcal{J}} = \{e\}$. Since for $w, q \in \mathcal{J}$

$$\begin{aligned}\langle wqw^{-1}q^{-1}, wqw^{-1}q^{-1} \rangle &= \langle w, wqw^{-1}q^{-1} \rangle + \langle q, wqw^{-1}q^{-1} \rangle \\ &+ \langle w^{-1}, wqw^{-1}q^{-1} \rangle + \langle q^{-1}, wqw^{-1}q^{-1} \rangle \\ &= \langle w, wqw^{-1}q^{-1} \rangle + \langle q, wqw^{-1}q^{-1} \rangle \\ &- \langle w, wqw^{-1}q^{-1} \rangle - \langle q, wqw^{-1}q^{-1} \rangle \\ &= 0.\end{aligned}$$

Then $wqw^{-1}q^{-1} = e$. This shows that all commutators of $\mathcal{J}$ are equal to e. So, we define inner products on abelian groups as follows.

Definition 2.5. Let $(\mathcal{J}, \langle .,. \rangle)$ be a semi-inner product abelian group, in which for $w \in \mathcal{J}$, $\langle w, w \rangle = 0$ implies $v = e$. Then $(\mathcal{J}, \langle .,. \rangle)$ is called an abelian inner product group.

In this paper it is supposed that every inner product group is abelian. Therefore, when we talk of an inner product group, we mean that there is an inner product on an abelian group.

Example 2.6. Let $\mathcal{J} = \mathbf{R}_+^*$ denote the group of positive real numbers with multiplication as the group operation. By

$$\langle w, q \rangle := (\log w)(\log q) \qquad (\forall w, q \in \mathcal{J}),$$

$\mathcal{J}$ will be an inner product group.

In the next result, we show what the relationship is between an inner product and an induced norm on groups, and we also show how an inner product group becomes a torsion free group.

Theorem 2.7. *Let $(\mathcal{J}, \langle .,. \rangle)$ be an inner product group. Then $\mathcal{J}$ is a torsion-free abelian normed group with the norm $\|.\| : \mathcal{J} \to \mathbb{R}$ for all $w \in \mathcal{J}$, defined by*

$$\|w\| = \sqrt{\langle w, w \rangle}.$$

Proof. Let us first show that

$$|\langle w, q \rangle| \leq \|w\|.\|q\| \qquad (\forall w, q \in \mathcal{J}). \tag{2.1}$$

Let w and q be arbitrary in $\mathcal{J}$. If $q = e$ then the inequality is true. For $w, q \neq e$ and $n \in \mathbb{N}$, the positivity of the inner product shows that

$$\begin{aligned}0 \leq \langle wq^{-n}, wq^{-n} \rangle &= \langle w, wq^{-n} \rangle + \langle q^{-n}, wq^{-n} \rangle \\ &= \langle q^n w^{-1}, w^{-1} \rangle + \langle q^n w^{-1}, q^n \rangle \\ &= -2n\langle w, q \rangle + \langle w, w \rangle + n^2 \langle q, q \rangle.\end{aligned}$$

Then

$$\langle q,q\rangle n^2 - 2\langle w,q\rangle n + \langle w,w\rangle \geq 0.$$

Let $a = \langle q,q\rangle$, $b = -2\langle w,q\rangle$ and $c = \langle w,w\rangle$. Then the equation becomes $an^2 + bn + c \geq 0$. This is a quadratic equation for $n \in \mathbb{N}$ with real coefficients. Since this polynomial takes only non-negative values, its discriminate $b^2 - 4ac$ must be non-positive

$$4\langle w,q\rangle^2 - 4\langle w,w\rangle\langle q,q\rangle \leq 0.$$

This implies that

$$|\langle w,q\rangle| \leq \sqrt{\langle w,w\rangle}\sqrt{\langle q,q\rangle} = \|w\|.\|q\| \qquad (\forall w,q \in \mathcal{J}).$$

The (2.1) inequality is called Cauchy-Schwartz inequality for groups. We next claim $(\mathcal{J}, \langle .,. \rangle)$ is a normed group. The positivity principle of the norm is clear. Since $\langle w,w\rangle = \langle w^{-1}, w^{-1}\rangle$, we have

$$\|w\|^2 = \langle w,w\rangle = \langle w^{-1}, w^{-1}\rangle = \|w^{-1}\|^2.$$

Now, we show that $\|.\|$ satisfies the triangle inequality. For all $w,q \in \mathcal{J}$ we have

$$\begin{aligned}\|wq\|^2 &= \langle wq,wq\rangle = \langle w,wq\rangle + \langle q,wq\rangle \\ &= \langle w,w\rangle + \langle w,q\rangle + \langle q,w\rangle + \langle q,q\rangle \\ &= \|w\|^2 + 2\langle w,q\rangle + \|q\|^2.\end{aligned}$$

