Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(2)(2022), 448-460 (Research paper)

Continuous $*$-controlled frames in Hilbert C^*-modules

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Abstract. In the present paper, we introduce the notion of continuous $*$-frames and $*$-C-controlled frames in Hilbert C^*-modules. We present some results of continuous frames in the view of $*$-C-controlled frames in Hilbert C^*-modules. Also we define $*$-(C, C')-frames and investigate multiplier operators for these frames.

Keywords: Hilbert C^*-module, $*$-controlled frame, continuous $*$-controlled frame, multiplier operator.

2020 Mathematics subject classification: 42C15; Secondary 06D22.

1. Introduction

In 1952, the concept of discrete frames for Hilbert spaces was introduced by Duffin and Schaeffer [9] to study some problems in non-harmonic Fourier series. Then Daubecheies, Grassman and Mayer [8] reintroduced and developed them. Various generalizations of frames e.g. frames of subspaces, wavelet frames, g-frames, weighted and controlled frames have developed, [5, 17, 23, 24]. Frame theory has used in many fields such as filter bank theory, image processing, etc. we refer to [6] for an introduction to frame theory in Hilbert spaces and its applications. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure were proposed

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Received: 28 June 2021
Revised: 25 July 2022
Accepted: 11 August 2022

by G. Kaiser [13] and independently by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames. Also, controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces, see [3].

Frank and Larson [10] presented a general approach to the frame theory in Hilbert C^*-modules. We refer the readers for a discussion of frames in Hilbert C^*-modules to Refs. [2, 11, 14]. As Hilbert C^*-module, it is a generalization of Hilbert spaces in that it allows the inner product to take values in a C^*-algebra rather than the field of complex numbers. There are many differences between Hilbert C^*-modules and Hilbert spaces. For example, we know that any closed subspace in a Hilbert space has an orthogonal complement, but it is not true for Hilbert C^*-module. Thus it is more difficult to make a discussion the theory of Hilbert C^*-modules than that of Hilbert spaces in general. We refer the readers to [15, 18] for more details on Hilbert C^*-modules. The theory of frames has extended from Hilbert spaces to Hilbert C^*-modules [10, 12, 16, 19, 20, 21, **?**, 22].

The paper is presented as follows. First, we recall the basic definitions and some notations about Hilbert C^*-modules, and we also give some properties of them so that we will use them in the later sections. In Section 2, we introduce the notion of continuous $*$-frames and continuous $*$-C-controlled frames in Hilbert C^*-modules. We present some results of frames in the view of continuous $*$-controlled frames. In section 3, we define continuous $*$-(C, C')-controlled frames and then we investigate multiplier operators for continuous $*$-(C, C')-controlled Bessel maps.

First, we recall some definitions and basic properties of Hilbert C^*-modules. Throughout this paper, $\mathcal{A}$ shows a C^*-algebra.

Definition 1.1. A pre-Hilbert module over C^*-algebra $\mathcal{A}$ is a complex vector space U which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:
(i) $\langle f, f \rangle \geq 0$,
(ii) $\langle f, f \rangle = 0$ iff $f = 0$,
(iii) $\langle f, g \rangle^* = \langle g, f \rangle$,
(iv) $\langle af, g \rangle = a\langle f, g \rangle$,
for all $f, g \in U$ and $a, b \in \mathcal{A}$.

A pre-Hilbert $\mathcal{A}$-module U is called Hilbert $\mathcal{A}$-module if U is complete with respect to the topology determined by the norm $\|f\| = \|\langle f, f \rangle\|^{\frac{1}{2}}$.

