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Nonuniform Dual Wavelets Associated with the Linear Canonical Transform

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Abstract. A generalization of Mallatâs classical multiresolution analysis, based on the theory of spectral pairs, was considered in two articles by Gabardo and Nashed. In this setting, the associated translation set is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. In this paper, we are interested in the dual wavelets whose construction depends on nonuniform multiresolution analysis associated with linear canonical transform. Here we prove that if the translates of the scaling functions of two multiresolution analyses in linear canonical transform settings are biorthogonal, so are the wavelet families which are associated with them. Under mild assumptions on the scaling functions and the wavelets, we also show that the wavelets generate Riesz bases.

Keywords: Nonuniform, Biorthogonal, Scaling function,Linear Canonical Transform.

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1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing

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discrete wavelet systems. A multiresolution analysis is an increasing family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0. The function φ is called the *scaling function* or the *father wavelet*. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of multiresolution analysis to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. But in all these cases, the translation set is always a group. Recently, Gabardo and Nashed in[21, 22] defined a multiresolution analysis associated with a translation set $\{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N-1, r$ is an odd integer and r, N are relatively prime, a discrete set which is not necessarily a group. They call this an *NUMRA*. As, the case $N = 1$ reduces to the standard definition of *MRA* with dyadic dilation. NUMRA with multiplicity D, is called *NUMRA-D* that generalizes a particular case of a result of Calogero and Garrigos [17] on biorthogonal *MRA's* of multiplicity D in nonstandard setup. A study with respect to *NUMRA* has been done by many authors in the references [2, 3, 4, 5, 6, 28, 29].

The concept of biorthogonal wavelets plays an important role in applications. We refer to [16, 17, 18, 19, 25, 26] for various aspects of this theory on $\mathbb{R}$. For the higher dimensional situation on $\mathbb{R}^n$, we refer to the article [26].

In the early 1970s, a promising linear integral transform with three free parameters, namely, linear canonical transform was independently introduced by Collins [20] in paraxial optics, and Moshinsky, and Quesne [27] in quantum mechanics, to study the conservation of information and uncertainty under linear maps of phase space. The LCT provides a unified treatment of the generalized Fourier transforms in the sense that it is an embodiment of several well-known integral transforms including the Fourier transform, fractional Fourier transform, Fresnel transform, scaling operations and so on [1, 23, 24, 30]. Over a couple of decades, the application areas for LCT have been growing at an exponential rate and is as such befitting for investigating deep problems in time-frequency analysis, filter design, phase retrieval problems, pattern recog- nition, radar analysis, holographic three-dimensional television, quantum physics, and many more. Apart from applications, the theoretical skeleton of LCT has likewise been extensively studied and investigated including the convolution theorems, sampling theorems, Poisson sum- mation formulae, uncertainty principles, shift-invariant theory and so on. For more about LCT and their applications, we allude to [7, 8, 9, 10, 11, 12, 13, 14, 15]. In this article we construct dual wavelets which depend on Nonuniform Multiresolution Analysis associated with linear canonical transform. We show that if ϕ and $\tilde{\phi}$ are the scaling functions of two multiresolution analyses (LCT-MRAs) such that their translates are biorthogonal, then the associated families of wavelets are also biorthogonal. Under mild decay conditions on the scaling functions and the wavelets, we also show that the wavelets generate Riesz bases for $L^2\big(\mathbb{R}\big)$.

The article is organized as follows. In section 2, we give a brief introduction about LCT nonuniform wavelets on $\mathbb{R}$. In section 3, we find necessary and sufficient conditions for the translates of a function to form a Riesz basis for its closed linear span. In the last section, we prove that the wavelets associated with dual MRAs are biorthogonal and generate Riesz bases for $L^2(\mathbb{R})$.

2. Preliminaries

In mathematics, a unimodular matrix M is a square integer matrix having determinant $+1$ or -1. For the sake of simplicity, we consider the second order matrix $\mathfrak{M}_{2\times2} = (A, B, C, D)$ with its transpose defined by $\mathfrak{M}^T_{2\times2} = (A, B, C, D)^T$. Let us first introduce the definition of Linear Canonical Transform.

Definition 2.1. The linear canonical transform of any $f \in L^2(\mathbb{R})$ with respect to the unimodular matrix $\mathfrak{M}_{2\times2} = (A, B, C, D)$is defined by

$$\mathcal{L}[f](\zeta) = \begin{cases} \int_{\mathbb{R}} f(t)\mathcal{K}_{\mathfrak{M}}(t,\zeta)dt & B \neq 0 \\ \\ \sqrt{D}\exp\frac{CD\zeta^2}{2} f(D\zeta) & B = 0. \end{cases} \tag{2.1}$$

where $\mathcal{K}_{\mathfrak{M}}(t,\zeta)$ is the kernel of linear canonical transform and is given by

$$\mathcal{K}_{\mathfrak{M}}(t,\zeta) = \frac{1}{\sqrt{2\pi\iota B}}\exp\left\{\frac{\iota(At^2 - 2t\zeta + D\zeta^2)}{2B}\right\}, \quad B \neq 0$$

It is here noted that for the case $B = 0$, the LCT defined by equation (2.1) corresponds to a chirp multiplication operation and is therefore of no particular interest to us. As such, in the rest of the article, we will keep our focus on the case when $B \neq 0$. It is here worth noticing that the phase-space transform (2.1) is lossless if and only if the matrix $\mathfrak{M}$ is unimodular; that is, $AD - BC = 1$. Several special transforms can be obtained from the linear canonical transform (2.1). For example, for $\mathfrak{M} = (1, B, 0, 1)$, gives the Fresnel transform, for $\mathfrak{M} = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$ the LCT yields us the fractional Fourier transform whereas for $\mathfrak{M} = (0, 1, -1, 0)$, we reach at the classical Fourier transform. Moreover, Bi-lateral Laplace, Gauss-Weierstrass, and Bargmann transform are also its special cases.