So, the Cauchy-Schwartz inequality for groups implies that

$$\|wq\|^2 \leq \|w\|^2 + 2\|w\|\|q\| + \|q\|^2 = (\|w\| + \|q\|)^2.$$

Hence,

$$\|wq\| \leq \|w\| + \|q\|.$$

It remains to prove that $\mathcal{J}$ is a torsion-free abelian group. Let $w \in \mathcal{J}$ such that $ord(w) = n$ and $n \neq 0$. Then $\mathbb{N}$-homogeneous property of the induced group-norm implies that

$$\|w^n\| = n\|w\| = \|e\| = 0.$$

Since $n \neq 0$, then $\|w\| = 0$ and $w = e$. □

Lemma 2.8. *Let $(\mathcal{J}, \langle .,. \rangle)$ be an inner product group. Then the group-norm induced by the inner product is $\mathbb{N}$-homogeneous and*

$$\|wq\|^2 + \|wq^{-1}\|^2 = 2(\|w\|^2 + \|q\|^2) \qquad (\forall w,q \in \mathcal{J}).$$

Proof. $\mathbb{N}$-homogeneity is clear. To prove the parallelogram law, we have

$$\begin{aligned}\|wq\|^2 + \|wq^{-1}\|^2 &= \langle wq, wq\rangle + \langle wq^{-1}, wq^{-1}\rangle \\ &= \|w\|^2 + \|q\|^2 + \langle w, q\rangle + \langle q, w\rangle \\ &+ \|w\|^2 + \|q\|^2 - \langle w, q\rangle - \langle q, w\rangle \\ &= 2(\|w\|^2 + \|q\|^2).\end{aligned}$$

□

Now it's time to define a Hilbert group because in the next section we are going to look at fixed points in Hilbert groups.

Definition 2.9. Let $(\mathcal{J}, \langle ., .\rangle)$ be an inner product group. Then $\mathcal{J}$ is called a Hilbert group if $\mathcal{J}$ is complete with respect to the norm induced by the inner product.

Example 2.10. Let $\mathcal{J} = (R^n, +)$. Then $\mathcal{J}$ with the inner product $\langle v, s\rangle = \sum_{j=1}^{n} v_j s_j$ is a Hilbert group.

3. Some fixed points in Hilbert groups

According to the new definition of Hilbert groups, as were introduced in the previous section, in this section, we investigate some fixed points in the closed and midconvex subgroups of a Hilbert group.

Theorem 3.1. *Let C be a closed subgroup of a Hilbert group H. Suppose that $\psi, \rho : C \to C$ are two continuous contractions of C, then $(\psi\rho^n(c))$ converges to $\psi(z)$. If ψ, ρ satisfying the*

$$\|\psi\rho(c)(\psi\rho(t))^{-1}\|^2 \leq \kappa\|\psi(c)\psi(t)^{-1}\|^2,$$

for all $c, t \in C$ and $0 \leq \kappa < 1$, then ρ has a fixed point in C.

Proof. We first prove that $\lim_{n\to\infty} \|\psi\rho^{n+1}(c)(\psi\rho^n(c))^{-1}\|^2 = 0$. Since

$$\|\psi\rho^{n+1}(c)(\psi\rho^n(c))^{-1}\|^2 \leq \kappa^n\|\psi\rho(c)\psi(c)^{-1}\|^2,$$

tending n to infinity, we have $\|\psi\rho^{n+1}(c)(\psi\rho^n(c))^{-1}\|^2 = 0$. Now, we claim that $(\psi\rho^n(c))_{n=1}^{\infty}$ is bounded. Suppose that $(\psi\rho^n(c))_{n=1}^{\infty}$ is unbounded. Then there exist $(n(i))_{i=1}^{\infty}$ is a sequence such that $n(1) = 1$ and for all $i \in \mathbb{N}$, $n(i+1)$ is minimal. Hence