If $\mathcal{A}$ is a C^*-algebra, then it is a Hilbert $\mathcal{A}$-module with respect to the inner product

$$\langle a, b \rangle = ab^*, \quad (a, b \in \mathcal{A}).$$

Example 1.2. Let $l^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ of elements of a C^*-algebra $\mathcal{A}$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ is convergent in $\mathcal{A}$. Then $l^2(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with respect to the pointwise operations and inner product is defined by

$$\langle \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \rangle = \sum_{n=1}^{\infty} a_n b_n^*.$$

Let U and V be two Hilbert C^*-modules. The set of all bounded $\mathcal{A}$-module maps from U to V is denoted by $Hom_{\mathcal{A}}(U,V)$ and we set $Hom_{\mathcal{A}}(U,U) = End_{\mathcal{A}}(U)$.
Let $T \in Hom_{\mathcal{A}}(U,V)$, T is called adjointable if there exists a map $T^* \in Hom_{\mathcal{A}}(V,U)$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x \in U, y \in V$. The set of all adjointable operators from U to V is denoted by $Hom^*_{\mathcal{A}}(U,V)$ and we set $Hom^*_{\mathcal{A}}(U,U) = End^*_{\mathcal{A}}(U)$. We consider $GL(U)$ as the set of all bounded linear invertible operators with the bounded inverse.

2. Continuous $*$-C-controlled frames

In this section, we introduce continuous controlled frames in Hilbert C^*-modules with C^*-valued bounds, and then we give some results for these frames. We assume that $\mathcal{A}$ is an unital C^*-algebra and, U is a Hilbert $\mathcal{A}$-module.

Let $\mathcal{Y}$ be a Banach space, $(\mathcal{X}, \mu)$ a measure space, and $f : \mathcal{X} \to \mathcal{Y}$ a measurable function. The integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions (see [7, 25]). Since every C^*-algebra and Hilbert C^*-module is a Banach space, hence we can use this integral in these spaces.

Definition 2.1. A mapping $F : \mathcal{X} \to U$ is called a continuous $*$-frame for U if

(i) F is weakly-measurable, i.e, for any $f \in U$, the mapping $\mathcal{X} \longmapsto \langle f, F(\mathcal{X}) \rangle$ is measurable on $\mathcal{X}$.

(ii) There exist strictly nonzero elements A, B in $\mathcal{A}$ such that

$$A\langle f, f \rangle A^* \leq \int_{\mathcal{X}} \langle f, F(x) \rangle \langle F(x), f \rangle d\mu(x) \leq B \langle f, f \rangle B^*, \quad (f \in U). \quad (2.1)$$

Definition 2.2. Let $C \in GL(U)$. A mapping $F : \mathcal{X} \to U$ is called a continuous $*$-C-controlled frame for U if

$(C1)$ F is weakly-measurable, i.e, for any $f \in U$, the mapping $\mathcal{X} \longmapsto \langle f, F(\mathcal{X}) \rangle$ is measurable on $\mathcal{X}$.

$(C2)$ There exist strictly nonzero elements A, B in $\mathcal{A}$ such that

$$A\langle f,f\rangle A^* \leq \int_{\mathcal{X}} \langle f, F(x)\rangle\langle CF(x), f\rangle d\mu(x) \leq B\langle f,f\rangle B^*, \quad (f \in U). \tag{2.2}$$

If $A = B$, the continuous $*$-C-controlled frame is called tight, and if $A = B = 1$ it is called a continuous Parseval $*$-C-controlled frame.
The map F is called a continuous $*$-C-controlled Bessel map with bound B if it has the upper bound condition in (2.2).

Remark 2.3. If $C = \mathcal{I}$, the identity operator on U, then the continuous $*$-C-controlled frame is continuous $*$-frame.
Also, the set of all continuous C-controlled frames can be considered as a subset of the family of Continuous $*$-C-controlled frames. For this, let F be a continuous C-controlled frame for the Hilbert C^*-module U with positive real bounds A, B. Then for $f \in U$, we have

$$(\sqrt{A})1_{\mathcal{A}}\langle f,f\rangle(\sqrt{A})1_{\mathcal{A}} \leq \int_{\mathcal{X}} \langle f, F(x)\rangle\langle CF(x), f\rangle \leq (\sqrt{B})1_{\mathcal{A}}\langle f,f\rangle(\sqrt{B})1_{\mathcal{A}}. \tag{2.3}$$

Hence, F is a continuous $*$-C-controlled frame with C^*-algebra valued bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$.

