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Q-soft R-submodules and their properties

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Abstract. The purpose of this paper is to define the concept of Q-soft R- submodules over commutative rings and discuss their relationship with R- submodules. Next, We describe some of the their basic preoperties and discuss the master properties of them. Later, we introduce the concepts of sum, intersection and external direct sum of them and study some types of separation axioms of them. Finally, we investigate them under homomorphisms of R- submodules.

Keywords: Theory of modules, submodules, maps of modules, fuzzy set theory, direct sum.

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1. Introduction

To deal with the complicated problems involving uncertainties in economics, engineering, environmental science, medical science and social science, methods of classical mathematics can not be successfully used. Alternatively, mathematical theories such as probability theory, fuzzy set theory, rough set theory, vague set theory and the interval mathematics were established by researchers to deal with uncertainties appearing in the above fields. These methods also have some inherent

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difficulties. To overcome these kinds of difficulties, Molodtsov [7] introduced the concept of soft sets. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applicable to many different fields. At present, works on soft set theory are progressing rapidly. We refer to [1, 3, 4, 5, 9] for very recent works on soft algebraic structures as operations on soft sets, soft ideals on BCK/BCI-algebras, soft ideals on soft semirings, soft rings, soft modules, respectively. Solairaju and Nagarajan [8] introduced a new algebraic structure Q-fuzzy group in 2008. in this paper, we focus on introducing basic notions of Q-soft R- submodules over a commutative ring. We obtain some characterrizations of Q-soft R- submodules and some results of Q-soft R- submodules under homomorphism. Also we define Q-soft external direct sum and prove that the direct sum of two Q-soft R- submodules is also Q-soft R- submodule.

2. Q-soft R- submodules

Definition 2.1. (See [6]) Let R be a ring. A commutative group $(M, +)$ is called a left R-module or a left module over R with respect to a mapping

$$. : R \times M \to M$$

if for all $r, s \in R$ and $m, n \in M$,
(1) $r.(m + n) = r.m + r.n$,
(2) $r.(s.m) = (rs).m$,
(3) $(r + s).m = r.m + s.m$.
If R has an identity 1 and if $1.m = m$ for all $m \in M$, then M is called a unitary or unital left R-module.
A right R-module can be defined in a similar fashion.

Definition 2.2. (See [2]) Let Q be a non-empty set. For any set A, a Q-soft set μ over U is a set, defined by a function μ, representing a mapping $\mu : A \times Q \to P(U)$, such that $\mu(x, q) = \emptyset$ if $x \notin A$. A Q-soft set over U can also be represented by the set of ordered pairs $\mu = \{(x, \mu(x, q)) \mid x \in A, \mu(x, q) \in P(U)\}$. From here on, "$Q$-soft set" will be used without over U.

Definition 2.3. (See [2]) Let μ and ν be Q-soft sets of set A.Then,
(1) μ is called an empty Q-soft subset, if $\mu(x, q) = \emptyset$ for all $(x, q) \in A \times Q$,
(2) μ is called a $A \times Q$-universal soft set, if $\mu(x, q) = U$ for all $(x, q) \in A \times Q$,
(3) the set $Im(\mu) = \{\mu(x, q) : (x, q) \in A \times Q\}$ is called image of μ,
(4) μ is a Q-soft subset of ν, if $\mu(x, q) \subseteq \nu(x, q)$ for all $(x, q) \in A \times Q$,
(5) μ and ν are soft equal, if and only if $\mu(x, q) = \nu(x, q)$ for all $(x, q) \in A \times Q$,

(6) the set $(\mu \cup \nu)(x,q) = \mu(x,q) \cup \nu(x,q)$ for all $(x,q) \in A \times Q$ is called union of μ and ν,
(7) the set $(\mu \cap \nu)(x,q) = \mu(x,q) \cap \nu(x,q)$ for all $(x,q) \in A \times Q$ is called intersection of μ and ν.