$$\|\psi\rho^{n(i+1)}(c)(\psi\rho^{n(i)}(c))^{-1}\|^2 > 1$$

and $\|\psi\rho^m(c)(\psi\rho^{n(i)}(c))^{-1}\|^2 \leq 1$ for all $m = n(i)+1, n(i)+2, ..., n(i+1)+1$. But

$$\begin{aligned} 1 < \|\psi\rho^{n(i+1)}(c)(\psi\rho^{n(i)}(c))^{-1}\|^2\| &\leq \|\psi\rho^{n(i+1)}(c)(\psi\rho^{n(i+1)-1}(c))^{-1}\|^2 \\ &+ \|\psi\rho^{n(i+1)-1}(c)(\psi\rho^{n(i)}(c))^{-1}\|^2 \\ &\leq \|\psi\rho^{n(i+1)}(c)(\psi\rho^{n(i+1)-1}(c))^{-1}\|^2 + 1. \end{aligned}$$

Then $\|\psi\rho^{n(i+1)}(c)(\psi\rho^{n(i)}(c))^{-1}\|^2 \to 1$ as $i \to \infty$. So $\|\psi\rho^{n(i+1)}(c)(\psi\rho^{n(i)})^{-1}(c)\|^2 \leq \kappa\|\psi\rho^{n(i+1)-1}(c)(\psi\rho^{n(i)-1})^{-1}(c)\|^2$ contradicition.
Since

$$\|\psi\rho^m(c)(\psi\rho^n(c))^{-1}\|^2 \leq \kappa^n\|\psi\rho^{m-n}(c)\psi(c)^{-1}\|^2,$$

for $0 \leq \kappa < 1$, then $\|\psi\rho^m(c)(\psi\rho^n(c))^{-1}\|^2 = 0$. Therefore $(\psi\rho^n(c))_{n=1}^{\infty}$ is a Cauchy sequence and $\psi\rho^n(c) = \psi(z)$. Hence

$$\|\psi\rho^{n+1}(c)(\psi\rho(z))^{-1}\| \leq \kappa\|\psi\rho^n(c)(\psi(z))^{-1}\| \to 0,$$

for $0 \leq \kappa < 1$. So, $\rho(z) = z$. □

In the following results, we use the concept of midconvexity to find fixed points in the closed and midconvex sets of a Hilbert group. But first, we need to define a midconvex set.

Definition 3.2. [2] Let $\mathcal{J}$ be a group. A subset C of $\mathcal{J}$ is called $\frac{1}{2}$-convex (or midconvex), if for every $c, t \in C$ there exists an element $z \in C$, denoted by $(ct)^{\frac{1}{2}}$, such that $z^2 = ct$.

The next lemma shows how under certain conditions there exists at least one fixed point on a closed subset of a Hilbert group.

Lemma 3.3. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group and C be a nonempty closed subgroup of $\mathcal{J}$ and let $\psi : C \to C$ be a mapping such that satisfying*

$$\|\psi(c)\psi(t)^{-1}\| \leq \eta\left[\|c\psi(c)^{-1}\| + \|t\psi(t)^{-1}\|\right],$$

for all $c, t \in C$ and $0 \leq \eta < 1$. If for arbitrary point $c_0 \in C$ there exists $t_0 \in C$ such that $\|t_0\psi(t_0)^{-1}\| \leq r_1\|c_0\psi(c_0)^{-1}\|$ and $\|t_0c_0^{-1}\| \leq r_2\|c_0\psi(c_0)^{-1}\|$, where $0 \leq r_1 < 1$ and $r_2 > 0$, then ψ has at least one fixed point.

Proof. For an arbitrary element $c_0 \in C$ define a sequence $(c_n)_{n=1}^{\infty} \subset C$ such that

$$\|\psi(c_{n+1})c_{n+1}^{-1}\| \leq r_1\|\psi(c_n)c_n^{-1}\|,$$

and

$$\|c_{n+1}c_n^{-1}\| \leq r_2\|\psi(c_n)c_n^{-1}\|,$$

for $n = 1, 2, ...$ It is easy to see that $(c_n)_{n=1}^{\infty}$ is a Cauchy sequence, since

$$\|c_{n+1}c_n^{-1}\| \leq r_2\|\psi(c_n)c_n^{-1}\| \leq r_2r_1^n\|\psi(c_0)c_0^{-1}\|.$$