The inversion formula corresponding to linear canonical transform (2.1) is defined by

$$f(t) = \int_{\mathbb{R}} \mathcal{L}[f](\zeta)\overline{\mathcal{K}_{\mathfrak{M}}(t,\zeta)}d\zeta.$$

Morever the well known Parsevel's formula of the linear canonical transform (2.1) may be stated as

$$\langle \mathcal{L}[f], \mathcal{L}[g] \rangle = \langle f, g \rangle, \quad \text{for all} \quad f, g, L^2(\mathbb{R}).$$

Definition 2.2. Given a real uni-modular matrix $\mathfrak{M} = (A, B, C, D)$ and integers $N \geq 1$ and r odd with $1 \leq r \leq 2N - 1$ such that r and N are relatively prime, an associated linear canonical nonuniform multiresolution analysis (abbreviated LCT-NUMRA) is a collection $\{V_j^{\mathfrak{M}} : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:

(1) $V_j^{\mathfrak{M}} \subset V_{j+1}^{\mathfrak{M}}$ for all $j \in \mathbb{Z}$;
(2) $\bigcup_{j\in\mathbb{Z}} V_j^{\mathfrak{M}}$ is dense in $L^2(\mathbb{R})$;
(3) $\bigcap_{j\in\mathbb{Z}} V_j^{\mathfrak{M}} = \{0\}$;
(4) $f(t) \in V_j^{\mathfrak{M}}$ if and only if $f(2Nt)\, e^{-i\pi A\left(1-(2N)^2\right)t^2/B} \in V_{j+1}^{\mathfrak{M}}$ for all $j \in \mathbb{Z}$;
(5) There exists a function ϕ in $V_0^{\mathfrak{M}}$ such that $\left\{\phi_{0,\lambda}^{\mathfrak{M}}(t) = \phi(t - \lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$, is a complete orthonormal basis for $V_0^{\mathfrak{M}}$.

Since $\phi \in V_0^{\mathfrak{M}} \subset V_1^{\mathfrak{M}}$ and the collection $\left\{\phi_{1,\lambda}^{\mathfrak{M}} : \lambda \in \Lambda\right\}$ is an orthonormal basis in $V_1^{\mathfrak{M}}$, hence, the function $\phi \in V_1^{\mathfrak{M}}$ has the Fourier expansion as

$$\phi(t) = \sum_{\lambda\in\Lambda} a_\lambda\, \phi_{1,\lambda}^{\mathfrak{M}}(t) = \sqrt{2N} \sum_{\lambda\in\zeta} a_\lambda \phi\big(2Nt - \lambda\big)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}, \tag{2.2}$$

where

$$a_\lambda = \int_{\mathbb{R}} \phi(t)\, e^{-i\pi A t^2/B}\, \overline{\phi_{1,\lambda}^{\mathfrak{M}}(t)}\, dt \quad \text{and} \quad \sum_{\lambda\in\Lambda} |a_\lambda|^2 < \infty. \tag{2.3}$$

Implementing the linear canonical transform on both sides of above equation, we have,

$$\mathcal{L}_M\big[\phi(t)\big](\zeta) = \widehat{\phi}\left(\frac{\zeta}{B}\right) = \Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)\widehat{\phi}\left(\frac{\zeta}{2NB}\right), \tag{2.4}$$

where

$$\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{B}\right) = \frac{1}{\sqrt{2N}}\sum_{\lambda\in\Lambda} a_\lambda^{\mathfrak{M}}\, e^{-2\pi i\lambda\zeta/B}. \tag{2.5}$$

For each $j\in\mathbb{Z}$ and real matrix $M=(A,B,C,D)$, the LCT wavelet subspace $W_j^{\mathfrak{M}}$ is defined as the orthogonal complement of $V_j^{\mathfrak{M}}$ in $V_{j+1}^{\mathfrak{M}}$, so that $W_j^{\mathfrak{M}}\perp V_j^{\mathfrak{M}}$. It is clear from the conditions (1), (2) and (3) of the Definition 2.2 that

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j^{\mathfrak{M}}.$$

Definition 2.3. A set of functions $\big\{\psi_1^{\mathfrak{M}},\psi_2^{\mathfrak{M}},\ldots,\psi_{2N-1}^{\mathfrak{M}}\big\}$ in $L^2\big(\mathbb{R}\big)$ will be called a *set of basic wavelets* associated with a given LCT-NUMRA if the family of functions $\Big\{\psi_\ell(t-\lambda)e^{-\frac{-\iota\pi A}{B}(t^2-\lambda^2)} : 1\le\ell\le 2N-1, \lambda\in\Lambda\Big\}$ forms an orthonormal basis for $W_0^{\mathfrak{M}}$.

Assume that there exists $(2N-1)$ functions $\big\{\psi_1^{\mathfrak{M}},\psi_2^{\mathfrak{M}},\ldots,\psi_{2N-1}^{\mathfrak{M}}\big\}$ in $L^2(\mathbb{R})$ such that their translates by the elements of Λ and dilations by the integer powers of $2N$ form a Riesz basis of $W_j^{\mathfrak{M}}$, i.e.,

$$W_j^{\mathfrak{M}} = \overline{\text{Span}}\Big\{\psi_{\ell,j,\lambda}^{\mathfrak{M}} : 1\le\ell\le 2N-1,\ \lambda\in\Lambda\Big\},\ j\in\mathbb{Z} \tag{2.6}$$

where

$$\psi_{\ell,j,\lambda}^{\mathfrak{M}}(t) = (2N)^{j/2}\,\psi_\ell\big((2N)^j t-\lambda\big)\,e^{-\frac{i\pi A}{B}(t^2-\lambda^2)},\ 1\le\ell\le 2N-1, \lambda\in\Lambda. \tag{2.7}$$

Since the closed subspace $V_1^{\mathfrak{M}}$ can be decomposed as $V_1^{\mathfrak{M}} = V_0^{\mathfrak{M}}\oplus W_0^{\mathfrak{M}}$, so we have $\psi_\ell^{\mathfrak{M}}(t)\in W_0^{\mathfrak{M}}\subseteq V_1^{\mathfrak{M}}$, for $1\le\ell\le 2N-1$ and every fixed $M=(A,B,C,D)$, and as a consequence, there exist a sequence $\big\{b_{\ell,\lambda}\big\}_{\lambda\in\Lambda}$ with $\sum_{\lambda\in\Lambda}\big|b_{\ell,\lambda}\big|^2<\infty$ such that

$$\psi_{\ell,0,0}^{\mathfrak{M}}(t) = \sqrt{2N}\sum_{\lambda\in\Lambda} b_{\ell,\lambda}\,\phi\big(2Nt-\lambda\big)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}, \tag{2.8}$$

which has an equivalent form in the LCT domain

$$\widehat{\psi_\ell^{\mathfrak{M}}}\left(\frac{\zeta}{B}\right) = \Lambda_k^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)\widehat{\phi}\left(\frac{\zeta}{2NB}\right), \tag{2.9}$$

where

$$\Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{B}\right) = \frac{1}{\sqrt{2N}}\sum_{\lambda\in\Lambda} b_{\ell,\lambda}^{\mathfrak{M}}\, e^{-2\pi i\lambda\zeta/B}. \tag{2.10}$$

A Schauder basis (x_n) for any Hilbert space H is a Riesz Basis if it is equivalent to an orthonormal basis. Obviously any orthonormal basis is a Riesz basis. Mathematically we can say that if H is a seperable Hilbert space, then (x_n) is a Riesz basis if and only if every $x \in H$ can be uniquely expressed as $x = \sum_{n\in\mathbb{N}} k_n x_n$ and there exists positive constants A and B, which we call as Riesz constants, such that

$$A \sum_{n\in\mathbb{N}} |k_n|^2 \leq \| \sum_{n=1}^{\infty} k_n x_n \|^2 \leq B \sum_{n\in\mathbb{N}} |k_n|^2$$

A kinder notion to the orthonormal basis is Riesz basis. When we substitute Riesz basis for orthonormal basis in the definition of an MRA, we get Riesz MRA.