Example 2.4. Let $U = \mathcal{A} = l^2(\mathbb{C})$ with the $\mathcal{A}$-inner product $\langle \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\rangle = \{a_n\overline{b_n}\}_{n\in\mathbb{N}}$. Consider the linear operator $C : U \to U$ defined as $C\{a_n\}_{n\in\mathbb{N}} = \{\alpha a_n\}_{n\in\mathbb{N}}$, where $\alpha \in \mathbb{R}^+$. Let $(\mathcal{X}, \mu)$ be a measure space in which $\mathcal{X} = [0,1]$ and μ is the Lebesgue measure. Suppose

$$\begin{aligned} F : \mathcal{X} &\to U \\ x &\longmapsto \{\sqrt{3}(\tfrac{1}{2} + \tfrac{1}{n})x\}_{n\in\mathbb{N}}. \end{aligned}$$

If $f = \{a_n\}_{n\in\mathbb{N}} \in U$, then we see that

$$\begin{aligned} \int_{\mathcal{X}} \langle f, F(x)\rangle\langle CF(x), f\rangle d\mu(x) &= \int_{[0,1]} \{\sqrt{3}(\frac{1}{2} + \frac{1}{n})a_n\}_{n\in\mathbb{N}}\{\sqrt{3}\alpha(\frac{1}{2} + \frac{1}{n})\overline{a_n}\}_{n\in\mathbb{N}} x^2 d\mu(x) \\ &= \alpha\{(\frac{1}{2} + \frac{1}{n})^2\}_{n\in\mathbb{N}}\{|a_n|^2\}_{n\in\mathbb{N}} \\ &= \sqrt{\alpha}\{\frac{1}{2} + \frac{1}{n}\}_{n\in\mathbb{N}}\langle\{a_n\}_{n\in\mathbb{N}}, \{a_n\}_{n\in\mathbb{N}}\rangle\sqrt{\alpha}\{\frac{1}{2} + \frac{1}{n}\}_{n\in\mathbb{N}}. \end{aligned}$$

Therefore F is a continuous tight $*$-C-frame with bound $\sqrt{\alpha}\{\frac{1}{2} + \frac{1}{n}\}_{n\in\mathbb{N}}$.

Example 2.5. Let $U = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} : a, b \in \mathbb{C} \right\}$, and $\mathcal{A} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C} \right\}$ which is a C^*-algebra. We define the inner product

$$\begin{aligned} \langle .,. \rangle : U \times U &\to \mathcal{A} \\ (M, N) &\longmapsto M(\overline{N})^t. \end{aligned}$$

This inner product makes U a C^*-module on $\mathcal{A}$. We consider a measure space $(\mathcal{X} = [0,1], \mu)$ whose μ is the Lebesgue measure. Suppose $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, in which $\alpha, \beta > 1$. We define
$C : U \to U$ as $U(M) = AM$ for any $M \in U$. Also $F : \mathcal{X} \to U$ is defined by $F(x) = \begin{pmatrix} \sqrt{3}x & 0 & 0 \\ 0 & 0 & \sqrt{3}x \end{pmatrix}$, for any $x \in \mathcal{X}$. It is clear that C is a positive bounded linear operator, in fact, $\|C\| \leq \max(\alpha, \beta)$, and $C^{-1}(M) = A^{-1}M$, hence $C \in GL(U)$. We have

$$\begin{aligned}
\int_{[0,1]} \langle f, F(x)\rangle \langle CF(x), f\rangle d\mu(x) &= \int_{[0,1]} \begin{pmatrix} 3x^2\alpha|a|^2 & 0 \\ 0 & 3x^2\beta|b|^2 \end{pmatrix} d\mu(x) \\
&= \int_{[0,1]} 3x^2 \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} d\mu(x) \\
&= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}.
\end{aligned}$$

Therefore F is a continuous tight $*$-C-controlled frame with bound $\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}$.