Example 2.4. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and let $Q = \{p, q\}, A = \{x_1, x_2, x_3, x_4, x_5\}$.
Let μ, ν, ξ and π be Q-soft sets of set A. Define

$$\mu = \{((x_1,p),\{u_1,u_2,u_3\}),((x_1,q),\{u_3,u_4,u_5\}),((x_2,p),\{u_1,u_5\}),((x_2,q),\{u_3,u_4\})\},$$

$$\nu = \{((x_2,p),\{u_1,u_2\}),((x_2,q),\{u_1,u_3\}),((x_3,p),\{u_2,u_4\}),((x_3,q),\{u_1,u_5\})\}$$

and

$$\xi = \{((x_4,p),U),((x_4,q),U)\}, \pi = \{((x_5,p),\{\}),((x_5,q),\{\})\}.$$

Then for all $(x,r) \in A \times Q$ we have

$$(\mu \cup \nu)(x,r) = \{((x_1,p),\{u_1,u_2,u_3\}),((x_1,q),\{u_3,u_4,u_5\}),((x_2,p),\{u_1,u_2,u_5\}), \\ ((x_2,q),\{u_1,u_3,u_4\}),((x_3,p),\{u_2,u_4\}),((x_3,q),\{u_1,u_5\})\}$$

and $(\mu \cap \nu)(x,r) = \{((x_2,p),\{u_1\}),((x_2,q),\{u_3\})\}$. Also $\xi = U$ and $\pi = \emptyset$.

Remark 2.5. The difinition of classical subset is not valid for the Q-soft subset. For example, let $U = \{u_1, u_2, u_3, u_4, u_5\}, Q = \{q\}, A = \{x_1, x_2, x_3\}$. Let μ and ν be Q-soft sets of set A. If

$$\mu = \{((x_1,q),\{u_1,u_2\}),((x_2,q),\{u_4\})\}$$

and

$$\nu = \{((x_1,q),\{u_1,u_2,u_3\}),((x_2,q),\{u_4,u_5\}),((x_3,q),\{u_1\})\},$$

then $\mu \subseteq \nu$ as Q-soft subset, but $\mu \nsubseteq \nu$ as classical subset.

Throughout this work, Q is a non-empty set, U refers to an initial universe set and $P(U)$ is the power set of U.

Definition 2.6. Let R be a ring and M be a (left) R-module. A function $\mu : M \times Q \to P(U)$ is a Q-soft R-submodule of M if for all $x, y \in M, q \in Q$ and $r \in R$ the following conditions are satisfied:
(1) $\mu(x+y,q) \supseteq \mu(x,q) \cap \mu(y,q)$,
(2) $\mu(rx,q) \supseteq \mu(x,q)$,
(3)$\mu(0,q) = U$.

Denote by $QSM(R)$, the set of all Q-soft R-submodules of M.

Example 2.7. Let $R = (\mathbb{Z}, +, -)$ be a ring. It is easy to show that $M = \mathbb{Z}$ is an R-module. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$. For all $q \in Q$ define $\mu : M \times Q \to P(U)$ as

$$\mu(x,q) = \begin{cases} \{u_3\} & \text{if } x \in \{\pm 1, \pm 3, ...\} \\ \{u_3, u_4, u_5\} & \text{if } x \in \{\pm 2, \pm 4, ...\} \\ U & \text{if } x = 0 \end{cases}$$

now we can say that $\mu \in QSM(R)$.

Proposition 2.8. *If R is a field, then condition (2) in Definition 2.6 is equivalent to the condition $\mu(rx, q) = \mu(x, q)$ for any $x \in M, q \in Q, (r \neq 0) \in R$.*

Proof. Let $x \in M, q \in Q, (r \neq 0) \in R$. Then $\mu(rx, q) \supseteq \mu(x, q) = \mu((\frac{1}{r}).rx, q) \supseteq \mu(rx, q)$ and so $\mu(rx, q) = \mu(x, q)$. □

Definition 2.9. Let $\mu_1, \mu_2 \in QSM(R)$. The intersection of μ_1 and μ_2 is denoted by $\mu_1 \cap \mu_2$ and defined by $\mu_1 \cap \mu_2)(x, q) = \mu_1(x, q) \cap \mu_2(x, q)$ for all $x \in M, q \in Q$.

Proposition 2.10. *Let $\mu_1, \mu_2 \in QSM(R)$. Then $(\mu_1 \cap \mu_2) \in QSM(R)$.*

Proof. Let $x, y \in M, q \in Q$ and $r \in R$.
(1)

$$\begin{aligned} (\mu_1 \cap \mu_2)(x + y, q) &= \mu_1(x + y, q) \cap \mu_2(x + y, q) \\ &\supseteq \mu_1(x, q) \cap \mu_1(y, q) \cap \mu_2(x, q) \cap \mu_2(y, q) \\ &= \mu_1(x, q) \cap \mu_2(x, q) \cap \mu_1(y, q) \cap \mu_2(y, q) \\ &= (\mu_1 \cap \mu_2)(x, q) \cap (\mu_1 \cap \mu_2)(y, q) \end{aligned}$$

thus

$$(\mu_1 \cap \mu_2)(x + y, q) \supseteq (\mu_1 \cap \mu_2)(x, q) \cap (\mu_1 \cap \mu_2)(y, q).$$

