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On generalized statistical limit points for triple sequence in random 2-normed spaces

Verda Gürdal [1]
[1] Suleyman Demirel University, Department of Mathematics, East Campus, 32260, Isparta, Turkey

Abstract. In this article we introduce the notion $\mathcal{I}_3$-cluster points, and investigate the relation between $\mathcal{I}_3$-cluster points and limit points of triple sequences in the topology induced by random 2-normed spaces and prove some important results.

Keywords: t-norm, random 2-normed space, ideal convergence, triple sequence, F-topology.

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1. Introduction

Statistical convergence for real sequence was first introduced by Fast [4] in 1951. Since then statistical convergence was investigated by more and more researchers. The concept of $\mathcal{I}$-convergence, and interesting generalization of statistical convergence [4], was first presented by Kostyrko et al. [17] with use of the ideal $\mathcal{I}$ of subsets of the set of natural numbers $\mathbb{N}$ and further studies done in [23]. The study of ideal convergence in triple sequence has been initiated by Şahiner and Tripathy [34]. More analysis in this field and more implications of these statistical convergence and ideal convergence can be seen in [1, 2, 3, 13, 24, 25]

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Menger [18] introduced the notion of probabilistic metric spaces, which is an interesting and important generalization of metric spaces, the study of these spaces was under the name of statistical metric. The idea of Menger was to use distribution function instead of non-negative real numbers as values of the metric. In this theory, the notion of distance has a probablistic nature. Namely, the distance between two points x and y is represented by a distribution function F_{xy}; and for $\varepsilon > 0$, the value $F_{xy}(\varepsilon)$ is interpreted as the probability that the distance from x to y is less than ε. In fact the probabilistic theory has become an area of active research for the last fourthly years. An important family of probabilistic metric spaces are probabilistic normed spaces. The notion of probabilistic normed spaces was introduced in [28] and [29] and since then several generalizations and applications of this notion have been investigated by various authors, namely [9, 10, 11, 12, 19, 20, 22, 26, 27, 30, 31, 33]. Further it was extended to random/probabilistic 2-normed spaces by Golet [8] using the concept of 2-norm of Gähler [7].

In this article, we will give informations about triple sequences and new results in the last section. New results are related to study the concept of $\mathcal{I}$-cluster points and ordinary limit points for triple sequences in random 2-normed spaces and their main properties.

2. Preliminaries

In this section, we recall some basic definitions and notations which form the background of the present work.

The notion of statistical convergence is based on the asymptotic density of the subsets of the set $\mathbb{N}$ of positive integers. In [5] an axiomatic approach is given for introducing the notion of density of sets $K \subseteq \mathbb{N}$.

Let K be a subset of the set of natural numbers $\mathbb{N}$. We denote by K_n the number of elements of the set K which are less or equal to $n \in \mathbb{N}$. Also $|K_n|$ denotes the cardinality of the set K_n. The natural (asymptotic) density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K_n|$$

whenever the limit exists. We recall also that $\delta(\mathbb{N} \setminus K) = 1 - \delta(K)$. If x is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we write that x_k satisfies P for almost all k (a.a.k). It is said that a sequence $x = (x_n)_{n\in\mathbb{N}}$ is statistically convergent to a point L if for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0.$$

In this case, we write $st\text{-}\lim x_k = L$ and S denotes the set of all statistically convergent sequences.

The notion of statistical convergence was further generalized in the paper [17] using the notion of an ideal of subsets of the set $\mathbb{N}$. We say that a non-empty family of sets $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if $\mathcal{I}$ is hereditary (i.e. $B \subset A \in \mathcal{I} \Rightarrow B \in \mathcal{I}$) and additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ for which $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$ is called a non-trivial ideal. A non-trivial ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$. If not otherwise stated in the sequel $\mathcal{I}$ will denote an admissible ideal. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be any ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \backslash A\}$ called the filter associated with the ideal $\mathcal{I}$, is a filter on $\mathbb{N}$.

Definition 2.1. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence. For more information about $\mathcal{I}$-convergent, see the references in [23].