Remark 2.6. Let $C \in GL(U)$. It is straightforward to see that a mapping $F : \mathcal{X} \to U$ is a continuous $*$-C-controlled Bessel map in Hilbert C^*-module E if and only if the operator $S_C f = \int_{x\in\mathcal{X}} \langle f, F(x)\rangle CF(x) d\mu(x)$ is well defined and there exists a strictly nonzero element B in $\mathcal{A}$ such that

$$\int_{x\in\mathcal{X}} \langle f, F(x)\rangle \langle CF(x), f\rangle d\mu(x) \leq B\langle f, f\rangle B^*, \quad (f \in U).$$

By using Remark 2.6 we have the following definition.

Definition 2.7. Let $C \in GL(E)$, and let the mapping $F : \mathcal{X} \to U$ be a continuous $*$-C-controlled Bessel map in Hilbert C^*-module U. The operator $S_C f = \int_{x\in\mathcal{X}} \langle f, F(x)\rangle CF(x) d\mu(x)$ is called a continuous $*$-C-controlled frame operator.

In proof of Theorem 2.9, we need the following lemma.

Lemma 2.8. *Let $F : \mathcal{X} \to U$ be a continuous $*$-frame with $*$-bounds A, B and continuous$*$-frame operator S, i.e. $Sf = \int_{x\in\mathcal{X}} \langle f, F(x)\rangle F(x) d\mu(x)$. Then S is a positive, bounded, selfadjoint and invertible operator, and*

$$\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2.$$

Proof. It is obvious that S is positive. For $f, g \in U$, we have

$$\begin{aligned}\langle Sf, g\rangle &= \Big\langle \int_{x\in\mathcal{X}} \langle f, F(x)\rangle F(x) d\mu(x), g\Big\rangle \\ &= \int_{x\in\mathcal{X}} \langle f, F(x)\rangle \langle F(x), g\rangle d\mu(x) \\ &= \int_{x\in\mathcal{X}} \langle f, F(x)\rangle \langle g, F(x)\rangle^* d\mu(x) \\ &= \Big\langle f, \int_{x\in\mathcal{X}} \langle g, F(x)\rangle F(x) d\mu(x)\Big\rangle \\ &= \langle f, Sg\rangle.\end{aligned}$$

Therefore S is selfadjoint. Similar to the proof of [22, Theorem 2.6], we conclude that S_C is invertible. Now, by (2.1) we get

$$A\langle f, f\rangle A^* \leq \langle Sf, f\rangle \leq B\langle f, f\rangle B^*.$$

Hence $\langle f, f\rangle \leq A^{-1}\langle Sf, f\rangle \leq (A^*)^{-1}$ and $\langle Sf, f\rangle \leq B\langle f, f\rangle B^*$, so we deduce that

$$\|A^{-1}\|^{-2}\|\langle f, f\rangle\| \leq \|\langle Sf, f\rangle\| \leq \|B\|^2\|\langle f, f\rangle\|,$$

for each $f \in U$. Therefore

$$\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2.$$

This completes the proof. □

Now, we get the following theorem.

Theorem 2.9. *Let $C \in GL^+(U)$, and let the mapping $F : \mathcal{X} \to U$ be a continuous $*$-C-controlled frame in Hilbert C^*-module U. Then continuous $*$-C-controlled frame operator S_C is invertible, and $\|A^{-1}\|^{-2} \leq \|S_C\| \leq \|B\|^2$.*

Proof. From the definition of S and S_C, we have $S_C = CS$. Using [15] and Lemma 2.8, S_C is $\mathcal{A}$-linear and bounded. Similar to the proof of [22, Theorem 2.6], we conclude that S_C is invertible and

$$\|A^{-1}\|^{-2} \leq \|S_C\| \leq \|B\|^2.$$

□

We get the following corollary.