3. Translates of Riesz Bases

Let us begin this section with a necessary condition for the translates of a function to be linearly independent

Lemma 3.1. *Let* $\phi^{\mathfrak{M}}, \tilde{\phi}^{\mathfrak{M}} \in L^2(\mathbb{R})$ *be given. Then* $\left\{\phi^{\mathfrak{M}}_{0,\lambda}(t) = \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$, *is biorthogonal to* $\left\{\tilde{\phi}^{\mathfrak{M}}_{0,\lambda}(t) = \tilde{\phi}(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$ *if and only if*

$$\sum_{\lambda\in\Lambda} \hat{\phi}\left(\frac{\zeta}{B} + \lambda\right) \overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{B} + \lambda\right)} = 1 \quad a.e\ \zeta \in \mathbb{R}.$$

Proof. For $\lambda, \sigma \in \Lambda$, it follows that $\left\langle \phi^{\mathfrak{M}}_{0,\lambda}, \tilde{\phi}^{\mathfrak{M}}_{0,\sigma} \right\rangle = e^{-\frac{i\pi A}{B}(\lambda^2-\sigma^2)}\delta_{\lambda,\sigma} \Leftrightarrow \left\langle \phi^{\mathfrak{M}}, \tilde{\phi}^{\mathfrak{M}}_{0,\sigma} \right\rangle = e^{\frac{i\pi A}{B}(\sigma^2)}\delta_{0,\sigma}$. Now in LCT domain we have

$$\delta_{\lambda,\sigma} = \frac{1}{B}\int_{\mathbb{R}} \widehat{\varphi}\left(\frac{\zeta}{B}\right) \overline{\widehat{\tilde{\varphi}}\left(\frac{\zeta}{B}\right)}\, e^{\frac{2\pi i\zeta}{B}(\lambda-\sigma)}\, d\zeta,$$

$$\delta_{0,\sigma} = \frac{1}{B}\int_{\mathbb{R}} \widehat{\varphi}\left(\frac{\zeta}{B}\right) \overline{\widehat{\tilde{\varphi}}\left(\frac{\zeta}{B}\right)}\, e^{-\frac{2\pi i\zeta\sigma}{B}}\, d\zeta.$$

Since $\left\{e^{-\frac{2\pi i\zeta\sigma}{B}} : \sigma \in \Lambda\right\}$ is an orthonormal basis of $L^2\left[0, \frac{B}{2}\right)$,using this fact we obtain the desired result. □

We now provide a sufficient condition for the translates of a function to be linearly independent.

Lemma 3.2. *Let $\phi^{\mathfrak{M}} \in L^2(\mathbb{R})$. Suppose there exists two constants $P, Q > 0$ such that*

$$P \leq \sum_{\lambda\in\Lambda} \left|\hat{\phi}\left(\frac{\zeta}{B}+\lambda\right)\right|^2 \leq Q \quad for\ a.e\ \zeta \in \mathbb{R}. \tag{3.1}$$

Then $\{\phi^{\mathfrak{M}}_{0,\lambda}(t) = \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\}$,*is linearly independent.*

Proof. For the proof of the Lemma, it is sufficient to find another function say $\tilde{\phi}^{\mathfrak{M}}$ whose translates are biorthogonal to $\phi^{\mathfrak{M}}$. Let us define the function $\tilde{\phi}^{\mathfrak{M}}$ by

$$\hat{\tilde{\phi}}(\frac{\zeta}{B}) = \frac{\hat{\phi}(\frac{\zeta}{B})}{\sum_{\lambda\in\Lambda}\left|\hat{\phi}(\frac{\zeta}{B}+\lambda)\right|^2}.$$

By equation (3.1), $\tilde{\phi}^{\mathfrak{M}}$ is well defined. Now

$$\begin{aligned}\sum_{\sigma\in\Lambda}\hat{\phi}\left(\frac{\zeta}{B}+\sigma\right)\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{B}+\sigma\right)} &= \sum_{\sigma\in\Lambda}\hat{\phi}\left(\frac{\zeta}{B}+\sigma\right)\frac{\overline{\hat{\phi}\left(\frac{\zeta}{B}+\sigma\right)}}{\sum_{\lambda\in\Lambda}\left|\hat{\phi}\left(\frac{\zeta}{B}+\lambda+\sigma\right)\right|^2} \\ &= \frac{\sum_{\sigma\in\Lambda}\left|\hat{\phi}\left(\frac{\zeta}{B}+\sigma\right)\right|^2}{\sum_{\nu\in\Lambda}\left|\hat{\phi}\left(\frac{\zeta}{B}+\nu\right)\right|^2} \\ &= 1.\end{aligned}$$

Applying Lemma 3.1, it follows that$\{\phi^{\mathfrak{M}}_{0,\lambda}(t) = \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\}$is linearly independent. This completes the proof of the Lemma. □

Lemma 3.3. *Suppose that the scaling function $\phi^{\mathfrak{M}}_{0,\lambda}$ satisfies inequality (3.1). Also let $f = \sum_{\lambda\in\Lambda} a_\lambda \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}$, where $f \in span\{\phi(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\}$ and $\{a_\lambda\}$ is a finite sequence. Define*

$$\Lambda^{\mathfrak{M}}_0\left(\frac{\zeta}{B}\right) = \frac{1}{\sqrt{2N}}\sum_{\lambda\in\Lambda} a^{\mathfrak{M}}_\lambda\, e^{-2\pi i\lambda\zeta/B}.$$

then

$$P\int_0^{B/2}\left|\Lambda^{\mathfrak{M}}_0\left(\frac{\zeta}{B}\right)\right|^2 d\zeta \leq \|f\|_2^2 \leq Q\int_0^{B/2}\left|\Lambda^{\mathfrak{M}}_0\left(\frac{\zeta}{B}\right)\right|^2 d\zeta.$$