(2)

$$(\mu_1 \cap \mu_2)(rx, q) = \mu_1(rx, q) \cap \mu_2(rx, q) \supseteq \mu_1(x, q) \cap \mu_2(x, q) = (\mu_1 \cap \mu_2)(x, q).$$

(3)

$$(\mu_1 \cap \mu_2)(0, q) = \mu_1(0, q) \cap \mu_2(0, q) = U \cap U = U.$$

Thus $(\mu_1 \cap \mu_2) \in QSM(R)$. □

Corollary 2.11. *Let $\{\mu_i \mid i \in I_n = 1, 2, ..., n\} \subseteq QSM(R)$. Then so is $\bigcap_{i \in I_n} \mu_i$.*

Definition 2.12. (See [6]) Let M be an R-module and N be a nonempty subset of M. Then N is called a submodule of M if N is a subgroup of M and for all $r \in R, a \in N$, we have $ra \in N$.

Proposition 2.13. *Let $\mu \in QSM(R)$. Then for any $\alpha \in P(U)$ with $\alpha \subseteq U$, set*

$$M_\alpha = \{x \mid x \in M, \mu(x,q) \supseteq \alpha\}$$

is a submodule of the module M and and μ is Q-soft R-submodule of M_α.

Proof. Let $x, y \in M_\alpha$ and $r \in R$. Then
(1) $\mu(x+y,q) \supseteq \mu(x,q) \cap \mu(y,q) = \alpha \cap \alpha = \alpha$ and so $x+y \in M_\alpha$.
(2) From $\mu(rx,q) \supseteq \mu(x,q) = \alpha$ we get $rx \in M_\alpha$.
(3) Finally, $\mu(0,q) = U \supseteq \alpha$ and $0 \in M_\alpha$.
Hence M_α is an R-submodule of M. The second part of the statement is obvious. □

Proposition 2.14. *Let M be an R-module and N be a subset of M. Let for all $\alpha \in P(U)$ if $\alpha \supseteq U$, then $\alpha = U$. If $\mu : N \times Q \to P(U)$ be the function as*

$$\mu(x,q) = \begin{cases} U & \text{if } x \in N \\ \emptyset & \text{if } x \notin N \end{cases}$$

then $\mu \in QSM(R)$ if and only if N is a submodule of M.

Proof. Let $\mu \in QSM(R)$ and we prove that N is a submodule of M. Let $x, y \in N \subseteq M$ and $q \in Q, r \in R$. Now

$$\mu(x+y,q) \supseteq \mu(x,q) \cap \mu(y,q) = U \cap U = U$$

so $\mu(x+y,q) = U$. Thus $x+y \in N$.
Since $\mu(rx,q) \supseteq \mu(x,q) = U$ so $\mu(rx,q) = U$ and then $rx \in N$.
Finally $\mu(0,q) = U$ means that $0 \in N$. Hence N is a submodule of M.
Conversely, let N is a submodule of M and we prove that $\mu \in QSM(R)$.
Suppose $x, y \in M, q \in Q$ and we have the following conditions
(1) If $x, y \in N$, then

$$\mu(x+y,q) = U \supseteq U = U \cap U = \mu(x,q) \cap \mu(y,q).$$

(2) If $x \notin N$ and $y \in N$, then

$$\mu(x+y) \supseteq \emptyset = \emptyset \cap U = \mu(x,q) \cap \mu(y,q).$$

(3) Finally, if $x, y \notin N$, then

$$\mu(x+y) \supseteq \emptyset = \emptyset \cap \emptyset = \mu(x,q) \cap \mu(y,q).$$

Hence from (1)-(3) we have that

$$\mu(x+y) \supseteq \mu(x) \cap \mu(y).$$

Now let $x \in M$ and $r \in R$. Then
(1) If $x \in N$, then $rx \in N$ and so $\mu(rx,q) = U = \mu(x,q)$.
(2) If $x \notin N$, then $rx \notin N$ and so $\mu(rx,q) = \emptyset = \mu(x,q)$.
Hence from (1) and (2) we obtain $\mu(rx) \supseteq \mu(x)$.
Finally, since $0 \in N$ we have $\mu(0,q) = U$.
Thus $\mu \in QSM(R)$. □

Definition 2.15. Let f be a mapping from R-module M into R-module N. Let $\mu \in QSM(R)$ and $\nu \in QSN(R)$. Then all $y \in N, q \in Q$ define

$$f(\mu)(y,q) = \begin{cases} \sup\{\mu(x,q) \mid x \in M, f(x) = y\} & \text{if } f^{-1}(x) \neq \emptyset \\ \emptyset & \text{if } f^{-1}(x) = \emptyset \end{cases}$$

Also $\forall x \in M,\ f^{-1}(\nu)(x,q) = \nu(f(x),q)$.