We now recall the following basic concepts from [32, 34] which will be needed throughout the paper.

A function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (or complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j, k, l \geq n_0$. A triple sequence (x_{jkl}) is said to be bounded if there exists $M > 0$ such that $|x_{jkl}| < M$ for all $j, k, l \in \mathbb{N}$. We denote the space of all bounded triple sequences by ℓ^3_∞.

Definition 2.2. A subset K of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{j,k,l\to\infty} \frac{|K_{jkl}|}{jkl}$$

exists, where the vertical bars denote the number of (j, k, l) in K such that $p \leq j$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{jkl})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3\left(\{(j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{jkl} - L| \geq \varepsilon\}\right) = 0.$$

As can be seen from the following example a st_3-convergent sequence does not need to be bounded.

Example 2.3. Let

$$(x_{jkl}) = \begin{cases} jkl & , \quad j, k, l \text{ are cubes} \\ 4 & , \quad \text{otherwise.} \end{cases}$$

Then $st_3\text{-}\lim x_{jkl} = 4$ but (x_{jkl}) is neither convergent in Pringsheim's sense nor bounded.

If $(x_{j_m k_n l_o})_{m,n,o\in\mathbb{N}}$ is a sub-sequence of the triple sequence $x = (x_{jkl})$ of real numbers and $\mathcal{M} = \{(_{j_m k_n l_o}) : m, n, o \in \mathbb{N}\}$, we abbreviate $(x_{j_m k_n l_o})_{m,n,o\in\mathbb{N}}$ by $(x)_{\mathcal{M}}$. Now $\delta_3(\mathcal{M}) = 0$, then $(x)_{\mathcal{M}}$ is said to be a thin sub-sequence of the sequence x and $(x)_{\mathcal{M}}$ is called a non-thin sub-sequence of x if $\mathcal{M}$ does not have triple natural density zero i.e., either $\delta_3(\mathcal{M})$ is a positive number or $\mathcal{M}$ fails to have triple natural density.

Definition 2.4. Let $\mathcal{I}_3$ be an admissible ideal on $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, then a triple sequence (x_{jkl}) is said to be $\mathcal{I}_3$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{jkl} - L| \geq \varepsilon\} \in \mathcal{I}_3.$$

In this case, one writes $\mathcal{I}_3\text{-}\lim x_{jkl} = L$.

Remark 2.5. (i) Let $\mathcal{I}_3(f)$ be the family of all finite subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Then $\mathcal{I}_3(f)$ convergence coincides with the convergence of triple sequences in [32].

(ii) Let $\mathcal{I}_3(\delta) = \{A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_3(\delta)$ convergence coincides with the statistical convergence in [32].

Example 2.6. Let $\mathcal{I} = \mathcal{I}_3(\delta)$. Define the triple sequence (x_{jkl}) by

$$(x_{jkl}) = \begin{cases} 1 & , \quad j,k,l \text{ are cubes} \\ 4 & , \quad \text{otherwise.} \end{cases}$$

Then for every $\varepsilon > 0$

$$\delta(\{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{jkl} - 4| \geq \varepsilon\}) \leq \lim_{p,q,r} \frac{\sqrt{p}\sqrt{q}\sqrt{r}}{pqr} = 0.$$

This implies that $\mathcal{I}\text{-}\lim x_{jkl} = 4$. But, the triple sequence (x_{jkl}) is not convergent to 4.

Throughout the chapter we consider the ideals of $2^{\mathbb{N}}$ by $\mathcal{I}$; the ideals of $2^{\mathbb{N}\times\mathbb{N}}$ by $\mathcal{I}_2$ and the ideals of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by $\mathcal{I}_3$.