Proof. Since $f(t)=\sum_{\lambda\in\Lambda} a_\lambda \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}$

Taking LCT on both sides we have,

$$\mathcal{L}_M\big[f(t)\big](\zeta)=\widehat{f}\left(\frac{\zeta}{B}\right)=\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{B}\right)\widehat{\phi}\left(\frac{\zeta}{B}\right)$$

Now by Parseval's relation associated to LCT

$$\begin{aligned}\int_{\mathbb{R}}\big|f(t)\big|^2dt &= \int_{\mathbb{R}}\left|\widehat{f}\left(\frac{\zeta}{B}\right)\right|^2 d\zeta\\ &= \int_{\mathbb{R}}\left|\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{B}\right)\right|^2\left|\widehat{\phi}\left(\frac{\zeta}{B}\right)\right|^2 d\zeta\\ &= \int_0^{B/2}\left|\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{B}\right)\right|^2\sum_{\lambda\in\Lambda}\left|\hat{\phi}\left(\frac{\zeta}{B}+\lambda\right)\right|^2 d\zeta\end{aligned}$$

.

Hence, using inequality (3.1), we get the desired result. □

we are now ready to prove the main result of this section which shows that how the translates of a biorthogonal function forms the Riesz basis.

Theorem 3.4. *Let* $\big\{\phi_{0,\lambda}^{\mathfrak{M}}(t)=\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\big\}$*, be a Riesz basis for its closed linear span. Assume that there exists a function* $\big\{\tilde{\phi}_{0,\lambda}^{\mathfrak{M}}(t)=\tilde{\phi}(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\big\}$*, which is biorthogonal to* $\big\{\phi_{0,\lambda}^{\mathfrak{M}}(t)=\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\big\}$*,. Then for every* $f\in\overline{span}\Big\{\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\Big\}$*, we have*

$$f=\sum_{\lambda\in\Lambda}\Big\langle f,\tilde{\phi}(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\Big\rangle\,\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}; \tag{3.2}$$

and there exists constants $P,Q>0$ *such that*

$$P\big\|f\big\|_2^2\leq\sum_{\lambda\in\Lambda}\left|\left\langle f,\hat{\tilde{\phi}}(\frac{\zeta}{B}-\lambda)\right\rangle\right|^2\leq Q\big\|f\big\|_2^2. \tag{3.3}$$

Proof. We first prove (3.2) and (3.3) for any $f\in\operatorname{span}\big\{\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\big\}$ and then generalize it to $\overline{\operatorname{span}}\big\{\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\big\}$. Assume that $f\in\operatorname{span}\big\{\phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2}:\lambda\in\Lambda\big\}$, then there exists a finite sequence $\{a_\lambda : \lambda \in \Lambda\}$ such that $f = \sum_{\lambda\in\Lambda} a_\lambda \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}$. Using biorthogonality, we obtain

$$\begin{aligned}
&\left\langle f, \tilde{\phi}(t-\sigma)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle \\
&= \left\langle \sum_{\lambda\in\Lambda} a_\lambda \varphi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2}, \tilde{\phi}(t-\sigma)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} \right\rangle \\
&= \sum_{\lambda\in\Lambda} a_\lambda \left\langle \phi(t-\lambda)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2}, \tilde{\phi}(t-\sigma)\, e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle \\
&= a_\lambda.
\end{aligned}$$

This proves (3.2). In order to prove (3.3), we make use of Lemma 3.3 to get

$$Q^{-1}\big\|f\big\|_2^2 \leq \int_0^{B/2} \left|\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{B}\right)\right|^2 d\zeta \leq P^{-1}\big\|f\big\|_2^2.$$

Therefore, using the Plancherel formula associated with LCT and the fact that $a_\lambda = \left\langle f, \tilde{\phi}_{0,\lambda}^{\mathfrak{M}}\right\rangle$, we have

$$\int_0^{B/2} \big|\Lambda_0^{\mathfrak{M}}(\frac{\zeta}{B})\big|^2 d\zeta = \sum_{\lambda\in\Lambda} \big|a_\lambda\big|^2 = \sum_{\lambda\in\Lambda} \left|\left\langle f, \tilde{\phi}(t-\lambda) e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2.$$

This proves (3.3). We now generalize the results to $\overline{\text{span}}\left\{\phi(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$. Let us first prove (3.3). For $f \in \overline{\text{span}}\left\{\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$, there exists a sequence $\{f_m : m \in \mathbb{Z}\}$ in span $\left\{\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda\right\}$ such that $\|f_m - f\|_2 \to 0$ as $m \to \infty$. Thus for each $\lambda \in \Lambda$, we have

$$\left\langle f_m, \tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle \to \left\langle f, \tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle \quad \text{as } m \to \infty.$$

So the result holds for each f_m. Therefore,

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \left|\left\langle f, \tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2 &= \sum_{\lambda\in\Lambda} \lim_{m\to\infty} \left|\left\langle f_m, \tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2 \\
&= \lim_{m\to\infty} \sum_{\lambda\in\Lambda} \left|\left\langle f_m, \tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2 \\
&\leq Q \lim_{m\to\infty} \big\|f_m\big\|_2^2 \\
&= Q\big\|f\big\|_2^2.
\end{aligned}$$

Thus, the upper bound holds in (3.3). Further, we have

$$\begin{aligned}&\left\{\sum_{\lambda\in\Lambda}\left|\left\langle f_m,\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2\right\}^{1/2}\\ &\leq\left\{\sum_{\lambda\in\Lambda}\left|\left\langle f_m-f,\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2\right\}^{1/2}\\ &+\left\{\sum_{\lambda\in\Lambda}\left|\left\langle f,\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2\right\}^{1/2}.\end{aligned}$$

As the upper bound in (3.3) holds for $f_m - f$ and lower bound for each f_m, we get

$$P^{1/2}\big\|f_m\big\|_2\leq Q^{1/2}\big\|f_m-f\big\|_2+\left(\sum_{\lambda\in\Lambda}\left|\left\langle f_m,\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2\right)^{1/2},$$

from which, it follows that

$$P\big\|f\big\|_2^2\leq\sum_{\lambda\in\Lambda}\left|\left\langle f,\tilde{\phi}(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}\right\rangle\right|^2.$$

