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Iterative reconstruction of continuous g-fusion frames in Hilbert spaces

Vahid Sadri [1] and Gholamreza Rahimlou [2]
[1] Department of Mathematics, Faculty of Tabriz Branch, Technical and Vocational University (TVU), Tehran , Iran
[2] Department of Mathematics, Faculty of Tabriz Branch, Technical and Vocational University (TVU), Tehran , Iran

Abstract. Regarding the applications of the fusion frames and generalization of them in data proceeding, their iterative is of particular importance when one of their members is deleted. In this note, a method for reconstruction of continuous generalized fusion frames and the error operator with its upper bound are presented. Also, the approximation operator for these frames will be introduced.

Keywords: Perseval frame, c-frame, cg-fusion frame, Bochner integrable function.

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1. Introduction and perliminaries

Robustence of Parseval fusion frames under erasures have been employed by Bodmann and et. al. in [3] for optimal transmission of quantum states and packet encoding. After them, Kutyniok and et. al. in [12] presented fusion frames which are optimally resilient against noise and erasure for random signals and further Casazza and Kutyniok in [6] studied this topic and they presented sufficient conditions on the robustness

[1] Corresponding author: vsadri@tvu.ac.ir
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of a fusion frame with respect to erasures of subspaces. In this paper, we focus on the study of those topics on continuous generalized fusion frames and we prove some new results about these frames.

Throughout this paper, (Ω, μ) is a measure space with positive measure μ, H is a Hilbert space, $\mathbb{H}$ is the collection of all closed subspaces of H, $\{H_\omega\}_{\omega\in\Omega}$ is a collection of Hilbert spaces, π_V is the orthogonal projection from H onto a closed subspace V and $\mathcal{B}(H, K)$ is the set of all bounded linear operators from H to K. If $H = K$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$.

Definition 1.1 (c-frame)**.** [2] Let $F : \Omega \to H$ be a weakly measurable (i.e., for all $h \in H$, the mapping $\omega \mapsto \langle F(\omega), h\rangle$ is measurable). Then the map F is called a continuous frame (or briefly c-frame) for H with respect to Ω, if there exist $0 < A \leq B < \infty$ such that for each $h \in H$,

$$A\|h\|^2 \leq \int_\Omega |\langle h, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|h\|^2. \tag{1.1}$$

The constants A and B are called frame bounds. If $A = B$, we say that F is an A-tight c-frame, and if $A = B = 1$, it is called a Parseval c-frame. Also, if the right hand of (1.1) holds, it is called a continuous Bessel mapping.

Definition 1.2 (cg-fusion frame)**.** [9] Let $F : \Omega \to \mathbb{H}$ such that for each $h \in H$ the mapping $\omega \to \pi_{F(\omega)}(h)$ is measurable (i.e. is weakly measurable), $\Lambda := \{\Lambda_\omega \in \mathcal{B}(F(\omega), H_\omega), \ \omega \in \Omega\}$ and $v : \Omega \to \mathbb{R}^+$ be a measurable function. Then $(F, \Lambda, v)_\Omega$ is called a continuous g-fusion frame (or cg-fusion frame) for H if there exist $0 < A \leq B < \infty$ such that for each $h \in H$,

$$A\|h\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu(\omega) \leq B\|h\|^2. \tag{1.2}$$

If the right hand of (1.2) holds, we say $(F, \Lambda, v)_\Omega$ a cg-fusion Bessel with the bound B. If $A = B$, then we say $(F, \Lambda, v)_\Omega$ a tight cg-fusion frame and $(F, \Lambda, v)_\Omega$ is called a Parseval g-fusion frame whenever $A = B = 1$.

Example 1.3. We attend to the Hilbert space $H = \mathbb{R}^2$ with standard base $\{e_1, e_2\}$. The set

$$\Omega := \{\omega \in \mathbb{R}^2 \, : \, \|\omega\| \leq 1\}$$

equipped with Lebesgue measure λ forms a measure space. Suppose that B_1 and B_2 is a partition of Ω where $\lambda(B_2) \geq \lambda(B_1) > 1$. We put

$\mathbb{H} = \{W_1, W_2\}$ which $W_1 = \operatorname{span}\{e_1\}$ and $W_2 = \operatorname{span}\{e_2\}$. Define

$$F : \Omega \longrightarrow \mathbb{H},$$

$$F(\omega) = \begin{cases} W_1, & \omega \in B_1 \\ W_2, & \omega \in B_2, \end{cases}$$

$$v : \Omega \longrightarrow [0, \infty),$$

$$v(\omega) = \begin{cases} \frac{1}{\sqrt{\lambda(B_1)}}, & \omega \in B_1 \\ \frac{1}{\sqrt{\lambda(B_2)}}, & \omega \in B_2. \end{cases}$$