Definition 2.7. ([7]) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies (**i**) $\|x, y\| = 0$ if and only if x and y are linearly dependent; (**ii**) $\|x, y\| = \|y, x\|$; (**iii**) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$; (**iv**) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x,y\| = |x_1 y_2 - x_2 y_1|\,, \quad x = (x_1, x_2)\,, \quad y = (y_1, y_2)\,.$$

Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|x,y\| \geq 0$ and $\|x, y+\alpha x\| = \|x,y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Also, if x, y and z are linearly dependent, then $\|x, y+z\| = \|x,y\| + \|x,z\|$ or $\|x, y-z\| = \|x,y\| + \|x,z\|$. Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for every $y \in X$.

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar [29].

Definition 2.8. Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0,1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2.9. A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutive, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 2.10. Let X be a linear space of dimension greater than one, τ is a triangle, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:

(1) $F(x,y;t) = H_0(t)$ if x and y are linearly dependent, where $F(x,y;t)$ denotes the value of $F(x,y)$ at $t \in \mathbb{R}$,

(2) $F(x,y;t) \neq H_0(t)$ if x and y are linearly independent,

(3) $F(x,y;t) = F(y,x;t)$ for all $x, y \in X$,

(4) $F(\alpha x, y; t) = F(x, y; \frac{t}{|\alpha|})$ for every $t > 0, \alpha \neq 0$ and $x, y \in X$,

(5) $F(x+y, z; t) \geq \tau(F(x,z;t), F(y,z;t))$ whenever $x, y, z \in X$.

If (2.2.5) is replaced by

(6) $F(x+y, z; t_1 + t_2) \geq F(x,z;t_1) * F(y,z;t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_+$;

then $(X, F, *)$ is called a random 2-normed space (for short, RTN space).

Remark 2.11. Note that every 2-norm space $(X, \|.,.\|)$ can be made a random 2-normed space in a natural way, by setting

(i) $F(x,y;t) = H_0(t - \|x,y\|)$, for every $x, y \in X, t > 0$ and $a * b = \min\{a,b\}, a,b \in S$;

(ii) $F(x,y;t) = \frac{t}{t+\|x,y\|}$ for every $x, y \in X, t > 0$ and $a * b = ab$ for $a, b \in S$.

Let $(X, F, *)$ be an RTN space. Since $*$ is a continuous t-norm, the system of (ε, λ)-neighborhoods of θ (the null vector in X)

$$\{\mathcal{N}_\theta(\varepsilon,\lambda) : \varepsilon > 0,\ \lambda \in (0,1)\},$$

where

$$\mathcal{N}_\theta(\varepsilon,\lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\}.$$

determines a first countable Hausdorff topology on X, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice versa.

A triple sequence $x = (x_{jkl})$ in X is said to be F-convergence to $L \in X$ if for every $\varepsilon > 0$, $\lambda \in (0,1)$ and for each nonzero $z \in X$ there exists a positive integer N such that

$$x_{jkl}, z - L \in \mathcal{N}_\theta(\varepsilon,\lambda) \text{ for each } n \geq N$$

or equivalently,

$$x_{jkl}, z \in \mathcal{N}_L(\varepsilon,\lambda) \text{ for each } n \geq N.$$

In this case we write $F\text{-}\lim x_{jkl}, z = L$.

3. The Main results

In this section, we will examine the concept of $\mathcal{I}_3$-cluster points and $\mathcal{I}_3$-limit points sets of a given triple sequences in the topology induced by random 2-normed spaces.

Definition 3.1. Let $(X, F, *)$ be an RTN space, $\mathcal{I}_3$ be an admissible ideal and $x = (x_{jkl}) \in X$. An element $L \in X$ is said to be an $\mathcal{I}_3$-limit point of a triple sequence x with respect to the random 2-norm F (or $\mathcal{I}_F^3(x)$-limit point) if there is a set

$$\mathcal{M} = \{(j_m, k_m, l_m) : j_1 < j_2 < ...; k_1 < k_2 < ...; l_1 < l_2 < ...\}$$

of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\mathcal{M} \notin \mathcal{I}_3$ and $F\text{-}\lim_{m\to\infty} x_{j_m k_m l_m}, z = L$ for each nonzero z in X.

The set of all $\mathcal{I}_F^3$-limit points of x is denoted by $\mathcal{I}\left(\Lambda_F^3(x)\right)$.