This proves (3.3). Similarly, we can prove (3.2) for $f\in\overline{\text{span}}\left\{\phi(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2)}:\lambda\in\Lambda\right\}$ and the proof of the theorem is complete. □

4. Biorthogonal Nonuniform Wavelets related to LCT

Let $\{V_j^{\mathfrak{M}}:j\in\mathbb{Z}\}$ and $\{\tilde{V}_j^{\mathfrak{M}}:j\in\mathbb{Z}\}$ be biorthogonal LCT-NUMRA's with scaling functions $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$. Then there exists integral periodic functions Λ_0 and $\tilde{\Lambda}_0$ such that $\hat{\phi}(\frac{\zeta}{B})=\Lambda_0^{\mathfrak{M}}(\zeta/2NB)\,\hat{\phi}(\zeta/2NB)$ and $\hat{\tilde{\phi}}(\zeta/B)=\tilde{\Lambda}_0^{\mathfrak{M}}(\zeta/2NB)\,\hat{\tilde{\phi}}(\zeta/2NB)$. Suppose there exists integral periodic functions $\Lambda_\ell^{\mathfrak{M}}$ and $\tilde{\Lambda}_\ell^{\mathfrak{M}}, 1\leq\ell\leq 2N-1$ such that

$$\Lambda^{\mathfrak{M}}(\zeta/B)\overline{\tilde{\Lambda^{\mathfrak{M}}}(\zeta/B)}=1,\tag{4.1}$$

where

$$\Lambda^{\mathfrak{M}}(\zeta/B)=\begin{pmatrix}\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)&\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{1}{4N}\right)&\dots&\Lambda_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{2N-1}{4N}\right)\\ \Lambda_1^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)&\Lambda_1^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{1}{4N}\right)&\dots&\Lambda_1^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{2N-1}{4N}\right)\\ \vdots&\vdots&\ddots&\vdots\\ \Lambda_{2N-1}^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)&\Lambda_{2N-1}^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{1}{4N}\right)&\dots&\Lambda_{2N-1}^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{2N-1}{4N}\right)\end{pmatrix}$$

and

$$\Lambda^{\tilde{\mathfrak{M}}}(\zeta/B) = \begin{pmatrix} \tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right) & \tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{1}{4N}\right) & \cdots & \tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{2N-1}{4N}\right) \\ \tilde{\Lambda}_1^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right) & \tilde{\Lambda}_1^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{1}{4N}\right) & \cdots & \tilde{\Lambda}_1^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{2N-1}{4N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Lambda}_{2N-1}^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right) & \tilde{m}_{2N-1}\left(\frac{\zeta}{2NB}+\frac{1}{4N}\right) & \cdots & \tilde{\Lambda}_{2N-1}^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{2N-1}{4N}\right) \end{pmatrix}.$$

For $1 \le \ell \le 2N-1$, define the associated wavelets as $\psi_\ell^{\mathfrak{M}}$ and $\tilde{\psi}_\ell^{\mathfrak{M}}$ by

$$\hat{\psi}_\ell^{\mathfrak{M}}\left(\frac{\zeta}{B}\right) = \Lambda_\ell^{\mathfrak{M}}(\zeta/2NB)\,\hat{\phi}(\zeta/2NB)$$

and

$$\hat{\tilde{\psi}}_\ell^{\mathfrak{M}}\left(\frac{\zeta}{B}\right) = \tilde{\Lambda}_\ell^{\mathfrak{M}}(\zeta/2N)\,\hat{\tilde{\phi}}(\zeta/2NB).$$

Definition 4.1. A pair of LCT-NUMRA's $\{V_j^{\mathfrak{M}} : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j^{\mathfrak{M}} : j \in \mathbb{Z}\}$ with scaling functions $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$ respectively are said to be dual to each other if $\{\phi^{\mathfrak{M}}(\cdot - \lambda) : \lambda \in \Lambda\}$ and $\{\tilde{\phi}^{\mathfrak{M}}(\cdot - \lambda) : \lambda \in \Lambda\}$ are biorthogonal.

Definition 4.2. Let $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$ be scaling functions for dual LCT-NUMRA's. For each $j \in \mathbb{Z}$, define the operators $P_j^{\mathfrak{M}}$ and $\tilde{P}_j^{\mathfrak{M}}$ on $L^2(\mathbb{R})$ by

$$P_j^{\mathfrak{M}} f = \sum_{\lambda \in \Lambda} \langle f, \tilde{\phi}_{j,\lambda}^{\mathfrak{M}} \rangle \phi_{j,\lambda}^{\mathfrak{M}} \quad \text{and} \quad \tilde{P}_j^{\mathfrak{M}} f = \sum_{\lambda \in \Lambda} \langle f, \phi_{j,\lambda}^{\mathfrak{M}} \rangle \tilde{\phi}_{j,\lambda}^{\mathfrak{M}},$$

respectively. Here $\phi_{j,\lambda}^{\mathfrak{M}} = \delta_j \phi^{\mathfrak{M}}(\cdot - \lambda)$ and $\tilde{\phi}_{j,\lambda}^{\mathfrak{M}} = \delta_j \tilde{\phi}^{\mathfrak{M}}(\cdot - \lambda)$. Same is the case with $\psi_{j,\lambda}^{\mathfrak{M}}$ and $\tilde{\psi}_{j,\lambda}^{\mathfrak{M}}$. It is easy to verify that these operators are uniformly bounded on $L^2(\mathbb{R})$ and both the series are convergent in $L^2(\mathbb{R})$.

Remark 4.3. The operators $P_j^{\mathfrak{M}}$ and $\tilde{P}_j^{\mathfrak{M}}$ satisfy the following properties.

(a) $P_j^{\mathfrak{M}} f = f$ if and only if $f \in V_j^{\mathfrak{M}}$ and $\tilde{P}_j^{\mathfrak{M}} f = f$ if and only if $f \in \tilde{V}_j^{\mathfrak{M}}$.

(b) $\lim_{j \to \infty} \left\| P_j^{\mathfrak{M}} f - f \right\|_2 = 0$ and $\lim_{j \to -\infty} \left\| P_j^{\mathfrak{M}} f \right\|_2 = 0$ for every $f \in L^2(\mathbb{R})$.

We now show that the wavelets associated with dual MRAs are biorthogonal.