Proposition 2.16. *Let f be an epimorphism from R-module M into R-module N. If $\mu \in QSM(R)$, then $f(\mu) \in QSN(R)$.*

Proof. Let $y_1, y_2 \in N$.
(1)

$$\begin{aligned} f(\mu)(y_1+y_2,q) &= \sup\{\mu(x_1+x_2,q) \mid x_1,x_2 \in M, f(x_1)=y_1, f(x_2)=y_2\} \\ &\supseteq \sup\{\mu(x_1,q)\cap\mu(x_2,q) \mid x_1,x_2 \in M, f(x_1)=y_1, f(x_2)=y_2\} \\ &= \sup\{\mu(x_1,q) \mid x_1 \in M, f(x_1)=y_1\}\cap\sup\{\mu(x_2,q) \mid x_2 \in M, f(x_2)=y_2\} \\ &= f(\mu)(y_1,q) \cap f(\mu)(y_2,q). \end{aligned}$$

(2) Let $x \in M$ and $r \in R$.

$$\begin{aligned} f(\mu)(ry) &= \sup\{\mu(rx) \mid rx \in M, f(rx) = ry\} \\ &\supseteq \sup\{\mu(x) \mid x \in M, f(x) = y\} \\ &= f(\mu)(y). \end{aligned}$$

thus $f(\mu)(ry) \supseteq f(\mu)(y)$.

(3) $f(\mu)(0,q) = \sup\{\mu(0,q) \mid 0 \in M, f(0) = 0\} = U$.
Hence $f(\mu) \in QSN(R)$. □

Proposition 2.17. *Let f be an epimorphism from R-module M into R-module N. If $\nu \in QSN(R)$, then $f^{-1}(\nu) \in QSM(R)$.*

Proof. Let $x_1, x_2 \in M$. Then
(1)

$$\begin{aligned} f^{-1}(\nu)(x_1+x_2,q) &= \nu(f(x_1+x_2),q) \\ &= \nu(f(x_1)+f(x_2),q) \\ &\supseteq \nu(f(x_1),q)\cap\nu(f(x_2),q) \\ &= f^{-1}(\nu)(x_1,q)\cap f^{-1}(\nu)(x_2,q) \end{aligned}$$

so
$$f^{-1}(\nu)(x_1+x_2,q) \supseteq f^{-1}(\nu)(x_1,q)\cap f^{-1}(\nu)(x_2,q).$$
(2) Let $x \in M$ and $r \in R$. Then

$$f^{-1}(\nu)(rx,q) = \nu(f(rx),q) = \nu(rf(x),q) \supseteq \nu(f(x),q) = f^{-1}(\nu)(x,q).$$

(3) $f^{-1}(\nu)(0,q) = \nu(f(0),q) = \nu(0,q) = U.$
Hence $f^{-1}(\nu) \in QSM(R)$. □

3. Sum and external direct sum of two Q-soft R-submodules

Definition 3.1. The sum on Q-soft subsets μ_1 and μ_2 of an R-module M is defined as follows:

$$(\mu_1+\mu_2)(x,q) = \sup\{\mu_1(y,q)\cap\mu_2(z,q) \mid x = y+z \in M\}.$$

Proposition 3.2. *Let $\mu_1, \mu_2 \in QSM(R)$. Then $(\mu_1+\mu_2) \in QSM(R)$.*

Proof. (1) Let $x_1, x_2, y_1, y_2, z_1, z_2 \in M$ and $q \in Q$. Then

$(\mu_1+\mu_2)(x_1+x_2, q = \sup\{\mu_1(y_1+y_2,q)\cap\mu_2(z_1+z_2,q) \mid x_1+x_2 = y_1+y_2+z_1+z_2\}$

$\supseteq \sup\{\mu_1(y_1,q)\cap\mu_1(y_2,q)\cap\mu_2(z_1,q)\cap\mu_2(z_2,q) \mid x_1+x_2 = y_1+z_1+y_2+z_2\}$

$= \sup\{\mu_1(y_1,q)\cap\mu_2(z_1,q)\cap\mu_1(y_2,q)\cap\mu_2(z_2,q) \mid x_1+x_2 = y_1+z_1+y_2+z_2\}$