Corollary 2.10. *Let $C \in GL(U)$ be adjointable, and let S_C be the frame operator of continuous $*$-C-controlled Bessel map with bound B. Then*

$$\|\int_{\mathcal{X}} \langle f, F(x)\rangle \langle CF(x), f\rangle\| \leq \|B\|^2\|f\|^2, \quad (f \in U). \tag{2.4}$$

Proof. By Lemma 2.6, for any $f \in U$ we have

$$\| \int_{\mathcal{X}} \langle f, F(x) \rangle \langle CF(x), f \rangle \| = \| \langle S_C f, f \rangle \| \leq \| S_C \| \| f \|^2 \leq \| B \|^2 \| f \|^2.$$

□

Remark 2.11. Let $C \in GL(E)$. If $F : \mathcal{X} \to U$ is a continuous $*$-C-controlled frame in Hilbert C^*-module U, then there exist strictly positive numbers a, b such that

$$a\|f\|^2 \leq \| \int_{\mathcal{X}} \langle f, F(x) \rangle \langle CF(x), f \rangle \| \leq b\|f\|^2, \quad (f \in U). \tag{2.5}$$

3. Continuous $*$-(C, C')-controlled frames

In this section, we introduce a family of continuous $*$-controlled frames which contains the set of all continuous $*$-C-controlled frames. In all of this section, we assume that $C, C' \in GL(E)$.

Definition 3.1. The mapping $F : \mathcal{X} \to U$ is called a continuous $*$-(C, C')-controlled frame in Hilbert C^*-module U, if
$(CC1)$ $C'F$ is weakly-measurable, i.e, for any $f \in U$, the mapping $\mathcal{X} \longmapsto \langle f, C'F(\mathcal{X}) \rangle$ is measurable on $\mathcal{X}$.

$(CC2)$ There exists strictly nonzero elements A, B in $\mathcal{A}$ such that

$$A\langle f, f \rangle A^* \leq \int_{\mathcal{X}} \langle f, C'F(x) \rangle \langle CF(x), f \rangle d\mu(x) \leq B\langle f, f \rangle B^*, \quad (f \in U). \tag{3.1}$$

If $C' = I$, then the continuous $*$-(C, C')-controlled frame F is a continuous $*$-C-controlled frame, and so the family of continuous $*$-(C, C')-controlled frames contains all continuous $*$-C-controlled frames.

Remark 3.2. The continuous $*$-(C, C')-controlled frame operator is defined by

$$S_{CC'} f = \int_{\mathcal{X}} \langle f, C'F(x) \rangle CF(x) d\mu(x).$$

Now, if $C, C' \in GL^+(U)$, then as the proof of Lemma 2.8, we conclude that $S_{CC'}$ is an $\mathcal{A}$-linear and bounded operator.

Proposition 3.3. *Let $C \in End^*_{\mathcal{A}}(U)$ be a unitary operator. Then $F : \mathcal{X} \to U$ is a continuous $*$-frame if and only if it is a continuous $*$-(C, C)-controlled frame.*

Proof. Let $F : \mathcal{X} \to U$ be a continuous $*$-(C, C)-controlled frame with $*$-frame bounds A, B. Then, for all $f \in U$,

$$\begin{aligned}
A\langle f, f\rangle A^* &= A\langle C^*Cf, f\rangle A^* \\
&= A\langle Cf, Cf\rangle A^* \\
&\leq \int_{\mathcal{X}} \langle Cf, CF(x)\rangle\langle CF(x), Cf\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \langle f, F(x)\rangle\langle F(x), f\rangle d\mu(x),
\end{aligned}$$

hence

$$A\langle f, f\rangle A^* \leq \int_{\mathcal{X}} \langle f, F(x)\rangle\langle F(x), f\rangle d\mu(x).$$