Theorem 4.4. *Let $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$ be the scaling functions for dual LCT-NUMRA's and $\psi_\ell^{\mathfrak{M}}$ and $\tilde{\psi}_\ell^{\mathfrak{M}}, 1 \le \ell \le 2N-1$ be the associated wavelets satisfying (4.1). Then, we have the following* *(a)* $\left\{\psi^{\mathfrak{M}}_{\ell,0,\lambda} : \lambda \in \Lambda\right\}$ *is biorthogonal to* $\left\{\tilde{\psi}^{\mathfrak{M}}_{\ell,0,\sigma} : \sigma \in \Lambda\right\}$,

(b) $\left\langle \psi^{\mathfrak{M}}_{\ell,0,\lambda}, \tilde{\phi}^{\mathfrak{M}}_{0,\sigma}\right\rangle = \left\langle \tilde{\psi}^{\mathfrak{M}}_{\ell,0,\lambda}, \phi^{\mathfrak{M}}_{0,\sigma}\right\rangle$, *for all* $\lambda, \sigma \in \Lambda$.

Proof. we have

$$\begin{aligned}
&\sum_{p\in\mathbb{Z}} \hat{\psi}^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{B}+\frac{p}{2}\right)\overline{\hat{\tilde{\psi}}^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{B}+\frac{p}{2}\right)} \\
&= \sum_{p\in\mathbb{Z}}\left\{\Lambda^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)\hat{\phi}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)\right. \\
&\qquad\qquad \left.\times\overline{\tilde{\Lambda}^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)}\,\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)}\right\} \\
&= \sum_{s=0}^{2N-1}\sum_{p\in\mathbb{Z}}\left\{\Lambda^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)\hat{\phi}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)\right. \\
&\qquad\qquad \left.\times\overline{\tilde{\Lambda}^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)}\,\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)}\right\} \\
&= \sum_{s=0}^{2N-1}\left\{\Lambda^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{2NB}+\frac{s}{4N}\right)\overline{\tilde{\Lambda}^{\mathfrak{M}}_{\ell}\left(\frac{\zeta}{2NB}+\frac{s}{4N}\right)}\right\} \\
&= 1.
\end{aligned}$$

Hence, by Lemma 3.1, $\left\{\psi^{\mathfrak{M}}_{\ell,0,\lambda} : \lambda \in \Lambda\right\}$ is biorthogonal to $\left\{\tilde{\psi}^{\mathfrak{M}}_{\ell,0,\lambda} : \lambda \in \Lambda\right\}$. This proves part (a). To prove part (b), we have for, $\lambda, \sigma \in \Lambda$

$$\begin{aligned}
\left\langle \psi^{\mathfrak{M}}_{\ell,0,\lambda}, \tilde{\phi}^{\mathfrak{M}}_{0,\sigma}\right\rangle &= \left\langle \psi_\ell(t-\lambda)e^{-\frac{i\pi A}{B}(t^2-\lambda^2}, \tilde{\phi}(t-\sigma)e^{-\frac{i\pi A}{B}(t^2-\sigma^2}\right\rangle \\
&= e^{i\pi\frac{A}{B}(\lambda^2-\sigma^2)}\int_{\mathbb{R}}\psi_\ell(t-\lambda)\,\overline{\tilde{\phi}(t-\sigma)}\,dt
\end{aligned}$$

In LCT domain we have by Parseval's identity,

$$
\begin{aligned}
&= \frac{e^{i\pi\frac{A}{B}(\lambda^2-\sigma^2)}}{B}\int_{\mathbb{R}} \widehat{\psi_\ell}\left(\frac{\zeta}{B}\right)\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{B}\right)}\, e^{-\frac{2\pi i\zeta}{B}(\lambda-\sigma)}\,d\zeta.\\
&= \frac{e^{i\pi\frac{A}{B}(\lambda^2-\sigma^2)}}{B}\int_{\mathbb{R}} \Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)\hat{\phi}\left(\frac{\zeta}{2NB}\right)\\
&\overline{\tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}\right)}\,\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}\right)}e^{-\frac{2\pi i\zeta}{B}(\lambda-\sigma)}d\zeta\\
&= \frac{e^{i\pi\frac{A}{B}(\lambda^2-\sigma^2)}}{B}\int_0^{B/2}\sum_{p\in\mathbb{Z}}\left\{\Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)\hat{\phi}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)\right.\\
&\qquad \left.\times\overline{\tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)}\,\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)}\right\}e^{-\frac{2\pi i\zeta}{B}(\lambda-\sigma)}d\zeta\\
&= \frac{e^{i\pi\frac{A}{B}(\lambda^2-\sigma^2)}}{B}\int_0^{B/2}\sum_{s=0}^{2N-1}\sum_{p\in\mathbb{Z}}\left\{\Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)\right.\\
&\hat{\phi}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)\\
&\qquad \left.\times\overline{\tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)}\,\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{2}+\frac{s}{4N}\right)}\right\}e^{-\frac{2\pi i\zeta}{B}(\lambda-\sigma)}d\zeta\\
&= \frac{e^{i\pi\frac{A}{B}(\lambda^2-\sigma^2)}}{B}\int_0^{B/2}\sum_{s=0}^{2N-1}\left\{\Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{s}{4N}\right)\overline{\tilde{\Lambda}_0^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{s}{4N}\right)}\right\}\\
&e^{-\frac{2\pi i\zeta}{B}(\lambda-\sigma)}d\zeta\\
&= 0.
\end{aligned}
$$

The dual one can also be shown equal to zero in a similar manner. This proves part (b) and hence completes the proof of the theorem. □

we are now ready to construct the Riesz basis for $L^2(\mathbb{R})$, which is evident from the following results:

Theorem 4.5. *Let $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$ and $\psi_\ell^{\mathfrak{M}}$ and $\tilde{\psi}_\ell^{\mathfrak{M}}, 1 \le \ell \le 2N-1$ be as in Theorem 4.4. Let $\psi_0^{\mathfrak{M}} = \phi^{\mathfrak{M}}$ and $\tilde{\psi}_0^{\mathfrak{M}} = \tilde{\phi}^{\mathfrak{M}}$. Then for every $f \in L^2(\mathbb{R})$, we have*

$$P_1^{\mathfrak{M}} f = P_0^{\mathfrak{M}} f + \sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\langle f, \tilde{\psi}_{\ell,0,\lambda}^{\mathfrak{M}}\rangle \psi_{\ell,0,\lambda}^{\mathfrak{M}} \tag{4.2}$$

and

$$\tilde{P}_1^{\mathfrak{M}} f = \tilde{P}_0^{\mathfrak{M}} f + \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \langle f, \psi_{\ell,0,\lambda}^{\mathfrak{M}} \rangle \tilde{\psi}_{\ell,0,\lambda}^{\mathfrak{M}}. \tag{4.3}$$

where the series in (4.2) and (4.3) converges in $L^2(\mathbb{R})$.