Definition 3.2. Let $(X, F, *)$ be an RTN space, and $x = (x_{jkl}) \in X$. An element $L \in X$ is said to be an $\mathcal{I}_3$-cluster point of x with respect to the random 2-norm F (or $\mathcal{I}_F^3$-cluster point) if for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}_3.$$

The set of all $\mathcal{I}_F^3$-cluster points of x is denoted by $\mathcal{I}\left(\Gamma_F^3\left(x\right)\right)$.

Theorem 3.3. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}_3$ be an admissible ideal. Then for each triple sequence $x = (x_{jkl})$ of X we have $\mathcal{I}\left(\Lambda_F^3\left(x\right)\right) \subset \mathcal{I}\left(\Gamma_F^3\left(x\right)\right)$ and the set $\mathcal{I}\left(\Gamma_F^3\left(x\right)\right)$ is a closed set.*

Proof. Let $L \in \mathcal{I}\left(\wedge_F^3\right)$. Then there exists a set

$\mathcal{M} = \{(j_m, k_m, l_m) : j_1 < j_2 < ...; k_1 < k_2 < ...; l_1 < l_2 < ...\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$

such that $\mathcal{M} \notin \mathcal{I}_3$ and

$$F\text{-}\lim_{m \to \infty} x_{j_m k_m l_m}, z = L \tag{3.1}$$

for each nonzero z in X. Suppose $\varepsilon > 0$ be arbitrary. According to 3.1, for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X there exists a positive integer p_0 such that for $j,k,l \geq p_0$ we have $x_{jkl} - L, z \in \mathcal{N}_L(\varepsilon, \lambda)$. Hence

$$\begin{aligned}&\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} - L, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \\ &\supset \{j_{p_0+1}, j_{p_0+2}, ...; k_{p_0+1}, k_{p_0+2}, ...; l_{p_0+1}, l_{p_0+2}, ...\}\end{aligned}$$

and so

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} - L, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}_3,$$

which means that $L \in \mathcal{I}\left(\Gamma_F^3\left(x\right)\right)$.

Let $y \in \overline{\mathcal{I}\left(\Gamma_F^3\right)}$. Take $\varepsilon > 0$ and $\lambda \in (0,1)$. There exists $L \in \mathcal{I}\left(\Gamma_F^3\left(x\right)\right) \cap \mathcal{N}_\theta\left(y, \varepsilon, \lambda\right)$. Choose $\eta > 0$ such that $\mathcal{N}_\theta\left(L, \eta, \lambda\right) \subset \mathcal{N}_\theta\left(y, \varepsilon, \lambda\right)$. We obviously have

$$\begin{aligned}&\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : y - x_{jkl}, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \\ &\supset \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : L - x_{jkl}, z \in \mathcal{N}_\theta(\eta, \lambda)\}.\end{aligned}$$

Hence

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : y - x_{jkl}, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \notin \mathcal{I}_3$$

and $y \in \mathcal{I}\left(\Gamma_F^3\left(x\right)\right)$. This completes the proof of the theorem. □

Definition 3.4. Let $(X, F, *)$ be an RTN space and $x = \left(x_{jkl}\right)_{j,k,l \in \mathbb{N}} \in X$. An element $L \in X$ is said to be limit point of the triple sequence $x = (x_{jkl})$ with respect to the random 2-norm F if there is subsequence of the sequence x which converges to L with respect to the random 2-norm F. By $L_F^3\left(x\right)$, we denote the set of all limit points of the sequence $x = (x_{jkl})$ with respect to the random 2-norm F.