On the other hand

$$\begin{aligned}
\int_{\mathcal{X}} \langle f, F(x)\rangle\langle F(x), f\rangle d\mu(x) &= \int_{\mathcal{X}} \langle C^*Cf, F(x)\rangle\langle F(x), C^*Cf\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \langle Cf, CF(x)\rangle\langle CF(x), Cf\rangle d\mu(x) \\
&\leq B\langle Cf, Cf\rangle B^* \\
&= B\langle f, f\rangle B^*.
\end{aligned}$$

Therefore F is continuous $*$-frame with $*$-bounds A and B.

Conversely, assume that F is a continuous $*$-frame with $*$-bounds A and B. Then for all $f \in U$, we get

$$\begin{aligned}
\int_{\mathcal{X}} \langle f, CF(x)\rangle\langle CF(x), f\rangle d\mu(x) &= \int_{\mathcal{X}} \langle C^*f, F(x)\rangle\langle F(x), C^*f\rangle d\mu(x) \\
&\leq B\langle C^*f, C^*f\rangle B^* \\
&= B\langle f, f\rangle B^*,
\end{aligned}$$

and

$$\begin{aligned}
A\langle f, f\rangle A^* &= A\langle CC^*f, f\rangle A^* \\
&= A\langle C^*f, C^*f\rangle A^* \\
&\leq \int_{\mathcal{X}} \langle C^*f, F(x)\rangle\langle F(x), C^*f\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \langle f, CF(x)\rangle\langle CF(x), \rangle d\mu(x).
\end{aligned}$$

Therefore F is a continuous $*$-(C, C)-controlled frame with $*$-frame bounds A and B. $\square$

We know that $|a|^2 = a^*a$, for any a in C^*-algebra $\mathcal{A}$. Hence, in the following, we assume that $\mathcal{A}$ is a commutative C^*-algebra. The next result is a motivation to define a continuous controlled multiplier operator.

Theorem 3.4. *Let F and G be continuous $*$-(C,C)-controlled and $*$-(C',C')-controlled Bessel maps with $*$-frame bounds B and B', respectively. Let $m \in \mathcal{L}^\infty(\mathcal{X},\mu)$. The operator*

$$\begin{aligned} M_{m,CF,C'G} : U &\to U \\ f &\longmapsto \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle C'G(x) d\mu(x) \end{aligned}$$

is a well-defined bounded operator.

Proof. For any $f, g \in H$ we have

$$\begin{aligned} \|\langle M_{m,CF,C'G} f, g\rangle\| &= \| \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle\langle C'G(x), g\rangle d\mu(x)\| \\ &\leq \| \left(\int_{\mathcal{X}} |m(x)|^2|\langle f, CF(x)\rangle|^2 d\mu(x)\right)^{\frac{1}{2}} \left(\int_{\mathcal{X}} |\langle C'G(x), g\rangle|^2 d\mu(x)\right)^{\frac{1}{2}} \| \\ &\leq \|m\|_\infty \| \left(\int_{\mathcal{X}} \langle CF(x), f\rangle\langle f, CF(x)\rangle d\mu(x)\right)^{\frac{1}{2}} \| \\ &\quad \| \left(\int_{\mathcal{X}} \langle g, C'G(x)\rangle\langle C'G(x), g\rangle d\mu(x)\right)^{\frac{1}{2}} \| \\ &=\leq \|m\|_\infty \| \left(\int_{\mathcal{X}} \langle f, CF(x)\rangle\langle CF(x), f\rangle d\mu(x)\right)^{\frac{1}{2}} \| \\ &\quad \| \left(\int_{\mathcal{X}} \langle g, C'G(x)\rangle\langle C'G(x), g\rangle d\mu(x)\right)^{\frac{1}{2}} \| \\ &\leq \|m\|_\infty \|\|B\|\|B'\|\|f\|\|g\|. \end{aligned}$$

This shows that

$$\|M_{m,CF,C'G}\| \leq \|B\|\|B'\|.$$

Hence $M_{m,CF,C'G}$ is well-defined and bounded. □

Now, we give the concept of multipliers for continuous $*$-(C,C')-controlled Bessel maps.