Proof. For the sake of convenience, we will only prove (4.2), as (4.3) is an easy consequence. In particular, we will prove it in the weak sense only. For this, let $f, g \in L^2(\mathbb{R})$. Then, we have

$$\begin{aligned}
&\sum_{\ell=0}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \tilde{\psi}_{\ell,0,\lambda}^{\mathfrak{M}} \right\rangle \overline{\langle g, \psi_{\ell,0,\lambda}^{\mathfrak{M}} \rangle} \\
&= \sum_{\ell=0}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \tilde{\psi}_\ell(t-\lambda)\, e^{-i\pi\frac{A}{B}(t^2-\lambda^2)} \right\rangle \overline{\left\langle g, \psi_\ell(t-\lambda)\, e^{-i\pi\frac{A}{B}(t^2-\lambda^2)} \right\rangle}
\end{aligned}$$

Now in LCT domain after simplyfing above we have

$$= \sum_{\ell=0}^{2N-1} \int_0^{B/2} \left\{ \sum_{p\in\mathbb{Z}} \hat{f}\left(\frac{\zeta}{B}+\frac{p}{2}\right) \overline{\hat{\tilde{\psi}}_\ell\left(\frac{\zeta}{B}+\frac{p}{2}\right)} \right\}$$

$$\times \left\{ \sum_{q\in\mathbb{Z}} \overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q}{2}\right)} \hat{\psi}_\ell\left(\frac{\zeta}{B}+\frac{q}{2}\right) \right\} d\zeta$$

$$= \int_0^{B/2} \sum_{\ell=0}^{2N-1} \left\{ \sum_{p\in\mathbb{Z}} \hat{f}\left(\frac{\zeta}{B}+\frac{p}{2}\right) \overline{\tilde{\Lambda}_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)}\, \overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{4N}\right)} \right.$$

$$\left. \times \sum_{q\in\mathbb{Z}} \overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q}{2}\right)} \Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{q}{4N}\right) \hat{\phi}\left(\frac{\zeta}{2NB}+\frac{q}{4N}\right) \right\} d\zeta$$

$$= \int_0^{B/2} \sum_{\ell=0}^{2N-1} \left\{ \sum_{r=0}^{2N-1} \sum_{p'\in\mathbb{Z}} \hat{f}\left(\frac{\zeta}{B}+\frac{p'}{2}N+\frac{r}{2}\right) \overline{\tilde{\Lambda}_\ell^{\mathfrak{M}}\left(\frac{}{2NB}+\frac{r}{4N}+\frac{p'}{2}\right)} \right.$$

$$\times \overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{r}{4N}+\frac{p'}{2}\right)}$$

$$\times \sum_{s=0}^{2N-1} \sum_{q'\in\mathbb{N}_0} \overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q'}{2}N+\frac{s}{2}\right)} \Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{s}{4N}+\frac{q'}{2}\right)$$

$$\left. \times \hat{\phi}\left(\frac{\zeta}{2NB}+\frac{s}{4N}+\frac{q'}{2}\right) \right\} d\zeta$$

$$= \int_0^{B/2} \sum_{r=0}^{2N-1} \sum_{p'\in\mathbb{Z}} \sum_{s=0}^{2N-1} \sum_{q'\in\mathbb{N}_0}$$

$$\left\{ \sum_{\ell=0}^{2N-1} \overline{\tilde{\Lambda}_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{r}{4N}\right)} \Lambda_\ell^{\mathfrak{M}}\left(\frac{\zeta}{2NB}+\frac{s}{4N}\right) \right\}$$

$$\times \hat{f}\left(\frac{\zeta}{B}+\frac{p'}{2}N+\frac{r}{2}\right) \overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{r}{4N}+\frac{p'}{2}\right)}\, \overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q'}{2}N+\frac{s}{2}\right)}$$

$$\times \hat{\phi}\left(\frac{\zeta}{2NB}+\frac{s}{4N}+\frac{q'}{2}\right) d\zeta$$

$$= \int_0^{B/2} \sum_{p'\in\mathbb{Z}} \sum_{q'\in\mathbb{N}_0} \sum_{s=0}^{2N-1} \hat{f}\left(\frac{\zeta}{B}+\frac{p'}{2}N+\frac{s}{2}\right) \overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{s}{4N}+\frac{p'}{2}\right)}$$

$$\times \overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q'}{2}N+\frac{s}{2}\right)} \hat{\phi}\left(\frac{\zeta}{2NB}+\frac{s}{4N}+\frac{p'}{2}\right) d\zeta$$

$$= \sum_{s=0}^{2N-1} \int_0^{s+B/2} \sum_{p'\in\mathbb{Z}} \sum_{q'\in\mathbb{N}_0} \hat{f}\left(\frac{\zeta}{B}+\frac{p'}{2}N\right) \overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p'}{2}\right)}$$

$$\times \overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q'}{2}N\right)} \hat{\phi}\left(\frac{\zeta}{2NB}+\frac{p'}{2}\right) d\zeta.$$

Moreover, proceeding in same way we have,

$$\begin{aligned}
&\sum_{\lambda\in\Lambda}\left\langle f,\tilde{\phi}^{\mathfrak{M}}_{1,\lambda}\right\rangle\overline{\left\langle g,\phi^{\mathfrak{M}}_{1,\lambda}\right\rangle}\\
&=\int_0^{B/2}\sum_{p\in\mathbb{Z}}\hat{f}\left(\frac{\zeta}{B}+\frac{p}{2}N\right)\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{2}\right)}d\zeta\\
&\qquad\int_0^{B/2}\sum_{q\in\mathbb{Z}}\overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q}{2}N\right)}\hat{\phi}\left(\frac{\zeta}{2NB}+\frac{q}{2}\right)d\zeta\\
&=\int_0^{B/2}\sum_{p\in\mathbb{Z}}\hat{f}\left(\frac{\zeta}{B}+\frac{p}{2}N\right)\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{2}\right)}d\zeta\\
&\qquad\int_0^{1/2}\sum_{q\in\mathbb{Z}}\overline{\hat{g}\left(\frac{\zeta}{B}+\frac{q}{2}N\right)}\hat{\phi}\left(\frac{\zeta}{2NB}+\frac{q}{2}\right)d\zeta\\
&=\int_0^{B/2}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}}\hat{f}\left(\frac{\zeta}{B}+\frac{p}{2}N\right)\overline{\hat{\tilde{\phi}}\left(\frac{\zeta}{2NB}+\frac{p}{2}\right)\hat{g}\left(\frac{\zeta}{B}+\frac{q}{2}N\right)}\\
&\qquad\hat{\phi}\left(\frac{\zeta}{2N}+\frac{q}{2}\right)d\zeta.
\end{aligned}$$