Consider $h = (h_1, h_2) \in \mathbb{R}^2$, define

$$\Lambda_\omega : F(\Omega) \longrightarrow \mathbb{R},$$

$$\Lambda_\omega h = \begin{cases} h_1, & \omega \in B_1 \\ h_2, & \omega \in B_2. \end{cases}$$

Therefore, Λ_ω is bounded operator for any $\omega \in \Omega$ and for each $h = (h_1, h_2) \in \mathbb{R}^2$ we have

$$\pi_{F(\omega)} h = \begin{cases} (h_1, 0), & \omega \in B_1 \\ (0, h_2), & \omega \in B_2, \end{cases}$$

and

$$\Lambda_\omega \pi_{F(\omega)} h = \begin{cases} h_1, & \omega \in B_1 \\ h_2, & \omega \in B_2. \end{cases}$$

Now, we conclude that the mapping $\omega \to \pi_{F(\omega)}(h)$ is measurable and $(F, \Lambda, v)_\Omega$ is a Parseval cg-fusion frame for $\mathbb{R}^2$.

Assume that $\mathcal{K} := \oplus_{\omega \in \Omega} H_\omega$ and define $\mathscr{L}^2(\Omega, \mathcal{K})$ the class of all measurable mapping $\varphi : \Omega \to \mathcal{K}$ such that

$$\|\varphi\|_2^2 := \int_{\omega \in \Omega} \|\varphi(\omega)\|^2 \, d\omega < \infty.$$

Now, synthesis and the analysis operators of a cg-fusion Bessel are defined by

$$T_{F,\Lambda} : \mathscr{L}^2(\Omega, \mathcal{K}) \longrightarrow H,$$

$$\langle T_{F,\Lambda} \varphi, h \rangle = \int_\Omega v(\omega) \langle \pi_{F(\omega)} \Lambda_\omega^* \varphi(\omega), h \rangle \, d\mu(\omega),$$

where $h \in H$, $\varphi \in \mathscr{L}^2(\Omega, \mathcal{K})$ and

$$T_{F,\Lambda}^* : H \longrightarrow \mathscr{L}^2(\Omega, \mathcal{K}),$$

$$T_{F,\Lambda}^* h = v \Lambda_{(.)} \pi_{F(.)} h,$$

The following result shows a property between the synthesis and the analysis operators of two cg-fusion Bessel mappings.

Theorem 1.4. *Suppose that* $\{e_j\}_{j\in\mathbb{J}}$ *is an orthonormal basis for* H *where* $|J|<\infty$*. Let* $(F,\Lambda,v)_\Omega$ *and* $(F,\Theta,\tau)_\Omega$ *be two cg-fusion Bessel mappings for* H *with bounds* $B_{F,\Lambda}$ *and* $B_{F,\Theta}$*, where* $\Theta := \{\Theta_\omega \in \mathcal{B}(H,H_\omega)\}$ *and* $\omega\in\Omega$*. If* $\phi := T_{F,\Lambda}T^*_{F,\Theta}$*, then* ϕ *is a trace class operator (i. e.* $\operatorname{tr}(|\phi|)<\infty$*).*

Proof. Suppose that $\phi = U|\phi|$ is the polar decomposition of the operator ϕ, where $U\in\mathcal{B}(H)$ is a partial isometry, therefore $|\phi| = U^*T_{F,\Lambda}T^*_{F,\Theta}$. Then,

$$\begin{aligned}
\operatorname{tr}(|\phi|) &= \sum_{j\in\mathbb{J}}\langle|\phi|e_j,e_j\rangle\\
&= \sum_{j\in\mathbb{J}}\langle T_{F,\Lambda}T^*_{F,\Theta}e_j,Ue_j\rangle\\
&= \sum_{j\in\mathbb{J}}\int_{\omega\in\Omega} v(\omega)\tau(\omega)\langle\pi_{F(\omega)}\Lambda^*_\omega\Theta_\omega\pi_{F(\omega)}e_j,Ue_j\rangle\,d\mu(\omega)\\
&= \sum_{j\in\mathbb{J}}\int_{\omega\in\Omega} v(\omega)\tau(\omega)\big\langle\Theta_{(\omega)}\pi_{F(\omega)}e_j,\Lambda_{(\omega)}\pi_{F(\omega)}Ue_j\big\rangle\,d\mu(\omega)\\
&\leq \sum_{j\in\mathbb{J}}\int_{\omega\in\Omega}\big\|\tau(\omega)\Theta_{(\omega)}\pi_{F(\omega)}e_j\big\|.\big\|v(\omega)\Lambda_{(\omega)}\pi_{F(\omega)}Ue_j\big\|\,d\mu(\omega)\\
&\leq \sum_{j\in\mathbb{J}}\Big(\int_{\omega\in\Omega}\big\|\tau(\omega)\Theta_{(\omega)}\pi_{F(\omega)}e_j\big\|^2\,d\mu\Big)^{\frac{1}{2}}\Big(\int_{\omega\in\Omega}\big\|v(\omega)\Lambda_{(\omega)}\pi_{F(\omega)}Ue_j\big\|^2\,d\mu(\omega)\Big)^{\frac{1}{2}}\\
&\leq \sum_{j\in\mathbb{J}}\sqrt{B_{F,\Lambda}B_{F,\Theta}}\|Ue_j\|\\
&\leq \sqrt{B_{F,\Lambda}B_{F,\Theta}}\ \|U\||\mathbb{J}|<\infty.
\end{aligned}$$