Definition 3.5. Let F and G be continuous $*$-(C,C)-controlled and $*$-(C',C')-controlled Bessel maps for U, respectively. Let $m \in \mathcal{L}^\infty(\mathcal{X},\mu)$. The operator

$$\begin{aligned} M_{m,CF,C'G} : U &\to U \\ f &\longmapsto \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle C'G(x) d\mu(x) \end{aligned}$$

is called the continuous $*$-(C, C')-controlled Bessel multiplier of F, G and m.

Let $M_{m,F,G}$ be the continuous $*$-controlled Bessel multiplier of F, G, where F and G are continuous $*$-controlled Bessel frames. We give the following result.

Corollary 3.6. *Let C and C' be unitary operators in $End^*_{\mathcal{A}}(U)$. Suppose F and G be continuous $*$-(C, C)-controlled and $*$-(C', C')-controlled Bessel maps with $*$-frame bounds B and B', respectively. Then*

$$M_{m,CF,C'G} = C' M_{m,F,G} C^*.$$

Proof. Using Proposition 3.3, F and G are continuous $*$-controlled Bessel maps. So we have

$$\begin{aligned}
\langle M_{m,CF,C'G} f, g\rangle &= \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle\langle C'G(x), g\rangle d\mu(x) \\
&= \int_{\mathcal{X}} m(x)\langle C^* f, F(x)\rangle\langle G(x), (C')^* g\rangle d\mu(x) \\
&= \langle M_{m,F,G} C^* f, (C')^* g\rangle \\
&= \langle C' M_{m,F,G} C^* f, g\rangle,
\end{aligned}$$

for all $f, g \in U$. Hence

$$M_{m,CF,C'G} = C' M_{m,F,G} C^*.$$

□

In the next result, we show that under some conditions a continuous $*$-(C, C)-controlled Bessel multiplier could be a positive (invertible) operator.

Proposition 3.7. *Let $C \in GL^+(U)$, and let $m(x) \geq \delta > 0$ a.e., then for any continuous $*$-(C, C)-controlled Bessel map F, the multiplier $M_{m,CF,CF}$ is a positive invertible operator.*

Proof. For any $f \in U$, we have

$$\begin{aligned}
\langle M_{m,CF,CF} f, f\rangle &= \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle\langle CF(x), f\rangle d\mu(x) \\
&= \int_{\mathcal{X}} m(x)|\langle f, CF(x)\rangle|^2 d\mu(x) \geq 0.
\end{aligned}$$

If $m(x) \geq \delta > 0$ *a.e.* and $\|m\|_\infty < \infty$, then

$$\begin{aligned}
0 &\leq \delta \int_{\mathcal{X}} \langle f, CF(x)\rangle\langle CF(x), f\rangle d\mu(x) \\
&\leq \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle\langle CF(x), f\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \langle f, Cm(x)^{\frac{1}{2}}F(x)\rangle\langle Cm(x)^{\frac{1}{2}}F(x), f\rangle d\mu(x) \\
&\leq \|m\|_\infty \int_{\mathcal{X}} \langle f, CF(x)\rangle\langle CF(x), f\rangle d\mu(x) \\
&\leq \|m\|_\infty B\langle f, f\rangle B^*.
\end{aligned}$$

Hence $G = m^{\frac{1}{2}}F$ is a continuous $*$-(C, C)-controlled Bessel map and $S_G = M_{m,CF,CF}$. Therefore the multiplier $M_{m,CF,CF}$ is a positive invertible operator. □