Combing above expressions , we get the desired result. □

Theorem 4.6. *Let $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$ and $\psi^{\mathfrak{M}}_{\ell}$ and $\tilde{\psi}^{\mathfrak{M}}_{\ell}, 1\leq\ell\leq 2N-1$ be as in Theorem 4.1. Then, for every $f\in L^2(\mathbb{R})$, we have*

$$f=\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left\langle f,\tilde{\psi}^{\mathfrak{M}}_{\ell,j,\lambda}\right\rangle\psi^{\mathfrak{M}}_{\ell,j,\lambda}=\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\langle f,\psi^{\mathfrak{M}}_{\ell,j,\lambda}\rangle\tilde{\psi}^{\mathfrak{M}}_{\ell,j,\lambda},\quad(4.4)$$

where the series converges in $L^2(\mathbb{R})$.

Proof. Using Remark 4.3 and Theorem 4.5, proof of Theorem 4.6 follows. □

Theorem 4.7. *Let $\phi^{\mathfrak{M}}$ and $\tilde{\phi}^{\mathfrak{M}}$ be the scaling functions for dual LCT-NUMRA's and $\psi^{\mathfrak{M}}_{\ell}$ and $\tilde{\psi}^{\mathfrak{M}}_{\ell}, 1\leq\ell\leq 2N-1$ be the associated wavelets satisfying the matrix condition (4.1). Then, the collection $\left\{\psi^{\mathfrak{M}}_{\ell,j,\lambda}:1\leq\ell\leq 2N-1, j\in\mathbb{Z},\lambda\in\Lambda\right\}$ and $\left\{\tilde{\psi}^{\mathfrak{M}}_{\ell,j,\lambda}:1\leq\ell\leq 2N-1, j\in\mathbb{Z},\lambda\in\Lambda\right\}$ are biorthogonal*

Proof. First we show that $\left\{\psi^{\mathfrak{M}}_{\ell,j,\lambda}:1\leq\ell\leq 2N-1, j\in\mathbb{Z},\lambda\in\Lambda\right\}$ and $\left\{\tilde{\psi}^{\mathfrak{M}}_{\ell,j,\lambda}:1\leq\ell\leq 2N-1, j\in\mathbb{Z},\lambda\in\Lambda\right\}$ are biorthogonal to each other. For this, we will show that for each ℓ, $1\leq\ell\leq 2N-1$ and $j\in\mathbb{Z}$,

$$\left\langle\psi^{\mathfrak{M}}_{\ell,j,\lambda},\tilde{\psi}^{\mathfrak{M}}_{\ell,j,\sigma}\right\rangle=e^{-i\pi\frac{A}{B}(\lambda^2-\sigma^2)}\delta_{\lambda,\sigma}.\quad(4.5)$$

We have in fact already proved (4.5) for $j = 0$. For $j \neq 0$, we have
$\left\langle \psi^{\mathfrak{M}}_{\ell,j,\lambda}, \tilde{\psi}^{\mathfrak{M}}_{\ell,j,\sigma} \right\rangle = \left\langle P^{\mathfrak{M}}_{-j}\psi^{\mathfrak{M}}_{\ell,0,\lambda}, P^{\mathfrak{M}}_{-j}\tilde{\psi}^{\mathfrak{M}}_{\ell,0,\sigma} \right\rangle = \left\langle \psi^{\mathfrak{M}}_{\ell,0,\lambda}, \tilde{\psi}^{\mathfrak{M}}_{\ell,0,\sigma} \right\rangle = e^{-i\pi \frac{A}{B}(\lambda^2-\sigma^2)}\delta_{\lambda,\sigma}$.
For fixed $\lambda, \sigma \in \Lambda$ and $j, j' \in \mathbb{Z}$ with $j < j'$, we claim that

$$\left\langle \psi^{\mathfrak{M}}_{\ell,j,\lambda}, \tilde{\psi}^{\mathfrak{M}}_{\ell',j',\sigma} \right\rangle = 0. \tag{4.6}$$

As $\psi^{\mathfrak{M}}_{\ell,0,\lambda} \in V^{\mathfrak{M}}_1$, hence $\psi^{\mathfrak{M}}_{\ell,j,\lambda} = P^{\mathfrak{M}}_{-j}\psi^{\mathfrak{M}}_{\ell,0,\lambda} \in V^{\mathfrak{M}}_{j+1} \subseteq V^{\mathfrak{M}}_{j'}$. Therefore, it is enough to show that $\tilde{\psi}^{\mathfrak{M}}_{\ell',j',\sigma}$ is orthogonal to every element of $V^{\mathfrak{M}}_{j'}$. Let $f \in V^{\mathfrak{M}}_{j'}$. Since $\left\{\phi^{\mathfrak{M}}_{j',\lambda} : \lambda \in \Lambda\right\}$ is a Riesz basis for $V^{\mathfrak{M}}_{j'}$, hence there exists an l^2-sequence $\left\{d_\lambda : \lambda \in \Lambda\right\}$ such that $f = \sum_{\lambda\in\Lambda} d_\lambda \phi^{\mathfrak{M}}_{j',\lambda}$ in $L^2(\mathbb{R})$. Using part (b) of Remark 4.3, we have

$$\left\langle \tilde{\psi}^{\mathfrak{M}}_{\ell',j',\sigma}, \phi^{\mathfrak{M}}_{j',\lambda} \right\rangle = \left\langle P^{\mathfrak{M}}_{-j'}\tilde{\psi}^{\mathfrak{M}}_{\ell',0,\sigma}, P^{\mathfrak{M}}_{-j'}\varphi_{0,\lambda} \right\rangle = 0.$$

Therefore,

$$\left\langle \tilde{\psi}^{\mathfrak{M}}_{\ell',j',\sigma}, f \right\rangle = \left\langle \tilde{\psi}^{\mathfrak{M}}_{\ell',j',\sigma}, \sum_{\lambda\in\Lambda} d_\lambda \phi^{\mathfrak{M}}_{j',\lambda} \right\rangle = \sum_{\lambda\in\Lambda} d_\lambda \left\langle \tilde{\psi}^{\mathfrak{M}}_{\ell',j',\sigma}, \phi^{\mathfrak{M}}_{j',\lambda} \right\rangle = 0.$$

This completes the proof. □

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