Now, there exists $z \in C$ such that $\lim_{n\to\infty} c_n = z$. Then

$$\begin{aligned}\|\psi(z)z^{-1}\| &\leq \|\psi(z)\psi(c_n)^{-1}\| + \|\psi(c_n)c_n^{-1}\| + \|c_n z^{-1}\| \\ &\leq \eta\left[\|z\psi(z)^{-1}\| + \|c_n\psi(c_n)^{-1}\|\right] + \|\psi(c_n)c_n^{-1}\| + \|c_n z^{-1}\|,\end{aligned}$$

and

$$\begin{aligned}\|\psi(z)z^{-1}\| &\leq \frac{\eta+1}{1-\eta}\|\psi(c_n)c_n^{-1}\| + \frac{1}{1-\eta}\|c_n z^{-1}\| \\ &\leq \frac{\eta+1}{1-\eta} r_1^n \|\psi(c_0)c_0^{-1}\| + \frac{1}{1-\eta}\|c_n z^{-1}\| \to 0,\end{aligned}$$

as $n \to \infty$. So, $\psi(z) = z$. □

Closed midconvex subsets of a hilbert group, some continuous mappings, and strictly increasing continuous functions on a Hilbert group lead us to find a fixed point, which we describe in detail in the next theorem.

Theorem 3.4. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group, C be a nonempty, closed and $\frac{1}{2}$-convex subgroup of $\mathcal{J}$ and let $\phi : C \to \mathbb{R}$ be a continuous mapping satisfying*

$$\eta := \inf_{c\in C} \phi(c) > -\infty,$$

for all $c \in C$. For some strictly increasing continuous function $\sigma : [0,\infty) \to [0,\infty)$ with $\sigma(0) = 0$, let

$$\phi((ct)^{\frac{1}{2}}) + \sigma(\|ct^{-1}\|) \leq \frac{1}{2}\phi(c) + \frac{1}{2}\phi(t),$$

for all $c, t \in C$. Then there is a unique $c \in C$ such that $\phi(c) = \eta$.

Proof. For $c, t \in C$ we have

$$\eta \leq \phi((ct)^{\frac{1}{2}}) \leq \frac{1}{2}\phi(c) + \frac{1}{2}\phi(t) - \sigma(\|ct^{-1}\|),$$

which implies

$$\sigma(\|ct^{-1}\|) \leq \frac{1}{2}\phi(c) + \frac{1}{2}\phi(t) - \eta.$$

It is immediate that ϕ has at most one minimizer in C. Let $(c_n)_{n=1}^{\infty} \subset C$ be a minimizing sequence of ϕ in C, meaning that

$$\phi(c_n) \to \eta.$$

It follows that $(c_n)_{n=1}^{\infty}$ is a Cauchy sequence, hence there is $c \in \mathcal{J}$ such that $c_n \to c$. As C is closed then $c \in C$, and since ϕ is continuous function, we have $\phi(c) = \eta$. □

Let ψ be a mapping on a closed midconvex subsets of a Hilbert group such that $\psi^2 = I$. Is there any unique fixed point for ψ? Under what conditions is a fixed pint found? The next theorem answers these questions.

Theorem 3.5. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group, C be a nonempty, closed and $\frac{1}{2}$-convex subgroup of $\mathcal{J}$ and let $\psi : C \to C$ be a mapping such that $\psi^2 = I$. If $\kappa < 1$ and for $c \in C$, the equation $z^2\psi(z)^{-1} = c$ has a solution in C, and*

$$\|\psi(c)\psi(t)^{-1}\| \leq \kappa \left[\|c\psi(c)^{-1}\| + \|t\psi(t)^{-1}\|\right],$$

for all $c, t \in C$, then ψ has a unique fixed point in C.

Proof. For $c \in C$, let $z = (c\psi(z))^{\frac{1}{2}}$. Then

$$\begin{aligned}\|z\psi(z)^{-1}\| = \|(c\psi(z)^{-1})^{\frac{1}{2}}\| &= \frac{1}{2}\|\psi^2(c)\psi(z)^{-1}\| \\ &\leq \frac{\kappa}{2}(\|c\psi(c)^{-1}\| + \|z\psi(z)^{-1}\|).\end{aligned}$$

Hence,

$$\|z\psi(z)^{-1}\| \leq \frac{\frac{\kappa}{2}}{1-\frac{\kappa}{2}}\|c\psi(c)^{-1}\|.$$