$= \sup\{\mu_1(y_1,q)\cap\mu_2(z_1,q) \mid x_1 = y_1+z_1\}\cap\sup\{\mu_1(y_2,q)\cap\mu_2(z_2,q) \mid x_2 = y_2+z_2\}$

$= (\mu_1+\mu_2)(x_1,q)\cap(\mu_1+\mu_2)(x_2,q).$

(2) Let $x, y, z \in M$ and $q \in Q, r \in R$.

$$\begin{aligned} (\mu_1+\mu_2)(rx,q) &= \sup\{\mu_1(ry,q)\cap\mu_2(rz,q) \mid rx = ry+rz\} \\ &\supseteq \sup\{\mu_1(y,q)\cap\mu_2(z,q) \mid x = y+z\} \\ &= (\mu_1+\mu_2)(x,q) \end{aligned}$$

then
$$(\mu_1+\mu_2)(rx,q) \supseteq (\mu_1+\mu_2)(x,q).$$

(3)

$$(\mu_1 + \mu_2)(0, q) = \sup\{\mu_1(0, q) \cap \mu_2(0, q) \mid 0 = 0 + 0\} = U \cap U = U.$$

Therefore $(\mu_1 + \mu_2) \in QSM(R)$. □

Corollary 3.3. *Let* $\{\mu_i \mid i \in I_n = 1, 2, ..., n\} \subseteq QSM(R)$. *Then so is* $\sum_{i \in I_n} \mu_i$.

Definition 3.4. Let μ_1 and μ_2 be two Q-soft R-submodules of M_1 and M_2 respectively. Define

$$\mu_1 \oplus \mu_2 : M_1 \oplus M_2 \to P(U)$$

by

$$(\mu_1 \oplus \mu_2)((x, y), q) = \mu_1(x, q) \cap \mu_2(y, q)$$

for all $x \in M_1, y \in M_2$ and $q \in Q$. Thus $\mu_1 \oplus \mu_2$ is called a Q-soft external direct sum of μ_1 and μ_2.

Proposition 3.5. *Let* $\mu_1 \in QSM_1(R)$ *and* $\mu_2 \in QSM_2(R)$. *Then*

$$(\mu_1 \oplus \mu_2) \in QS(M_1 \oplus M_2)(R).$$

Proof. Let $x_i \in M_1$ and $y_i \in M_2$ for $i = 1, 2$. If $q \in Q$ and $r \in R$, then
(1)

$$\begin{aligned}(\mu_1 \oplus \mu_2)((x_1, y_1) + (x_2, y_2), q) &= (\mu_1 \oplus \mu_2)((x_1 + x_2, y_1 + y_2), q)\\ &= \mu_1(x_1 + x_2, q) \cap \mu_2(y_1 + y_2, q)\\ &\supseteq \mu_1(x_1, q) \cap \mu_1(x_2, q) \cap \mu_2(y_1, q) \cap \mu_2(y_2, q)\\ &= (\mu_1 \oplus \mu_2)((x_1, y_1), q) \cap (\mu_1 \oplus \mu_2)((x_2, y_2), q)\end{aligned}$$

hence

$$(\mu_1 \oplus \mu_2)((x_1, y_1) + (x_2, y_2), q) \supseteq (\mu_1 \oplus \mu_2)((x_1, y_1), q) \cap (\mu_1 \oplus \mu_2)((x_2, y_2), q).$$

$$\begin{aligned}(\mu_1 \oplus \mu_2)(r(x_1, y_1), q) &= (\mu_1 \oplus \mu_2)((rx_1, ry_1), q)\\ &= \mu_1(rx_1, q) \cap \mu_1(rx_2, q)\\ &\supseteq \mu_1(x_1, q) \cap \mu_1(x_2, q)\\ &= (\mu_1 \oplus \mu_2)((x_1, y_1), q)\end{aligned}$$

thus

$$(\mu_1 \oplus \mu_2)(r(x_1, y_1), q) \supseteq (\mu_1 \oplus \mu_2)((x_1, y_1), q).$$

(3) $(\mu_1 \oplus \mu_2)((0, 0), q) = \mu_1(0, q) \cap \mu_2(0, q) = U \cap U = U.$
Thus $(\mu_1 \oplus \mu_2) \in QS(M_1 \oplus M_2)(R)$. □

Corollary 3.6. *Let* $\{\mu_i \mid i \in I_n = 1, 2, ..., n\} \subseteq QSM_{i\in I_n}(R)$. *Then*

$$\mu = \bigoplus_{i\in I_n} \mu_i \in QS(\bigoplus_{i\in I_n} M_i)(R).$$

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