It is obvious $\mathcal{I}\left(\Lambda_F^3(x)\right) \subseteq L_F^3(x)$, $\mathcal{I}\left(\Gamma_F^3(x)\right) \subseteq L_F^3(x)$: Take $L \in \mathcal{I}\left(\Gamma_F^3(x)\right)$, then $\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}_3$ for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X. If $L \notin L_F^3(x)$, then there is $\varepsilon' > 0$ such that $\mathcal{N}_L(\varepsilon', \lambda)$ contains only a finite number of elements of x in X. Then $\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_L(\varepsilon', \lambda)\} \in \mathcal{I}_3$, but it contradicts to $L \in \mathcal{I}\left(\Gamma_F^3(x)\right)$. Hence $x \in \mathcal{I}\left(\Gamma_F^3(x)\right)$. Thus $x \in L_F^3(x)$, and so $\mathcal{I}\left(\Gamma_F^3(x)\right) \subseteq L_F^3(x)$.

Definition 3.5. Let $\mathcal{I}_3$ be an admissible ideal in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $(X, F, *)$ be an RTN space. The triple sequence (x_{jkl}) in X is said to be $\mathcal{I}_F$-convergent to $L \in X$ with respect to the random 2-norm F if for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{I}_3.$$

Lemma 3.6. *Let $x = (x_{jkl})$ be a triple sequence in an RTN space $(X, F, *)$. If x is $\mathcal{I}_F$-convergent with respect to the random 2-norm F, then $\mathcal{I}\left(\Lambda_F^3(x)\right)$ and $\mathcal{I}\left(\Gamma_F^3(x)\right)$ are both equal to the singleton set $\{\mathcal{I}_F\text{-}\lim x_n, z\}$ for each nonzero z in X.*

Proof. Let $\mathcal{I}_F$-$\lim_n x_n, z = L_1$, where $L_1 \neq L_2$. Then there exist two subsets A and A', that is,

$$\mathcal{A} = \{(j_m, k_m, l_m) : j_1 < j_2 < ...; k_1 < k_2 < ...; l_1 < l_2 < ...\}$$

and

$$\mathcal{A}' = \{(p_m, q_m, r_m) : p_1 < p_2 < ...; q_1 < q_2 < ...; r_1 < r_2 < ...\}$$

of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\mathcal{A} \notin \mathcal{I}_3 \text{ and } F - \lim_{m\to\infty} x_{j_m k_m l_m}, z = L_2 \tag{3.2}$$

$$\mathcal{A}' \notin \mathcal{I}_3 \text{ and } F - \lim_{m\to\infty} x_{p_m q_m r_m}, z = L_1. \tag{3.3}$$

By (3.3), given $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, there exists $p_0 \in \mathbb{N}$ such that for $m > p_0$ we have $x_{p_m q_m r_m}, z \in \mathcal{N}_{L_1}(\varepsilon, \lambda)$. Hence,

$$\begin{aligned} \mathcal{A} &= \left\{(j_m, k_m, l_m) \in \mathcal{A}' : x_{p_m q_m r_m}, z \notin \mathcal{N}_{L_1}(\varepsilon, \lambda)\right\} \\ &\subset \left\{(p_m, q_m, r_m) \in \mathcal{A}' : p_1 < p_2 < ... < p_{p_0};\right. \\ &\qquad \left. q_1 < q_2 < ... < q_{p_0}; r_1 < r_2 < ... < r_{p_0}\right\}. \end{aligned}$$

Since $\mathcal{I}_3$ is an admissible ideal we have $\mathcal{A} \in \mathcal{I}_3$. If can choose the set

$$\mathcal{B} = \left\{(p_m, q_m, r_m) \in \mathcal{A}' : x_{p_m q_m r_m}, z \in \mathcal{N}_{L_1}(\varepsilon, \lambda)\right\} \notin \mathcal{I}_3.$$

On the other hand, $\mathcal{B} \in \mathcal{I}_3$, then $\mathcal{A} \cup \mathcal{B} = \mathcal{A}' \in \mathcal{I}_3$, which contradicts (3.3). Since $\mathcal{I}_F$-$\lim_n x_n, z = L_2$, we have that for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$,

$$\mathcal{C} = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \notin \mathcal{N}_{L_2}(\varepsilon, \lambda)\} \in \mathcal{I}_3.$$

Hence,

$$\mathcal{C}^c = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \in \mathcal{N}_{L_2}(\varepsilon,\lambda)\} \in \mathcal{F}(\mathcal{I}_3).$$

Since for every $L_1 \neq L_2$, we have $\mathcal{B}\cap\mathcal{C}^c = \emptyset$, $\mathcal{B} \subset \mathcal{C}$. Since $\mathcal{C} \in \mathcal{I}_3$ implies $\mathcal{B} \in \mathcal{I}_3$, this contradicts the fact that $\mathcal{B} \notin \mathcal{I}_3$. Hence $\mathcal{I}\left(\Lambda_F^3(x)\right) = L_2$.