Using the triangle inequality, we obtain

$$\|zc^{-1}\| \leq \frac{1}{2}\|\psi(z)c^{-1}\| \leq \frac{1}{2}(\|\psi(z)z^{-1}\| + \|zc^{-1}\|).$$

So,

$$\|zc^{-1}\| \leq \|z\psi(z)^{-1}\| \leq \kappa\|c\psi(c)^{-1}\|,$$

where $\eta = \frac{\frac{\kappa}{2}}{1-\frac{\kappa}{2}} < 1$.
For arbitrary $c_0 \in C$, we define a sequence $(c_n)_{n=1}^{\infty} \subset C$ in the following manner:

$$c_{n+1} = (c_n\psi(c_{n+1}))^{\frac{1}{2}}.$$

By Lemma 3.3, this sequence is converges to $s \in C$ and $\psi(s) = s$. It is obvious that s is unique. □

The next theorem specifies another fixed point for a mapping on a closed and midconvex subset of a Hilbert group.

Theorem 3.6. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group, C be a nonempty, closed and $\frac{1}{2}$-convex subgroup of $\mathcal{J}$ and let $\psi : C \to C$ be a mapping. If $2 \leq \kappa < 4$ and*

$$\|c\psi(c)^{-1}\| + \|t\psi(t)^{-1}\| \leq \kappa\|ct^{-1}\|,$$

for all $c, t \in C$, then ψ has at least one fixed point.

Proof. Let for arbitrary element $c_0 \in C$, a sequence $(c_n)_{n=1}^{\infty}$ be defined by

$$c_{n+1} = (c_n \psi(c_n))^{\frac{1}{2}} \quad (n = 0, 1, 2...).$$

Then we have

$$c_n \psi(c_n)^{-1} = c_n^2 c_n^{-1} \psi(c_n)^{-1} = (c_n c_{n+1}^{-1})^2,$$

and since the norm is $\mathbb{N}$-homogeneous, we have

$$\|c_n \psi(c_n)^{-1}\| = \|(c_n c_{n+1}^{-1})^2\| = 2\|c_n c_{n+1}^{-1}\|.$$

So, for $c = c_{n-1}$ and $t = c_n$, we obtain

$$2\|c_{n-1} c_n^{-1}\| + 2\|c_n c_{n+1}^{-1}\| \leq \kappa \|c_{n-1} c_n\|.$$

Therefore, $\|c_n c_{n+1}^{-1}\| \leq m \|c_{n-1} c_n^{-1}\|$, where $0 \leq m = \frac{\kappa - 2}{2} < 1$, as $2 \leq \kappa < 4$. Then $(c_n)_{n=1}^{\infty}$ is a Cauchy sequence in C and hence, it converges to some $z \in C$. Since

$$\|z \psi(c_n)^{-1}\| \leq \|z c_n^{-1}\| + \|c_n \psi(c_n)^{-1}\| = \|z c_n^{-1}\| + 2\|c_n c_{n+1}^{-1}\|,$$

then

$$\lim_{n \to \infty} \psi(c_n) = z.$$

Therefore, for $c = z$ and $t = c_n$, we have

$$\|z \psi(z)^{-1}\| + 2\|c_n c_{n+1}^{-1}\| \leq \kappa \|z c_n^{-1}\|.$$

Tending n to infinity, implies that $\psi(z) = z$. □

Corollary 3.7. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group, C be a nonempty, closed and $\frac{1}{2}$-convex subgroup of $\mathcal{J}$ and let $\psi : C \to C$ be a mapping. If $0 \leq \iota < 2$ and*

$$\|c \psi(t)^{-1}\| + \|t \psi(c)^{-1}\| \leq \iota \|c t^{-1}\|,$$

for all $c, t \in C$. Then ψ has a fixed point.

Proof. For all $c, t \in C$, we have

$$\|c \psi(c)^{-1}\| + \|t \psi(t)^{-1}\| \leq \|c t^{-1}\| + \|t \psi(c)^{-1}\| + \|t c^{-1}\| + \|c \psi(t)^{-1}\|.$$

Thus,

$$\|c \psi(c)^{-1}\| + \|t \psi(t)^{-1}\| \leq \iota \|c t^{-1}\| + 2\|c t^{-1}\|.$$

Therefore, we conclude that ψ satisfies Theorem 3.6 with $\kappa = \iota + 2$. □

Theorem 3.8. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group, C be a nonempty, closed and $\frac{1}{2}$-convex subgroup of $\mathcal{J}$ and let $\psi : C \to C$ be a mapping. If $2 \leq \kappa < 5$ and*

$$\|\psi(c) \psi(t)^{-1}\| + \|c \psi(c)^{-1}\| + \|t \psi(t)^{-1}\| \leq \kappa \|c t^{-1}\|, \tag{3.1}$$

for all $c, t \in C$, then ψ has at least one fixed point.