□

In the next theorem, we show a relation between ordinary c-frames with cg-fusion frames.

Theorem 1.5. *For each* $\omega\in\Omega$*, let* $\Lambda := \{\Lambda_\omega\in\mathcal{B}(H,H_\omega)\}$ *and* $v:\Omega\to\mathbb{R}^+$ *be a measurable function. Let* $F_\omega : X\to H_\omega$ *be a c-frame for* H_ω *with bounds* A_ω *and* B_ω *where* (X,ν) *is a measure space with positive measure* ν*. Define*

$$\begin{aligned}
\mathcal{F} :&\Omega\longrightarrow\mathbb{H},\\
\mathcal{F}(\omega) &= \overline{span}\{\Lambda^*_\omega F_\omega(x)\}_{x\in X},
\end{aligned}$$

and suppose that

$$0<A:=\inf_{\omega\in\Omega}A_\omega\leq B:=\sup_{\omega\in\Omega}B_\omega<\infty.$$

If the mappings $\omega \mapsto \pi_{\mathcal{F}(\omega)}(h)$ *and* $(\omega, x) \mapsto \langle v(\omega)\Lambda_\omega^* F_\omega(x), h\rangle$ *are weakly measurable for each* $h \in H$*, then the following assertions are equivalent:*

(I) *The mapping*

$$\Gamma : \Omega \times X \longrightarrow H,$$
$$\Gamma(\omega, x) = v(\omega)\Lambda_\omega^* F_\omega(x)$$

is a c-frame for H *with respect to* $\Omega \times X$*.*

(II) $(\mathcal{F}, \Lambda, v)_\Omega$ *is a cg-fusion frame for* H*.*

(III) *For any* $\omega \in \Omega$ *and each orthonormal bases* $\{e_{\omega j}\}_{j\in \mathbb{J}_\omega}$ *for* H_ω*, the mapping*

$$\Xi : \Omega \times \mathbb{J}_\omega \longrightarrow H,$$
$$\Xi(\omega, j) = v(\omega)\Lambda_\omega^* e_{\omega j}$$

is a c-frame for H *with respect to* $\Omega \times \mathbb{J}_\omega$*.*

Proof. Suppose that item (I) holds with frame bounds C and D. For each $h \in H$, we have

$$\begin{aligned}
A\int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{\mathcal{F}(\omega)}(h)\|^2\, d\mu(\omega) &\leq \int_\Omega A_\omega v^2(\omega)\|\Lambda_\omega \pi_{\mathcal{F}(\omega)}(h)\|^2\, d\mu(\omega) \\
&\leq \int_\Omega \int_X |\langle v(\omega)\Lambda_\omega \pi_{\mathcal{F}(\omega)}(h), F_\omega(x)\rangle|^2\, d\nu(x)\, d\mu(\omega) \\
&= \int_\Omega \int_X |\langle \pi_{\mathcal{F}(\omega)}(h), v(\omega)\Lambda_\omega^* F_\omega(x)\rangle|^2\ d\nu(x) d\mu(\omega) \\
&= \int_\Omega \int_X |\langle h, v(\omega)\Lambda_\omega^* F_\omega(x)\rangle|^2\, d\nu(x)\, d\mu(\omega) \\
&\leq D\|h\|^2.
\end{aligned}$$

This means that $(\mathcal{F}, \Lambda, v)_\Omega$ is a cg-fusion Bessel for H with the bound $\dfrac{D}{A}$. Similerly, $\dfrac{C}{B}$ is the lower frame bound for $(\mathcal{F}, \Lambda, v)_\Omega$. For the opposite case, assume that $(\Lambda, \mathcal{F}, v)_\Omega$ is a cg-fusion frame with bounds C and D.