Suppose that $\mathcal{I}_F$-$\lim_n x_n, z = L_1$, where $L_1 \neq L_2$. By definition, for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$, we get

$$\mathcal{A}_1 = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \in \mathcal{N}_{L_2}(\varepsilon,\lambda)\} \notin \mathcal{I}_3$$

and

$$\mathcal{A}_2 = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_n, z \in \mathcal{N}_{L_1}(\varepsilon,\lambda)\} \notin \mathcal{I}_3.$$

For $L_1 \neq L_2$, we have $\mathcal{A}_1\cap\mathcal{A}_2 = \emptyset$ and so $\mathcal{A}_2 \subset \mathcal{A}_1^c$. Also, $\mathcal{I}_F$-$\lim_n x_n, z = L_2$ implies that

$$\mathcal{A}_1^c = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \notin \mathcal{N}_{L_2}(\varepsilon,\lambda)\} \in \mathcal{I}_3.$$

Hence $\mathcal{A}_2 \in \mathcal{I}_3$, which is a contradiction to $\mathcal{A}_2 \notin \mathcal{I}_3$.We have $\mathcal{I}\left(\Lambda_F^3(x)\right) = \mathcal{I}\left(\Gamma_F^3(x)\right) = L_2$. This completes the proof of the lemma. □

Theorem 3.7. *Let $(X,F,*)$ be an RTN space and $x = (x_{jkl})$ and $y = (y_{jkl})$ be triple sequences in X such that*

$$\mathcal{A} = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl} \neq y_{jkl}\} \in \mathcal{I}_3.$$

Then $\mathcal{I}\left(\Lambda_F^3(x)\right) = \mathcal{I}\left(\Lambda_F^3(y)\right)$ and $\mathcal{I}\left(\Gamma_F^3(x)\right) = \mathcal{I}\left(\Gamma_F^3(y)\right)$.

Proof. Let $\mathcal{A} \in \mathcal{I}_3$ and $\varepsilon > 0$ be given. If $L \in \mathcal{I}\left(\Lambda_F^3(x)\right)$, then there is a subset

$$\mathcal{M} = \{(j_m,k_m,l_m) : j_1 < j_2 < ...; k_1 < k_2 < ...; l_1 < l_2 < ...\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$$

such that $\mathcal{M} \notin \mathcal{I}_3$ and $F - \lim_{m\to\infty} x_{j_m k_m l_m}, z = L$ for each nonzero z in X. Given $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $N \in \mathbb{N}$ such that $x_{j_m k_m l_m}, z \in \mathcal{N}_L(\varepsilon,\lambda)$ for $m > N$ and nonzero $z \in X$. Since

$$\mathcal{B}_1 = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (j,k,l) \in \mathcal{M} \wedge x_{jkl} \neq y_{jkl}\} \in \mathcal{I}_3,$$

then

$$\mathcal{B}_2 = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (j,k,l) \in \mathcal{M} \wedge x_{jkl} = y_{jkl}\} \notin \mathcal{I}_3.$$