Proposition 3.8. *Let F and G be continuous $*$-(C, C)-controlled and $*$-(C', C')-controlled Bessel maps for U, respectively. Let $m \in \mathcal{L}^\infty(\mathcal{X}, \mu)$. Then*

$$M^*_{m,CF,C'G} = M_{\overline{m},C'G,CF}.$$

Proof. For all $f, g \in U$ we have

$$\begin{aligned}
\langle f, M^*_{m,CF,C'G}g\rangle &= \langle M_{m,CF,C'G}f, g\rangle \\
&= \int_{\mathcal{X}} m(x)\langle f, CF(x)\rangle\langle C'G(x), g\rangle d\mu(x) \\
&= \int_{\mathcal{X}} \langle f, CF(x)\rangle\langle C'G(x), g\rangle m(x) d\mu(x) \\
&= \int_{\mathcal{X}} \Big\langle f, \overline{m(x)}\langle g, C'G(x)\Big\rangle CF(x)\rangle d\mu(x) \\
&= \Big\langle f, \int_{\mathcal{X}} \overline{m(x)}\langle g, C'G(x)\rangle CF(x) d\mu(x)\Big\rangle \\
&= \langle f, M_{\overline{m},C'G,CF}g\rangle.
\end{aligned}$$

□

Remark 3.9. Let $(\mathcal{X}, \mu)$ be a measure space and let F be a continuous $*$-(C, C)-Bessel mapping from $\mathcal{X}$ to U. Then, the operator

$$T_F : \mathcal{L}^2(\mathcal{X}, \mu) \to U$$

weakly defined by

$$\langle T_F m, f\rangle = \int_{\mathcal{X}} m(x)\langle F(x), f\rangle d\mu(x), \quad (f \in U)$$

is well defined, linear and bounded operator. It's adjoint is given by

$$T_F^* : U \to \mathcal{L}^2(\mathcal{X}, \mu), \quad T_F^* f(x) = \langle f, F(x) \rangle \quad (x \in \mathcal{X}).$$

The operator T_F is called the synthesis operator and T_F^* is called the analysis operator of F. It is easy to see that $S_{CC'} = T_{C'F} T_{CF}^*$.

Let F and G be continuous $*$-(C, C) and $*$-(C', C')-controlled Bessel maps for U, respectively. Then one easily shows that

$$M_{m,CF,C'G} = T_{C'G} D_m T_{CF}^*,$$

where $D_m : \mathcal{L}^2(\mathcal{X}, \mu) \to \mathcal{L}^2(\mathcal{X}, \mu)$ defined by $(D_m \phi)(x) = m(x)\phi(x)$. It has been proved that if $m \in \mathcal{L}^\infty(\mathcal{X}, \mu)$, then D_m is bounded and $\|D_m\| = \|m\|_\infty$.
Balazs, Bayer, and Rahimi, using Lebesgue's Dominated Convergence Theorem, have shown in [4, Theorem 3.7] that under some conditions $M_{m,F,G} = T_F D_m T_F^*$ is compact for continuous Bessel mappings F and G for U with respect to $(\mathcal{X}, \mu)$. On the framework of continuous $*$-(C, C')-controlled Bessel maps in Hilbert C^*-modules we give the next problem.

Conclusion

In the present paper, the concept of continuous $*$-frames and continuous $*$-C-controlled frames have been given, then some well-known results of continuous frames are extended to $*$-continuous frames. At the end of this paper, continuous $*$-(C, C')-controlled frames have been investigated. Especially, the concept of multipliers for continuous $*$-(C, C')-controlled Bessel maps is defined, and some results of multipliers from continuous Bessel maps to continuous $*$-(C, C')-controlled Bessel maps have extended.

Open problem. Under what conditions $M_{m,CF,C'G}$ is compact for continuous $*$-controlled Bessel mappings F and G for U with respect to $(\mathcal{X}, \mu)$?

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