For each $h \in H$ we have

$$\begin{aligned}
&\int_\Omega \int_X |\langle h, v(\omega)\Lambda_\omega^* F_\omega(x)\rangle|^2 \, d\nu(x) \, d\mu(\omega) \\
&= \int_\Omega \int_X |\langle \pi_{\mathcal{F}(\omega)}(h), v(\omega)\Lambda_\omega^* F_\omega(x)\rangle|^2 \, d\nu(x) \, d\mu(\omega) \\
&= \int_\Omega \int_X v^2(\omega) |\langle \Lambda_\omega \pi_{\mathcal{F}(\omega)}(h), F_\omega(x)\rangle|^2 \, d\nu(x) \, d\mu(\omega) \\
&\leq \int_\Omega B_\omega v^2(\omega) \|\Lambda_\omega \pi_{\mathcal{F}(\omega)}(h)\|^2 \, d\mu(\omega) \\
&\leq BD\|h\|^2,
\end{aligned}$$

and it is easy to check that AC is a lower frame bound.

To prove the equivalence of (II) and (III), note that

$$\begin{aligned}
\int_\Omega v^2(\omega) \big\|\Lambda_\omega \pi_{\mathcal{F}(\omega)} h\big\|^2 \, d\mu(\omega) &= \int_\Omega v^2(\omega) \sum_{j\in\mathbb{J}_\omega} |\langle \Lambda_\omega \pi_{\mathcal{F}(\omega)} h, e_{\omega j}\rangle|^2 \, d\mu(\omega) \\
&= \int_\Omega \sum_{j\in\mathbb{J}_\omega} |\langle h, v(\omega)\pi_{\mathcal{F}(\omega)}\Lambda_\omega^* e_{\omega j}\rangle|^2 \, d\mu(\omega) \\
&= \int_\Omega \sum_{j\in\mathbb{J}_\omega} |\langle h, v(\omega)\Lambda_\omega^* e_{\omega j}\rangle|^2 \, d\mu(\omega).
\end{aligned}$$

□

2. Main results

Assume that $F, G : \Omega \to \mathbb{H}$ and $v : \Omega \to \mathbb{R}^+$. If $\Lambda := \{\Lambda_\omega \in \mathcal{B}(H, H_\omega),\ \omega \in \Omega\}$ and $\Theta := \{\Theta_\omega \in \mathcal{B}(H, H_\omega),\ \omega \in \Omega\}$ such that $v\pi_{G(.)}\Theta_{(.)}^*\varphi$ is a measurable functions for each $\varphi \in \mathscr{L}^2(\Omega, \mathcal{K})$, we define the approximation operator with respect to Λ and Θ as follows:

$$\Psi : H \longrightarrow H,$$

$$\Psi h = \int_\Omega v(\omega)\pi_{G(\omega)}\Theta_\omega^*\big(v(\omega)\Lambda_\omega \pi_{F(\omega)} h\big) \, d\mu(\omega).$$

Before we make a theorem, we need the following lamma which provides a condition for commutativity of an integral and inner product. Assume that Y is a Banach space. We say $f : X \to Y$ is a Bochner integrable function, if there exists a sequence of integrable simple functions $\{f_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$ a. e. and also

$$\lim_{n\to\infty} \int_X \|f_n(x) - f(x)\| \, d\mu(x) = 0.$$

Lemma 2.1. [13] *Let $f : \Omega \to H$ be a Bochner integrable function. Then for each $h \in H$ we have*

$$\int_\Omega \langle f(\omega), h\rangle \, d\mu(\omega) = \Big\langle \int_\Omega f(\omega)\, d\mu(\omega), h\Big\rangle.$$

Theorem 2.2. *Let $v\pi_{F(.)}\Lambda^*_{(.)}\varphi$ and $v\pi_{G(.)}\Theta^*_{(.)}\varphi$ be Bochner integrable functions for each $\varphi \in \mathscr{L}^2(\Omega, \mathcal{K})$, $C_1, C_2 > 0$ and $0 \leq \gamma < 1$ be real numbers such that for each $h \in H$ the following assertions holds:*

(I) $\int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu(\omega) \leq C_1\|h\|^2$;
(II) $\| \int_\Omega v(\omega)\pi_{G(\omega)}\Theta^*_\omega \varphi(\omega)\, d\mu(\omega)\|^2 \leq C_2\|\varphi\|_2^2$;
(III) $\|h - \Psi h\|^2 \leq \gamma\|h\|^2$.

Then $(F, \Lambda, v)_\Omega$ is a cg-fusion frame for H with bounds $C_2^{-1}(1-\gamma)^2$ and C_1. Also, $(G, \Theta, v)_\Omega$ is a g-fusion frame for H with bounds $C_1^{-1}(1-\gamma)^2$ and C_2.