Indeed, if $\mathcal{B}_2 \in \mathcal{I}_3$, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \in \mathcal{I}_3$, but $\mathcal{B} \notin \mathcal{I}_3$. Hence the subsequence $(y_{jkl})_{(j,k,l)\in\mathcal{B}_2}$ of the sequence $y = (y_{jkl})$ is convergent to L with respect to the random 2-norm F. This implies that $L \in \mathcal{I}\left(\Lambda_F^3(y)\right)$. Similarly we can show that $\mathcal{I}\left(\wedge_F^3(y)\right) \subset \mathcal{I}\left(\wedge_F^3(x)\right)$. Hence $\mathcal{I}\left(\Lambda_F^3(y)\right) = \mathcal{I}\left(\Lambda_F^3(x)\right)$. Now let $L \in \mathcal{I}\left(\Gamma_F^3(x)\right)$. Then

$$\mathcal{C}_1 = \{(j,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{jkl}, z \in \mathcal{N}_L(\varepsilon,\lambda)\} \notin \mathcal{I}_3$$

for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ and

$$\mathcal{C}_2 = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : (j,k,l) \in \mathcal{C}_1 \wedge x_{jkl} = y_{jkl}\} \notin \mathcal{I}_3.$$

Therefore, $\mathcal{C}_2 \subset \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : y_{jkl}, z \in \mathcal{N}_L(\varepsilon,\lambda)\}$. It shows that

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : y_{jkl}, z \in \mathcal{N}_L(\varepsilon,\lambda)\} \notin \mathcal{I}_3$$

i.e. $L \in \mathcal{I}\left(\Gamma_F^3(y)\right)$. Similarly, we can show that $\mathcal{I}\left(\Gamma_F^3(y)\right) \subset \mathcal{I}\left(\Gamma_F^3(x)\right)$ and hence $\mathcal{I}\left(\Gamma_F^3(x)\right) = \mathcal{I}\left(\Gamma_F^3(y)\right)$. This completes the proof of the theorem. □

The next theorem proves a strong connection between $\mathcal{I}_F^3$-cluster and limit points of a given triple sequence with respect to the random 2-norm F.

Definition 3.8. Let $(X,F,*)$ be an RTN space, $\mathcal{I}_3$ be an admissible ideal and $(x_{j_m k_n l_o})_{m,n,o\in\mathbb{N}}$ be a sub-sequence of the triple sequence $x = (x_{jkl})_{j,k,l\in\mathbb{N}}$. If $\mathcal{K} = \{(j_m k_n l_o) : m,n,o \in \mathbb{N}\} \in \mathcal{I}_3$, then the subsequence $x_{\mathcal{K}} = (x_{j_m k_n l_o})$ in X is called $\mathcal{I}_F^3$-thin subsequence of the triple sequence $x = (x_{jkl})$ in X. If $\mathcal{K} \notin \mathcal{I}_3$, then the subsequence $x_{\mathcal{K}}$ in X is called $\mathcal{I}_F^3$-nonthin subsequence of the triple sequence $x = (x_{jkl})$ in X.

It is clear that if L is a $\mathcal{I}_F^3$-limit point of $x \in X$, then there is a $\mathcal{I}_F^3$-nonthin subsequence $x_{\mathcal{K}}$ that convergent to L with respect to the random 2-norm F.

Definition 3.9. An admissible ideal $\mathcal{I}_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ is said to satisfy the condition (AP) if for every sequence $(A_s)_{s\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}_3$ there are sets $B_s \subset \mathbb{N}$, $s \in \mathbb{N}$, such that the symmetric difference $A_s \Delta B_s$ is a finite set for every $s \in \mathbb{N}$ and $\cup_{s\in\mathbb{N}} B_s \in \mathcal{I}_3$.

Theorem 3.10. *Let $(X,F,*)$ be an RTN space and $\mathcal{I}_3$ be an admissible ideal with property (AP) and $x = (x_{jkl})$ be a triple sequence in X. Then there is a sequence $y = (y_{jkl}) \in X$ such that $L_F^3(y) = \mathcal{I}\left(\Gamma_F^3(x)\right)$ and*

$$\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} \neq y_{jkl}\} \in \mathcal{I}_3.$$