Proof. Let for arbitrary element $c_0 \in C$, a sequence $(c_n)_{n=1}^{\infty}$ be defined by

$$c_{n+1} = (c_n\psi(c_n))^{\frac{1}{2}} \quad (n = 0, 1, 2...).$$

So,

$$c_n\psi(c_{n-1})^{-1} = (c_{n-1}\psi(c_{n-1}))^{\frac{1}{2}}\psi(c_{n-1})^{-1} = (c_{n-1}\psi(c_{n-1}^{-1})^{\frac{1}{2}}.$$

Then

$$2\|c_nc_{n+1}^{-1}\| - \|c_{n-1}c_n^{-1}\| \leq \|\psi(c_{n-1})\psi(c_n)^{-1}\|.$$

In (3.1), put $c = c_{n-1}$ and $t = c_n$, then we get

$$2\|c_nc_{n+1}^{-1}\| - \|c_{n-1}c_n^{-1}\| + 2\|c_{n-1}c_n^{-1}\| + \|c_nc_{n+1}^{-1}\| \leq \kappa\|c_{n-1}c_n^{-1}\|,$$

and so $\|c_nc_{n+1}^{-1}\| \leq \frac{\kappa-1}{4}\|c_{n-1}c_n^{-1}\|$. The sequence $(c_n)_{n=1}^{\infty}$ is a Cauchy sequence in C and hence, it converges to some $z \in C$. Since $\psi(c_n)$ also converges to z, we get $\|\psi(z)z^{-1}\| + \|z\psi(z)^{-1}\| \leq 0$ which implies $\psi(z) = z$. □

As we have seen, for some mappings on a closed and midconvex subset of a Hilbert group, various fixed points are reached. These fixed points are changed as the mapping conditions and coefficients are changed. In the next theorem, a fixed point is examined under specific conditions of the mapping.

Theorem 3.9. *Let $(\mathcal{J}, \|.\|)$ be a Hilbert group, C be a nonempty, closed and $\frac{1}{2}$-convex subgroup of $\mathcal{J}$ and let $\psi : C \to C$ be a mapping. If there exist real numbers a,b and κ such that*

$$0 \leq \kappa + |a| - 2b < 2(a + b);$$

and for all $c, t \in C$,

$$a\|\psi(c)\psi(t)^{-1}\| + b[\|c\psi(c)^{-1}\| + \|t\psi(t)^{-1}\|] \leq \kappa\|ct^{-1}\|, \tag{3.2}$$

then ψ has at least one fixed point.

Proof. Let for arbitrary element $c_0 \in C$, a sequence $(c_n)_{n=1}^{\infty}$ be defined by

$$c_{n+1} = (c_n\psi(c_n))^{\frac{1}{2}} \quad (n = 0, 1, 2...).$$

If $a \geq 0$, by puting $c = c_{n-1}$ and $t = c_n$ in (3.2), we obtain

$$2a\|c_nc_{n+1}^{-1}\| - |a|\|c_{n-1}c_n^{-1}\| + 2b[\|c_{n-1}c_n^{-1}\| + \|c_nc_{n+1}^{-1}\|] \leq \kappa\|c_nc_{n+1}^{-1}\|.$$

If $a < 0$, by using the inequality

$$\|c_n\psi(c_n)^{-1}\| + \|c_n\psi(c_{n-1})^{-1}\| \geq \|\psi(c_{n-1})\psi(c_n)^{-1}\|,$$

we obtain

$$\|c_nc_{n+1}^{-1}\| \leq \lambda\|c_{n-1}c_n^{-1}\|,$$

where $\lambda = \frac{|a|-2b+\kappa}{2(a+b)}$. As $0 \leq \lambda < 1$, the sequence $(c_n)_{n=1}^{\infty}$ is a Cauchy sequence in C and hence, it converges to some $z \in C$. Tending n to infinity, we get

$$a\|\psi(z)z^{-1}\| + b\|z\psi(z)^{-1}\| \leq 0.$$

Then, as $a + b > 0$, it follows that $\psi(z) = z$. □

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