Proof. Let $h \in H$, via (I) and (II), we get

$$\|\Psi h\|^2 \leq C_2\|v\Lambda_{(.)}\pi_{F(.)}h\|_2^2 = C_2 \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu(\omega) \leq C_2 C_1\|h\|^2.$$

Hence, Ψ is a bounded operator. So, by Neumann Theorem in Banach spaces and item (III), Ψ is invertible and $\|\Psi^{-1}\| \leq (1-\gamma)^{-1}$. Thus,

$$\begin{aligned}\|h\|^2 &= \|\Psi^{-1}\Psi h\|^2 \\ &\leq (1-\gamma)^{-2}\|\Psi h\|^2 \\ &\leq C_2 C_1 (1-\gamma)^{-2}\|h\|^2.\end{aligned}$$

So, we conclude that

$$C_2^{-1}(1-\gamma)^2\|h\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu(\omega) \leq C_1\|h\|^2,$$

and the first part is proved. For the second part, let $h \in H$ and we have by Lemma 2.1,

$$\begin{aligned}
&\Big(\int_\Omega v^2(\omega)\|\Theta_\omega \pi_{G(\omega)}(h)\|^2\, d\mu(\omega)\Big)^2\\
&\quad= \Big(\int_\Omega v^2(\omega)\langle \pi_{G(\omega)}\Theta^*_{G(\omega)}\Theta_\omega \pi_{G(\omega)}(h), h\rangle\, d\mu(\omega)\Big)^2\\
&\quad\le \Big(\Big\langle \int_\Omega v(\omega)\pi_{G(\omega)}\Theta^*_{G(\omega)}\big(v(\omega)\Theta_\omega \pi_{G(\omega)}h\big)\, d\mu(\omega), h\Big\rangle\Big)^2\\
&\quad\le \Big\| \int_\Omega v(\omega)\pi_{G(\omega)}\Theta^*_{G(\omega)}\big(v(\omega)\Theta_\omega \pi_{G(\omega)}h\big)\, d\mu(\omega)\Big\|^2 \|h\|^2\\
&\quad\le C_2 \|v\Theta_{()}\pi_{G()}h\|^2\|h\|^2\\
&\quad\le C_2\|h\|^2 \int_\Omega v^2(\omega)\|\Theta_\omega \pi_{G(\omega)}(h)\|^2\, d\mu(\omega).
\end{aligned}$$

Therefore,

$$\int_\Omega v^2(\omega)\|\Theta_\omega \pi_{G(\omega)}(h)\|^2\, d\mu(\omega) \le C_2\|h\|^2.$$

For the second inequality, let $\varphi \in \mathscr{L}^2(\Omega, \mathcal{K})$ and we have

$$\begin{aligned}
&\Big\| \int_\Omega v(\omega)\pi_{F(\omega)}\Lambda^*_\omega \varphi(\omega)\, d\mu(\omega)\Big\|^2\\
&\quad= \Big(\sup_{\|f\|=1} \Big|\Big\langle \int_\Omega v(\omega)\pi_{F(\omega)}\Lambda^*_\omega \varphi(\omega)\, d\mu(\omega), f\Big\rangle\Big|\Big)^2\\
&\quad= \Big(\sup_{\|f\|=1} \Big| \int_\Omega \big\langle v(\omega)\pi_{F(\omega)}\Lambda^*_\omega \varphi(\omega), f\big\rangle\, d\mu(\omega)\Big|\Big)^2\\
&\quad= \Big(\sup_{\|f\|=1} \Big| \int_\Omega \big\langle \varphi(\omega), v(\omega)\Lambda_\omega \pi_{F(\omega)} f\big\rangle\, d\mu(\omega)\Big|\Big)^2\\
&\quad\le \Big(\sup_{\|f\|=1} \int_\Omega \|\varphi(\omega)\|\|v(\omega)\Lambda_\omega \pi_{F(\omega)} f\|\, d\mu(\omega)\Big)^2\\
&\quad\le \|\varphi\|_2^2\Big(\sup_{\|f\|=1} \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} f\|^2\, d\mu(\omega)\Big)\\
&\quad\le C_1\|\varphi\|_2^2.
\end{aligned}$$

Now by similar argument and applying an approximation operator of the form

$$\Psi^* h = \int_\Omega v(\omega)\pi_{F(\omega)}\Lambda^*_\omega\big(v(\omega)\Theta_\omega \pi_{G(\omega)}h\big)\, d\mu(\omega),$$

we can establish Θ has the required properties. □

The next result is a generalization of Theorem 3.2 from [6] for cg-fusion frames.