Proof. If $\mathcal{I}\left(\Gamma_F^3(x)\right) = L_F^3(x)$, then $y = x$ and this case is trivial. Let $\mathcal{I}\left(\Gamma_F^3(x)\right)$ is a proper subset of $L_F^3(x)$. Then $L_F^3(x) \backslash \mathcal{I}\left(\Gamma_F^3(x)\right) \neq \emptyset$ and for each $L \in L_F^3(x) \backslash \mathcal{I}\left(\Gamma_F^3(x)\right)$. There is an $\mathcal{I}_F^3$-thin subsequence $(x_{j_m k_n l_o})_{m,n,o\in\mathbb{N}}$ of x such that $F - \lim_{m,n,o\to\infty} x_{j_m k_n l_o}, z = L$, i.e., given $\varepsilon > 0$, $\lambda \in (0,1)$ there exists a positive integer N such that $x_{j_m k_n l_o}, z \notin \mathcal{N}_L(\varepsilon,\lambda)$ for $m,n,o > N$ and nonzero $z \in X$. Hence there is a $\mathcal{N}_L(\varepsilon,\lambda)$ such that $\{(m,n,o) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{j_m k_n l_o}, z \in \mathcal{N}_L = \mathcal{N}_L(\delta,\lambda)\} \in \mathcal{I}_3$ for each $\delta > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. It is obvious that the collection of all $\mathcal{N}_L$'s is an open cover of $L_F^3(x) \backslash \mathcal{I}\left(\Gamma_F^3(x)\right)$. So by Covering Theorem there is a countable and mutually disjoint subcover $\{\mathcal{N}_s\}_{j=1}^{\infty}$

such that each $\mathcal{N}_s$ contains an $\mathcal{I}_F^3$-thin subsequence of $(x_{jkl}) \in X$. Now let

$$A_s = \{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl}, z \in \mathcal{N}_s = \mathcal{N}_s(\delta, \lambda), \ s \in \mathbb{N}\}$$

for each $\delta > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. It is clear that $A_s \in \mathcal{I}_3$ $(s = 1, 2, ...)$ and $A_r \cap A_s = \emptyset$. Then by (AP) property of $\mathcal{I}_3$ there is a countable collection $\{B_s\}_{s=1}^{\infty}$ of subsets of $\mathbb{N}$ such that $B = \cup_{s=1}^{\infty} B_s \in \mathcal{I}_3$ and $A_s \diagdown B$ is a finite set for each $s \in \mathbb{N}$. Let $M = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \backslash B = \{(j_m, k_n, l_o) : m, n, o \in \mathbb{N}\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Now the sequence $y = (y_{jkl}) \in X$ is defined by $y_{jkl} = x_{j_m k_n l_o}$ if $(m,n,o) \in B$ and $y_{jkl} = x_{jkl}$ if $(j,k,l) \in M$. Obviously, $\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{jkl} \neq y_{jkl}\} \subset B \in \mathcal{I}_3$, so by Theorem 3.7, $\mathcal{I}\left(\Gamma_F^3(y)\right) = \mathcal{I}\left(\Gamma_F^3(x)\right)$. Since $A_s \backslash B$ is finite set, the sequence $y_B = (y_{jkl})_{(j,k,l) \in B}$ has no limit point with respect to the random 2-norm F that is not also an $\mathcal{I}_F^3$-limit point of y, i.e., $L_F^3(y) = \mathcal{I}\left(\Gamma_F^3(y)\right)$. Therefore, we have proved $L_F^3(y) = \mathcal{I}\left(\Gamma_F^3(x)\right)$. $\square$

4. Conclusion

The theory of random normed (RN) spaces is important area of research in functional analysis. Much work has been done in this theory and it has many important applications in real world problems. This study aims to find out the use of the notions of $\mathcal{I}_3$-limt points, $\mathcal{I}_3$-cluster points and ordinary limit points of triple sequences for demonstrating some results in the area of random 2-normed space. With the help of its applications, we give the relationship between $\mathcal{I}_3$-cluster points and limit points in the topology induced by random 2-normed space (RTNS) and acquired meaningful results for these notions. The results acquired here are more common than corresponding results for normed spaces. It is expected that new results will help to understand deeply the concept of this new type of convergence on RTNS.

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