Theorem 2.3. *Let $(F, \Lambda, v)_\Omega$ be a cg-fusion frame for H with bounds A and B, and also $\Omega_1 \subset \Omega$ be a measurable subspace. Then the following statements hold.*

(I) *If $\{\Lambda_\omega\}_{\omega\in\Omega_1}$ is a cg-frame for H with the lower frame bound B and $v(\omega) > 1$ for each $\omega \in \Omega_1$, then*

$$\bigcap_{\omega\in\Omega_1} F(\omega) = \{0\}.$$

(II) *If $\{\Lambda_\omega\}_{\omega\in\Omega_1}$ is a tight cg-frame for H with the lower frame bound B, also $v(\omega) = 1$ for each $\omega \in \Omega_1$ and $\ker\{\Lambda_\omega\}_{\omega\in\Omega\setminus\Omega_1} = \{0\}$, then*

$$\bigcap_{\omega\in\Omega_1} F(\omega) \perp \operatorname{span}\{F(\omega)\}_{\omega\in\Omega\setminus\Omega_1} \ a.e.$$

(III) *If $C := \int_{\Omega_1} v^2(\omega)\|\Lambda_\omega\|^2 \, d\mu(\omega) < A$, then $(F, \Lambda, v)_{\Omega\setminus\Omega_1}$ is a cg-fusion frame for H with bounds $A - C$ and B.*

Proof. (I). For each $h \in \bigcap_{\omega\in\Omega_1} F(\omega)$, we have $\pi_{F(\omega)}h = h$ for every $\omega \in \Omega_1$. So,

$$B\|h\|^2 < \int_{\Omega_1} v^2(\omega)\|\Lambda_\omega h\|^2 \, d\mu(\omega) \leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}h\|^2 \, d\mu(\omega) \leq B\|h\|^2.$$

Thus, $h = 0$.

(II). For each $h \in \bigcap_{\omega\in\Omega_1} F(\omega)$, we get

$$\begin{aligned} B\|h\|^2 &= \int_{\Omega_1} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}h\|^2 \, d\mu(\omega) \\ &\leq \int_{\Omega_1} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}h\|^2 \, d\mu(\omega) + \int_{\Omega\setminus\Omega_1} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}h\|^2 \, d\mu(\omega) \\ &= \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}h\|^2 \, d\mu(\omega) \\ &\leq B\|h\|^2. \end{aligned}$$

Therefore,

$$\int_{\Omega\setminus\Omega_1} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}h\|^2 \, d\mu(\omega) = 0,$$

and this means that $h \perp \operatorname{span}\{F(\omega)\}_{\omega\in\Omega\setminus\Omega_1}$ a. e.

(III) The upper bound is clear. For the lower bound, if $h \in H$ we get

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_1} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} h\|^2 \, d\mu(\omega) \\
&\quad = \int_{\Omega} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} h\|^2 \, d\mu(\omega) - \int_{\Omega_1} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} h\|^2 \, d\mu(\omega) \\
&\quad \geq A\|h\|^2 - \int_{\Omega_1} v^2(\omega)\|\Lambda_\omega\|^2 \|h\|^2 \, d\mu(\omega) \\
&\quad = (A - C)\|h\|^2.
\end{aligned}$$

□

If the subspace Ω_1 (which is introduced in Theorem 2.3) is singleton, then we can get the following result.

Corollary 2.4. *Let $(F, \Lambda, v)_\Omega$ be a cg-fusion frame for H with bounds A and B. If there exists $\omega_0 \in \Omega$ such that the subspace $\{\omega_0\}$ is measueable and $v^2(\omega_0)\|\Lambda_{\omega_0}\|^2 < A$, then $(F, \Lambda, v)_{\Omega\setminus\{\omega_0\}}$ is a cg-fusion frame for H with bounds $A - v^2(\omega_0)\|\Lambda_{\omega_0}\|^2$ and B.*

The following is a generalization of Corollary 3.4 from [6].

Corollary 2.5. *Let $(F, \Lambda, v)_\Omega$ be a tight cg-fusion frame for H with a bound A and $\omega_0 \in \Omega$ such that the subspace $\{\omega_0\}$ is measueable. Then the following assertions are equivalent.*

(I) $\mu(\{\omega_0\})v^2(\omega_0)\|\Lambda_{\omega_0}\pi_{F(\omega_0)}\|^2 < A$.

(II) $(F, \Lambda, v)_{\Omega\setminus\{\omega_0\}}$ *is a cg-fusion frame for H .*

Proof. The proof of $(I) \Rightarrow (II)$ is evident from Corollary 2.4. For the opposite, assume that C is a lower frame bound of $(F, \Lambda, v)_{\Omega\setminus\{\omega_0\}}$. For each $0 \neq h \in H$, we have

$$\begin{aligned}
C\|h\|^2 &\leq \int_{\Omega\setminus\{\omega_0\}} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} h\|^2 \, d\mu(\omega) \\
&= \int_{\Omega} v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} h\|^2 \, d\mu(\omega) - \mu(\{\omega_0\})v^2(\omega_0)\|\Lambda_{\omega_0}\pi_{F(\omega_0)} h\|^2 \\
&= (A\|h\|^2 - \mu(\{\omega_0\})v^2(\omega_0)\|\Lambda_{\omega_0}\pi_{F(\omega_0)} h\|^2).
\end{aligned}$$

Hence,

$$0 < C \leq A - \mu(\{\omega_0\})v^2(\omega_0)\frac{\|\Lambda_{\omega_0}\pi_{F(\omega_0)} h\|^2}{\|h\|^2}.$$

Thus, we conclude that $A - \mu(\{\omega_0\})v^2(\omega_0)\|\Lambda_{\omega_0}\pi_{F(\omega_0)}\|^2 > 0$. □

In next result, we provide a new cg-fusion frame for the space H with by removing a number of members of a Parseval c-frame for H_ω.

Theorem 2.6. *Let $(F,\Lambda,v)_\Omega$ be a cg-fusion frame for H with bounds A and B. For each $\omega\in\Omega$, let $\mathcal{F}_\omega$ be a Parseval c-frame for H_ω which $\mathcal{F}_\omega\big|_{\Omega\setminus\Omega_1}$ is a c-frame for H_ω with the lower frame bound C_ω for each finite subspace measurable $\Omega_1\subset\Omega$ and all $\omega\in\Omega$. If $C:=\big(\min_{\omega\in\Omega}C_\omega\big)$ and*

$$\begin{aligned} G :&\Omega\longrightarrow \mathbb{H},\\ G(\omega)&=\overline{\operatorname{span}}\{\Lambda_\omega^*\mathcal{F}_\omega(x)\}_{x\in\Omega\setminus\Omega_1}, \end{aligned}$$

then $(G,\Lambda,v)_\Omega$ is a cg-fusion frame for H with bounds AC and B.

Proof. For each $h\in H$, we have

$$\begin{aligned} \int_\Omega v^2(\omega)\|\Lambda_\omega\pi_{G(\omega)}h\|^2\,d\mu(\omega) &= \int_\Omega v^2(\omega)\int_\Omega |\langle\Lambda_\omega\pi_{G(\omega)}h,\mathcal{F}_\omega(x)\rangle|^2\,d\mu(x)\,d\mu(\omega)\\ &\geq \int_\Omega v^2(\omega)\int_{\Omega\setminus\Omega_1} |\langle\pi_{G(\omega)}h,\Lambda_\omega^*\mathcal{F}_\omega(x)\rangle|^2\,d\mu(x)\,d\mu(\omega)\\ &= \int_\Omega v^2(\omega)\int_{\Omega\setminus\Omega_1} |\langle\pi_{F(\omega)}h,\Lambda_\omega^*\mathcal{F}_\omega(x)\rangle|^2\,d\mu(x)\,d\mu(\omega)\\ &= \int_\Omega v^2(\omega)\int_{\Omega\setminus\Omega_1} |\langle\Lambda_\omega\pi_{F(\omega)}h,\mathcal{F}_\omega(x)\rangle|^2\,d\mu(x)\,d\mu(\omega)\\ &\geq \int_\Omega v^2(\omega)C_\omega\|\Lambda_\omega\pi_{F(\omega)}h\|^2\,d\mu(\omega)\\ &\geq C\int_\Omega v^2(\omega)\|\Lambda_\omega\pi_{F(\omega)}h\|^2\,d\mu(\omega)\\ &\geq AC\|h\|^2. \end{aligned}$$

The upper bound is clear. □

Now, we aim to study the approximation operator Ψ in finite case for H, similar to the method which has been presented in [6]. Suppose that $|\Omega|<\infty$ and $(F,\Lambda,v)_\Omega$ is a Parseval cg-fusion frame for H. For each $\omega_0\in\Omega$, we define

$$\begin{aligned} &\mathcal{D}_{\omega_0}:\mathscr{L}^2(\Omega,\mathcal{K})\longrightarrow\mathscr{L}^2(\Omega,\mathcal{K}),\\ &\mathcal{D}_{\omega_0}\varphi(\omega)=\delta_{\omega,\omega_0}\varphi(\omega_0). \end{aligned}$$

We define the associated 1-erasure reconstruction error $\mathcal{E}_1(F,\Lambda)$ to be

$$\mathcal{E}_1(F,\Lambda)=\max_{\omega\in\Omega}\mu(\{\omega\})v^2(\omega)\|\pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega\pi_{F(\omega)}\|.$$

Since

$$\|\pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega\pi_{F(\omega)}\|=\sup_{\|h\|=1}\|\pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega\pi_{F(\omega)}h\|\leq\|\Lambda_\omega\|^2,$$

therefore,

$$\mathcal{E}_1(F,\Lambda) = \max_{\omega\in\Omega} \mu(\{\omega\})v^2(\omega)\|\Lambda_\omega\|^2.$$

Lemma 2.7. *Let $F:\Omega \to H$ be a Perseval c-frame for H with* $\dim H = n$. *Then*

$$\int_\Omega \|F(\omega)\|^2\, d\mu(\omega) = \dim H.$$

Proof. Suppose that $\{e_j\}_{j=1}^n$ is an orthonormal basis for H. Since

$$n = \sum_{j=1}^n \int_\Omega |\langle e_j, F(\omega)\rangle|^2\, d\mu(\omega) = \int_\Omega \sum_{j=1}^n |\langle e_j, F(\omega)\rangle|^2\, d\mu(\omega) = \int_\Omega \|F(\omega)\|^2\, d\mu(\omega),$$

this completes the proof. □

Theorem 2.8. *Let $\Lambda_\omega F(\omega)$ be closed subspaces for all $\omega\in\Omega$, $|\Omega|<\infty$ with counting measure and (F,Λ,v) be a Parseval cg-fusion frame for finite dimensional H and also $|H_\omega|<\infty$ for each $\omega\in\Omega$. Then the following conditions are equivalent.*

(I) *The set (F,Λ,v) satisfies $\mathcal{E}_1(F,\Lambda) = \min_{\omega\in\Omega}\mathcal{E}_1(\mathcal{F}(\omega),\Lambda_\omega,v)$, where $(\mathcal{F}(\omega),\Lambda_\omega,v)_{\omega\in\Omega}$ is a Parseval cg-fusion frame for H with* $\dim\mathcal{F}(\omega) = \dim F(\omega)$ *for each $\omega\in\Omega$.*

(II) *For each $\omega\in\Omega$ we have*

$$v^2(\omega)\|\Lambda_\omega\|^2 = \frac{\dim H}{|\Omega|\dim[\Lambda_\omega F(\omega)]}.$$

Proof. Assume that $\{e_{\omega j}\}_{j\in\mathbb{J}_\omega}$ is a orthonormal basis for $\Lambda_\omega(F(\omega))$ for each $\omega\in\Omega$. Via Theorem 1.5, the mapping

$$\Omega\times\mathbb{J}_\omega \longrightarrow H,$$
$$(\omega,j)\longmapsto v(\omega)\Lambda_\omega^* e_{\omega j}$$

is a Parseval c-frame for H. Thus, by Lemma 2.7, we can write

$$\begin{aligned}\dim H &= \int_\Omega \sum_{1}^{\dim \Lambda_\omega F(\omega)} v^2(\omega)\|\Lambda_\omega^* e_{\omega j}\|^2\, d\mu(\omega)\\ &\le \int_\Omega \dim[\Lambda_\omega F(\omega)] v^2(\omega)\|\Lambda_\omega\|^2\, d\mu(\omega)\\ &= \sum_\omega \dim[\Lambda_\omega F(\omega)] v^2(\omega)\|\Lambda_\omega\|^2|\Omega|.\end{aligned}$$

Therefore, there exists $\omega\in\Omega$ such that

$$\dim H \le |\Omega|\dim[\Lambda_\omega F(\omega)]v^2(\omega)\|\Lambda_\omega\|^2.$$

Since the dimensions as well as the number of subspaces are fixed, we conclude that $\mathcal{E}_1(F,\Lambda)$ is minimal if and only if

$$\dim H = |\Omega| \dim[\Lambda_\omega F(\omega)] v^2(\omega)\|\Lambda_\omega\|^2 \ , \qquad (\forall \omega \in \Omega).$$

□

We can obtain the following result which can be derived from Theorem 2.8 and its proof.

Corollary 2.9. *Let $\Lambda_\omega F(\omega)$ be closed subspaces for all $\omega \in \Omega$, $|\Omega| < \infty$ with counting measure and (F,Λ,v) be a Parseval cg-fusion frame for finite dimensional H and also $|H_\omega| < \infty$ for each $\omega \in \Omega$. Then the following conditions are equivalent.*

(I) *The set (F,Λ,v) satisfies $\mathcal{E}_1(F,\Lambda) = \min_{\omega\in\Omega} \mathcal{E}_1(\mathcal{F}(\omega),\Lambda_\omega,v)$, where $(\mathcal{F}(\omega),\Lambda_\omega,v)_{\omega\in\Omega}$ is a Parseval cg-fusion frame for H with $\dim \mathcal{F}(\omega) = \dim F(\omega)$ for each $\omega \in \Omega$.*

(II) *For each $\omega \in \Omega$ we have*

$$v^2(\omega)\|\Lambda_\omega\|^2 = \frac{1}{|\Omega|} \ \textit{and} \quad \dim H = \dim \Lambda_\omega F(\omega).$$

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